ON A VOLTERRA INTEGRAL EQUATION WITH DEVIATING ARGUMENTS

Krishnan Balachandran

Department of Mathematics
Bharathiar University
Coimbatore 641 046
Tamil Nadu, India

ABSTRACT

We prove an existence theorem for nonlinear Volterra integral equation with deviating arguments without assuming the Lipschitz condition.

AMS Subject Classification: 45DO5.

Key Words: Existence theorem, nonlinear integral equation, fixed point theorem.

1. INTRODUCTION.

Several researchers have contributed to the theory of Volterra integral equations [5]. Recently Banas [3] proved an existence theorem for nonlinear Volterra integral equation with deviating argument without assuming the Lipschitz condition. Balachandran [1] [2] generalized the results to a more general class of nonlinear Volterra integral equations with deviating arguments. In this paper we shall prove an existence theorem without assuming the Lipschitz condition for nonlinear Volterra equations with deviating arguments. The result generalizes the previous result [3].

2. BASIC ASSUMPTIONS.

Let us introduce the following notations. Let $I = [0, \infty)$, $J = (0, \infty)$ and $E = \{(t, s) : 0 \leq s \leq t < \infty\}$. Let $p(t)$ be a given continuous function defined on the interval I with value in J. We will denote by $C_p = C(I, p(t); \mathbf{R}^n)$ the space of all continuous functions from I into $\mathbf{R}^n$ such that

$$\sup\{|x(t)|p(t) : t \geq 0\} < \infty.$$

It has been shown [6] that C_p forms the real Banach space with respect to the norm

$$||x|| = \sup\{|x(t)|p(t) : t \geq 0\}.$$

Consider the following nonlinear Volterra integral equation with deviating arguments

$$x(t) = \phi(t) + \int_0^t H(t, s, x(\alpha_1(s)), \ldots, x(\alpha_n(s)))ds \tag{1}$$

where x, ϕ and H are n-vectors. Assume the following conditions:

[0]Received: April 1989; Revised: January 1990

i. $H : E \times \mathbf{R}^{n^2} \to \mathbf{R}^n$ is continuous and there exist continuous functions $m : E \to J$, $a : I \to J$ and $b_i : I \to J$ for $i = 1, 2, \ldots, n$ such that

$$|H(t, s, x_1, \ldots, x_n)| \leq m(t, s) + a(t) \sum_{i=1}^{n} b_i(s)|x_i(s)|$$

for any $(t, s) \in E$ and $(x_1, \ldots, x_n) \in \mathbf{R}^{n^2}$.
Let $b(s) = \sum_{i=1}^{n} b_i(s)$ and $L(t) = \int_0^t a(s)b(s)ds$ and take an arbitrary number $M > 0$ and consider the space C_p with $p(t) = [a(t)\exp(ML(t) + t)]^{-1}$.

ii. $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is continuous and there exists a constant A such that

$$|\phi(t)| \leq Aa(t)\exp(ML(t)).$$

iii. There exists a constant $B \geq 0$ such that for any $t \in I$ the following inequality holds

$$\int_0^t m(t, s)ds \leq Ba(t)\exp(ML(t)).$$

iv. $\alpha_i : I \to I$ are continuous functions satisfying the condition $L(\alpha_i(t)) - L(t) \leq K_i$, where K_i, $i = 1, \ldots, n$ are positive constants.

v.

$$\frac{a(\alpha_i(t))}{a(t)} \leq M(1 - A - B)\exp(-MK_i), \quad i = 1, \ldots, n.$$

$$A + B < 1.$$

In what follows we will employ the following criterion of compactness of sets in the space C_p [4].

Lemma. Let Q be a bounded set in the space C_p. If all the functions belonging to Q are equicontinuous on each interval $[0, \eta]$ and $\lim_{\eta \to \infty} \sup\{|x(t)|p(t) : t \geq \eta\} = 0$ uniformly with respect to Q, then Q is relatively compact in C_p.

If $x \in C_p$, we will denote by $\omega(x, h)$ its modulus of continuity on the interval $[0, \eta]$ as

$$\omega(x, h) = \sup\{|x(t) - x(s)| : |t - s| \leq h, \ t, s \in [0, \eta]\}.$$

3. EXISTENCE THEOREM.

Theorem. Under the assumptions (i) to (v) the equation (1) has at least one solution x in the space C_p such that $|x(t) \leq a(t)\exp(ML(t))$ for any $t \geq 0$.

Proof. Consider the following transformation defined on the space C_p by

$$(Fx)(t) = \phi(t) + \int_0^t H(t, s, x(\alpha_1(s)), \ldots, x(\alpha_n(s)))ds. \tag{2}$$

Observe that our assumptions imply that $(Fx)(t)$ is continuous on I and define

$$G = \{x \in C_p : |x(t)| \leq a(t)\exp(ML(t))\}.$$

Obviously G is nonempty, bounded, closed and convex in the space C_p. Now we show that F maps the set G into itself. Take $x \in G$. Then from our assumptions we have

$$\begin{aligned}
|(Fx)(t)| &\leq Aa(t)\exp(ML(t)) + \int_0^t m(t,s) + a(t)\sum_{i=1}^n b_i(s)|x(\alpha_i(s))|ds \\
&\leq (A+B)a(t)\exp(ML(t)) + a(t)\int_0^t \sum_{i=1}^n b_i(s)a(\alpha_i(s))\exp(ML(\alpha_i(s)))ds \\
&\leq (A+B)a(t)\exp(ML(t)) \\
&\quad + a(t)\int_0^t \sum_{i=1}^n b_i(s)a(\alpha_i(s))\exp(M(L(\alpha_i(s)) - L(s)))\exp(ML(s))ds \\
&\leq (A+B)a(t)\exp(ML(t)) \\
&\quad + a(t)(1-A-B)\int_0^t M\sum_{i=1}^n b_i(s)\exp(ML(s))\exp(-MK_i)\exp(MK_i)a(s)ds \\
&= (A+B)a(t)\exp(ML(t)) + (1-A-B)a(t)\int_0^t Ma(s)b(s)\exp(ML(s))ds \\
&= a(t)\exp(ML(t)).
\end{aligned}$$

which shows that $FG \subset G$. Next we show that F is continuous on the set G. For this let us fix $h > 0$ and $x, y \in G$ such that $\|x - y\| \leq h$. Further take an arbitrary fixed $\eta > 0$. Then using the fact that the function H is uniformly continuous on $[0,\eta] \times [0,\eta] \times [-\gamma(\alpha_1(\eta)), \gamma(\alpha_1(\eta))] \times \ldots \times [-\gamma(\alpha_n(\eta)), \gamma(\alpha_n(\eta))]$, where $\gamma(\alpha_i(\eta)) = \max\{a(\alpha_i(s))\exp(ML(\alpha_i(s)) : s \in [0,\eta]\}$, we obtain for $t \in [0,\eta]$

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq \int_0^t |H(t,s,x(\alpha_1(s)),\ldots,x(\alpha_n(s))) - H(t,s,y(\alpha_1(s)),\ldots,y(\alpha_n(s)))|ds \\
\text{(3)} \qquad &\leq \beta(h)
\end{aligned}$$

where $\beta(h)$ is some continuous function such that $\lim_{h\to 0}\beta(h) = 0$. Let us take $t \geq \eta$. Then

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)|[a(t)\exp(ML(t)+t)]^{-1} &\leq [|(Fx)(t)| + |(Fy)(t)|][a(t)\exp(ML(t))]^{-1}e^{-t} \\
&\leq 2e^{-t}.
\end{aligned}$$

Hence for sufficiently large η we have

$$|(Fx)(t) - (Fy)(t)|p(t) \leq h \quad \text{for } t \geq \eta. \tag{4}$$

Thus in view of (3) and (4) we deduce that F is continuous on the set G.

Now we show that FG is relatively compact. In the set G, note that $|(Fx)(t)|p(t) \leq e^{-t}$ which implies that

$$\lim_{\eta\to\infty} \sup\{|(Fx)(t)|p(t) : t \geq \eta\} = 0 \tag{5}$$

uniformly with respect to $x \in G$.

Furthermore, let us fix $h > 0$, $\eta > 0$, $t, s \in [0,\eta]$ such that $|t - s| \leq h$. Then for $x \in G$, we get

$$|(Fx)(t) - (Fx)(s)| \leq |\phi(t) - \phi(s)| + |\int_0^t H(t,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u)))du$$

$$
\begin{aligned}
& - \int_0^s H(s,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u)))du| \\
\leq\ & \omega(\phi,h) + |\int_0^t H(t,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u)))du \\
& - \int_0^s H(s,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u)))du| \\
& + \int_0^s |H(t,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u))) \\
& - H(s,u,x(\alpha_1(u)),\ldots,x(\alpha_n(u)))|du \\
\leq\ & \omega(\phi,h) + h\max\{m(t,s) \\
& + a(t)\sum_{i=1}^{s} b_i(s)p(\alpha_i(s)) : 0 \leq s \leq t \leq \eta\} + \eta\omega(H,h)
\end{aligned}
$$

which approaches 0 as $h \to 0$, since $\lim_{h\to 0}\omega(\phi,h) = \lim_{h\to 0}\omega(H,h) = 0$.

We deduce that all the functions belonging to the set FG are equicontinuous on each interval $[0,\eta]$ and by (5), using the lemma we infer that FG is relatively compact. Then the Schauder fixed point theorem guarantees that F has a fixed point $x \in G$ such that $(Fx)(t) = x(t)$.

REFERENCES.

1. Balachandran, K., "An Existence Theorem for Nonlinear Volterra Integral Equation," *J. Math. Phy. Sci.*, to appear.

2. Balanchandran, K., 1989, "Existence of Solution for Nonlinear Volterra Integral Equation with Deviating Argument," *J. Math. Phy. Sci.*, 23:201–206.

3. Banas, J., 1986, "An Existence Theorem for Nonlinear Volterra Integral Equation with Deviating Argument," *Rendiconti del Circolo Matematico di Palermo*, Series 2:35:82–89.

4. Banas, J. and Goebel, K., *Measure of Noncompactness in Banach Spaces*, Marcel Dekker, Inc., New York, 1980.

5. Burton, T.A., *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

6. Corduneau, C., *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.