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STABILITY OF VOLTERRA SYSTEM WITH IMPULSIVE EFFECT[1]

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ABSTRACT

Sufficient conditions for uniform stability and uniform asymptotic stability of impulsive integrodifferential equations are investigated by constructing a suitable piecewise continuous Lyapunov-like functionals without the decresent property. A result which establishes no pulse phenomena in the given system is also discussed.

Key words: Uniform stability, asymptotic stability beating, Lyapunov functional, fundamental matrix, integral curves, surfaces.

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1. INTRODUCTION: The stability analysis of ordinary differential equations with impulsive effect has been the subject of many investigations [1, 2,4] in recent years and various interesting results are reported. However, much has not been developed in this direction of integro-differential equations with impulsive effect except for a few [3, 5] in which the impulsive integral inequalities are used. The purpose of this paper is to investigate sufficient conditions for uniform stability and uniform asymptotic stability of Linear integro-differential equations by employing the piecewise continuous Liapunov functional without the decrescent property. It is also proved that every solution of the integro-differential system meets any given surface exactly once and thus there exists no pulse phenomena in the system.

Let the hyper surfaces σ_k be defined by the equations

$$\sigma_k \equiv t = \tau_k(x),\ 0<\tau_1(x)<\ldots<\tau_k(x)<\ldots.$$

where $\tau_k(x)\to\infty$ as $k\to\infty$.

Pc^+ denote the class of piecewise continuous functions from

$R_+^2 \to R^{n^2}$ with discontinuities of the first kind at $t \neq \tau_k(x)$, $k = 1,2\ldots$ and left continuous at $t = \tau_k$.

Let $\tau_0(x) \equiv 0$ for $x \in R_+$ and

$$G_k = \{(t,x) \in I \times R^n: \tau_{k-1}(x) < t < \tau_k(x)\},\ k = 1,2\ldots$$

The function V: $I \times R^+ \to R$ belongs to class V_0 if:

(i) The function V is continuous on each of the sets G_k and $V(t,0) = 0$

(ii) For each k = 1,2... and $(t_0, x_0) \in G_k$ there exists finite limits

$$V(t_0 - 0,x_0) = \lim_{\substack{(t,x) \to (t_0,x_0) \\ (t,x) \in G_k}} V(t,x); \quad V(t_0,x_o) = \lim_{\substack{(t,x) \to (t_0,x_0) \\ (t,x) \in G_k}} V(t,x)$$

and $V(t_0 - 0,x_0) = V(t_0,x_0)$ is satisfied.

Also if $(t_0,x_0) \in G_k$ then $V(t_0 + 0, x_0) = V(t_0,x_0)$

Let $V \in V_0$ For $(t,x) \in \bigcup_{1}^{\infty} G_k$, D^+V is defined as

$$D^+V(t,x) = \lim_{h^- \to 0^+} \operatorname{Sup} \frac{1}{h} [V(t + h, x(t + h)) - V(t, x(t))]$$

2. Consider the impulsive integro-differential system

$$X' = (A(t)x + \int_{t_0}^{t} k(t,s)x(s)\,ds \qquad , t \neq \tau_k(x), k=1,2..$$

$$\Delta x|_{t=\tau_k}(x) = I_k(x) \qquad ,x(t_0) = x_0 \tag{2.1}$$

where $A \in PC^+ [R_+,R^{n2}]$, $K \in PC^+[R_+^2, R^{n2}]$, and $I_k(0) = 0$, $t \geq t_0$, k=1,2...

Let us consider:
$$\begin{aligned} x' &= A(t)x \qquad t \neq \tau_k(x) \\ \Delta x|_{t=\tau_k(x)} &= B_k(x) \end{aligned} \tag{2.2}$$

where $\det(I + B_k) \neq 0$.

Not let ϕ_k (t,s) be the fundamental matrix of the linear system

$$x' = A(t)x, \ (\tau_{k-1} < t < \tau_k) \tag{2.3}$$

Then the solution of the linear system (2.2) can be written in the form
$x(t,t_0,x_0) = \psi(t,t_0+0)x_0$, where

$$\psi(t,s) = \begin{cases} \phi_k(t,s) & \text{for } \tau_{k-1}<s<t<\tau_k \\ \phi_{k+1}(t,t_k)(I+B_k)\phi_k(t_k,s) & \text{for } \tau_{k-1}<s<\tau_k<t<\tau_{k+1} \\ \phi_k(t,t_k)(I+B_k)^{-1}\phi_{k+1}(t_k,s) & \text{for } \tau_{k-1}<s<\tau_k<t<\tau_{k+1} \end{cases}$$

The following Lemma gives sufficient conditions for the absence of beating.

Lemma 2.1: Let the following conditions be satisfied for $|x|<\rho$

(i) $|\phi_k(t,s)| \le \alpha \bar{e}^{\lambda(t-s)}$ for $0\le s<t<\infty$ for all k.

(ii) $|A(t)|\le\beta$ for $t\ge 0$.

(iii) $|(I + B_k)| \le \gamma$ where I is the identity matrix.

(iv) $|K(t,s)|| \le M\bar{e}^{\sigma(t-s)}$ where $M>0, \sigma>0$ for $0\le s\le t<\infty$

(v) There exists a number $\hbar>0$ such that

$$\sup_{\substack{0\le s\le 1 \\ |x|\le\hbar}} <\frac{\partial\tau_k}{\partial x}(x+sI_k(x))>\le 0,\ k=1,2\ldots$$

and $$\sup_{|x|<\hbar} \left|\frac{\partial\tau_k(x)}{\partial x}\right| \le N,\ k=1,2\ldots$$

(vii) $\left(\beta + \frac{M}{\sigma}\right)\rho N<1$

Then there exists a number $\rho \leq \bar{h}$ such that if x(t) is a solution of (2.1), which lies in the ball $\{x \in R^n : |x| \leq \rho\}$ for $0 \leq t \leq T, T>0$, then the integral curve $\{(t,x(t))): t \in [0,T]\}$ meets the hyper surface $t = \tau_k(x)$ exactly once.

<u>Proof:</u> Let $F(t,s) = A(t)x + \int_{t_0}^{t} K(t,s)x(s)\,ds$

If $|x| \leq \rho$ then from (2.1) and (i), (ii), (iii), and (iv) we get

$$\begin{aligned}
|F(t,s)| &\leq |A(t)x| + \int_{t_0}^{t} |K(t,s)|\,|x(s)|\,ds \\
&\leq \beta|x| + M\int_{t_0}^{t} \overline{e}^{\sigma(t-s)} \sup_{0 \leq s \leq T} |x(s)|\,ds \\
&\leq \beta|x| + M\rho\int_{t_0}^{t} \overline{e}^{\sigma(t-s)}\,ds \\
&< \left(\beta + \frac{M}{\sigma}\right)\rho
\end{aligned}$$

Now assume that some solution x(t) of (2.1) under the above assumptions meets some surface $t = \tau_k(x)$ more than once.

Let $t = t_j$ be the point at which the solution first meets the surface $t = \tau_k(x)$ for some j and again another closest hit at $t = t^*$ such that $t^* - t_j > 0$. Then we have

$t_j = \tau_k(x(t))$ and $t^* = \tau_k(x(t^*))$ where $t_0 < t_j < t^*$

Then the solution satisfies the integral equation

$$x(t) = x_j + I_k(x_j) + \int_{t_j}^{t} F(s,x(s))\,ds$$

Let $h = \int_{t_j}^{t^*} F(s,x(s))\, ds.$

Define the function $x(s) = \tau_k(x_j+I_k(x_j)+sh) + \tau_k(x_j+sI_k(x_j))$

for $s \in [0,1]$. Then by mean value theorem

$$x(1)-x(0) = \int_0^1 x'(s)\,ds$$

$$\begin{aligned} t^*-t_j &= \frac{\partial\tau_k}{\partial x}(x_j+I_k(x_j)+h)-\tau_k(x_j) \\ &= \int_0^1 <\frac{\partial\tau_k}{\partial x}(x_j+I_k(x_j) + sh)\,ds \cdot (I_k(x_j)+h) \\ &= \int_0^1 <\frac{\partial\tau_k}{\partial x}(x_j+I_k(x_j)+sh), h>ds \\ &\qquad +\int_0^1 <\frac{\partial\tau_k}{\partial x}(x_j+sI_k(x_j)), I_k(x_j)>ds \end{aligned} \tag{2.4}$$

Since we have $\left|\frac{\partial\tau_K(x)}{\partial x}\right| \leq N$ and $|F(s,x(s))| < \left(\beta+\frac{M}{\sigma}\right)\rho$

By Cauchy-Schwartz inequality the first integral on the right hand side of (2.4) satisfies

$$\int_0^1 < \frac{\partial\tau_k}{\partial x}(x_j) + I_k(x_j) + sh) ; h> ds \leq N\left(\beta+\frac{M}{\sigma}\right)\rho\,(t^*-t_j)$$

hence we have

$$\left[1-N\left(\beta+\frac{M}{\sigma}\right)\rho\right](t^*-t_j) \leq \int_0^1 < \frac{\partial\tau_k}{\partial x}(x_j+sI_k(x_j)), I_k(x_j) > ds$$

Since $\left(\beta + \frac{M}{\sigma}\right)\rho N<1,$ in view of hypothesis (v) this leads to contradiction which completes the proof of the lemma.

Define the matrix G(t) as

$$G(t)= \int_t^\infty \psi^\tau(s,t)\,\psi(s,t)\, ds \text{ where } \psi^\tau \text{ is the transpose of } \psi.$$

Clearly G(t) is symmetric. And define $W(t,x) = < G(t)x,x >^{1/2}$ and

$V \varepsilon V_0$ for $(t,x) \varepsilon (t_{k-1}, t_k) \times R^n$ as

$$V(t,x) = W(t,x) + \beta \int_{t_0}^{t} \int_{t}^{\infty} \|K(u,s)\| du \|x(s)\| ds.$$

Theorem 2.1: Assume the following conditions hold.

(i) $$L\|x\| \le \langle G(t)x,x\rangle^{\frac{1}{2}} \le \frac{1}{2\hat{M}}\|x\|$$

(ii) $$\|G(t)x\| \le \hat{R}\langle G(t)x,x\rangle^{\frac{1}{2}}$$

(iii) $$-\hat{M} + \beta\int_{t}^{\infty} \|K(u,t)\| \, du \le 0, \beta \ge \hat{R}$$

(iv) $\|x\| > \|x + I_k(x)\|$ and

$$\langle G(t)x,x\rangle^{\frac{1}{2}} > \langle G(t)(x + I_k(x)), (x + I_k(x))\rangle^{\frac{1}{2}}$$ where

L, $\hat{M}$, $\hat{R}$, and β are positive real numbers

Then the zero solution of (2.1) is uniformly stable.

Proof: Let $W(t,x) = \langle G(t)x,x\rangle^{\frac{1}{2}}$

$$W'(t,x) = \frac{\langle G'(t)x,x\rangle}{2\langle G(T)x,\rangle^{\frac{1}{2}}} + \frac{\langle 2G(t))x,x\rangle}{2\langle G(t)x,x\rangle^{\frac{1}{2}}}$$

we have
$$\frac{\partial\psi(s,t)}{\partial t} = -\psi(s,t)A(t)$$
$$\frac{\partial\psi^{\tau}(s,t)}{\partial t} = -A^T(t)\psi^T(s,t)$$

Hence $$G'(t) = -I - \int_{t}^{\infty} \left[\frac{\partial\psi^{t}(s,t)}{\partial t}\psi(s,t)\psi^T(s,t)\frac{\partial\psi(s,t)}{\partial t}\right]$$

Which implies $$G'(t) = -I - A^T(t)G(t) - G(t)A(t)$$

Hence

$$W'_{(2.1)}(t,X) = \frac{-\langle x,x\rangle}{2\langle G(t)x,x\rangle^{\frac{1}{2}}} + \frac{\langle G(t)x, \int_{t_0}^{t} K(t,s)x(s)\,ds}{\langle G(t)x,x\rangle^{\frac{1}{2}}}$$

$$\text{for } t \neq I_k, (t,x) \varepsilon \bigcup_1^{\infty} G_k$$

Now

$$V'_{(2.1)}(t,x) = \frac{-\langle x,x\rangle}{2\langle G(t)x,x\rangle^{\frac{1}{2}}} + \frac{\langle G(t)x, \int_{t_0}^{t} K(t,S)x(s)\,ds\rangle}{\langle G(t)x,x\rangle^{\frac{1}{2}}}$$

$$+ \beta \int_t^{\infty} \|K(u,t)\|du\|x(t)\| - \beta \int_t^{\infty} \|K(t,s)\|du\|x(s)\|ds \text{ for } t \neq \tau_k (t,x) \varepsilon \bigcup_1^{\infty} G_k$$

by (i) and (ii) we get

$$V'_{(2.1)}(t,x) \leq -\hat{M}\|x\| + \hat{K}\int_{t_0}^{t} \|K(t,s)\|\|x(s)\|ds$$

$$+\beta \int_t^{\infty} \|K(u,t)\|du\|x(t)\| - \beta \int_{t_0}^{t} \|K(t,s)\|du\|x(s)\|ds$$

Hence in view of assumption (iii) it follows that

$$V'_{(2.1)}(t,x(s)) \leq 0 \text{ for } t \neq \tau_k , (t,x) \varepsilon \bigcup_1^{\infty} G_k$$

This implies for t $\neq \tau_k$ that by hypothesis (iv)

$$L\|x(t)\| \leq V(t,x) \leq V(t_0,x_0) \leq W(t_0,x_0) \leq \frac{1}{2\hat{M}}\|x_0\|$$

this gives the uniform stability of (2.1)

Remark 2.1: In the above theorem it is not assumed the descresent property on V.

Theorem 2.2 Assume the following conditions hold for $\|x\| < \rho$

(i) $L\|x\| \leq \langle G(t)x, x\rangle^{\frac{1}{2}} < +\frac{1}{2\hat{M}}\|x\|$

(ii) $\|G(t)x\| < \hat{K}\langle G(t)x, x\rangle^{\frac{1}{2}}$

(iii) $\hat{\gamma} \leq \hat{M} - \beta \int_t^\infty \|K(u,t)\| du$ for some $\hat{\gamma} > 0, \beta > \hat{K}$

(iv) $\|x\| > \|x + I_k(x)\|$ and

$$\langle G(t)x, x\rangle^{\frac{1}{2}} > \langle G(t)(x + I_k(x)), (x + I_k(x))\rangle^{\frac{1}{2}}$$

where L, $\hat{M}$, $\hat{K}$, $\hat{\gamma}$, and β are positve real numbers.
Then the zero solution of (2.1) is uniformly asymptotically stable.

Proof: By Theorem 2.1 the zero solution of (2.1) is uniformly stable. Following the proof of Theorem 2.1 one obtains

$$V'(t,x) \leq -\hat{\gamma}\|x\| \text{ for } t \neq \tau_k, \|x\| < \rho \text{ and } (t,x) \varepsilon \overset{\infty}{\underset{1}{U}} G_k.$$
(2.1)

Let s be the number corresponding to ϵ in the definition of uniforms stability.

Take $T(\epsilon) = \dfrac{\left[\frac{1}{2\hat{M}}\right]}{\hat{\gamma}\delta}\|x_0\|$ where $x(t_0) = x_0$

We now claim that $\|x(t^*, t_0, x_0)\| \leq \delta$ for some $t^* \varepsilon [t_0, t_0 + \tau]$

Whenever $\|x(s)\| < \rho$ for $0 \leq s \leq t_0$.

For if $\|x(t, t_0, x_0)\| > \delta$ for all $t^* \varepsilon [t_0, t_0 + \tau]$, then
By hypotheses (i) and (iv)

$$0 < L\delta = L\|x(t, t_0, x_0)\| \leq V(t, x(t) \leq V(t_0, x_0) + \int_{t_0}^{t} V'_{2.1}(s, x(s))\, ds$$

$$\leq \left[\frac{1}{2\hat{M}}\right]\rho - \hat{\gamma} \int_{t_0}^{t} \|x(s)\| ds$$

put $t = t_0 + T$, then we get

$$0 < L\delta \le \left[\frac{1}{2\hat{M}}\right]\rho - T\gamma\delta$$

$$\le \left[\frac{1}{2\hat{M}}\right]\rho - \frac{\left[\frac{1}{2\hat{M}}\right]}{\gamma\delta}\rho\gamma\delta = 0$$

and thus we have a contradiction.

Hence there exits $t^* \epsilon [t_0, t_0 + \tau]$ such that $\|x(t^*, t_0, x_0)\| < \delta$.

By uniform stability it follows that $\|x(t, t_0, x_0\| < \epsilon$ for all $t > t^*$ or $t \ge t_0 + T$ which completes the proof.

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