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NEURAL NETWORKS WITH MEMORY[1]

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ABSTRACT

This paper is divided into four parts. Part 1 contains a survey of three neural networks found in the literature and which motivate this work. In Part 2 we model a neural network with a very general integral form of memory, prove a boundedness result, and obtain a first result on asymptotic stability of equilibrium points. The system is very general and we do not solve the stability problem. In the third section we show that the neural networks are very robust. The fourth section concerns simplification of the systems from the second part. Several asymptotic stability results are obtained for the simplified systems.

Key words: Neural networks, stability, memory.

AMS (MOS) subject classifications: 34K20.

1. INTRODUCTION

In this paper we consider neural networks with time delay and give conditions to ensure that solutions converge to the equilibrium points of corresponding systems without delays. The proofs are based on construction of Lyapunov functionals having derivatives which satisfy very strong relations. The work may be considered as extensions of results of Hopfield ([5], [6]), Han, Sayeh, and Zhang [4], and Marcus and Westervelt ([11], [12]).

The first model to be considered is that of Hopfield ([5; p. 2555], [6; p.3089]) and it may be described as follows. Hopfield states that most neurons are capable of generating a train of action potentials (propagating pulses of electrochemical activity) when the average potential across their membrane is held well above the normal resting value. For such neurons, u_i is taken to be the mean potential of a neuron from the total effect of its excitatory and inhibitory inputs. It is assumed that neuron i is connected to neuron j; Hopfield takes $V_i = g_i(u_i)$ (the input-output relation) as the short term average of the firing rate of cell i. He states that u_i will lag behind the instantaneous outputs V_j of the other cells because of the

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input capacitance C of the cell membranes, the transmembrane resistance R, and the finite impedance T_{ij}^{-1} between the output V_j and the cell body of cell i. These time lags are ignored by Hopfield, but are of fundamental interest to us for this work.

His system is

$$C_i(du_i/dt) = \sum_j T_{ij}V_j - u_i/R_i + I_i,\ u_i = g_i^{-1}(V_i). \tag{1}$$

All of these functions are evaluated at time t. Here, the input-output relation $V_i = g_i(u_i)$ is a sigmoidal function with $g_i(0) = 0$, $g_i'(x) > 0$, $g_i(u_i) \to \pm 1$ as $u_i \to \pm\infty$. (We have written $(d/dx)(g_i(x)) = g_i'(x)$ and we will sometimes denote $du_i/dt = u_i'$, but the context should clearly indicate the meaning.) The quantity $T_{ij}g_j(u_j)$ represents the electrical current input to cell i due to the present potential of cell j, and T_{ij} is the synapse efficacy. He assumes linear summing. Both $+$ and $-$ signs can occur in T_{ij}. The constant I_i represents any other fixed input to neuron i; Hopfield, as well as other investigators, frequently takes I_i to be zero. Unless a neuron is self connected, $T_{ii} = 0$; Marcus and Westervelt [12], as well as others, treat systems with self connections. If there are no self connections, then it is impossible for the matrix (T_{ij}) to be positive or negative definite, a condition we later require. But we later point out that, mathematically, we may always regard a neuron as being self connected owing to its own current potential. Hopfield [6; p. 3089] regards definiteness of (T_{ij}) as pathological.

If neuron i is connected to neuron j, then neuron j is connected to neuron i and it is frequently assumed, particularly by Hopfield, that $T_{ij} = T_{ji}$; but this is not always done and Hopfield discusses experiments in which that condition does not hold. There may even be skew symmetric cases [6; p.3090].

An electrical simulation model is also given by Hopfield [6; p. 3089].

The derivation of (1) by Hopfield is clear, soundly motivated, and highly interesting, but perhaps his most interesting contribution is the construction of a Lyapunov function

$$E = -(1/2)\sum_i \sum_j T_{ij}V_iV_j + \sum_i (1/R_i)\int_0^{V_i} g_i^{-1}(s)ds - \sum_i I_iV_i \tag{2}$$

whose derivative along a solution of (1) satisfies (for $T_{ij} = T_{ji}$)

$$dE/dt = -\sum_i (dV_i/dt)\Big(\sum_j T_{ij}V_j - (u_i/R_i) + I_i\Big)$$

or

$$dE/dt = -\sum_i (dV_i/dt)\,(du_i/dt) = -\sum_i c_i g_i'(u_i)(u_i')^2 \tag{3}$$

(here again $g' = dg/du_i$, while $u_i' = du_i/dt$), a useful and remarkable relation. At this point Hopfield [6; p. 3090] states (but does not prove) that E is bounded along a solution, so every solution converges to a point at which $du_i/dt = 0$ for every i. It is to be noted that E is not radially unbounded, so an independent proof of boundedness of solutions must be given. Such a proof is simple and will later be supplied.

Now E is a very interesting Lyapunov function. A later calculation shows that

$$\partial E/\partial u_i = -C_i u_i' g_i'(u_i)$$

so that (1) is actually

$$C_i u_i' = -(\partial E/\partial u_i)/g_i'(u_i), \tag{1)*$$

and that brings us to the second model of interest. Equation (1)* is almost a gradient system. This can also be inferred from the work of Cohen and Grossberg [2] which deals with a more general system than (1) and which they note [2; p. 818] is related to a gradient system.

Han, Sayeh, and Zhang [4] go a step further (as do many other investigators) and design a neural network based on an exact gradient system

$$X' = -grad V(X), \ ' = d/dt, \tag{4}$$

where $V(X)$ is a given Lyapunov function having minima at certain fixed points, called stored vectors, and $\partial V/\partial x_i$ is continuous. Then the derivative of V along (4) is

$$dV/dt = grad\ V(X) \cdot (-grad\ V(X)) = -\ \|grad\ V(X)\|^2, \tag{5}$$

where $\|\cdot\|$ is the Euclidian length.

They also consider a concrete example of (4) in the form

$$dx_i/dt = -\sum_{s=1}^{M} [(x_i - x_i^s)/\sigma_s^2] exp\Big(-\sum_{j=1}^{N} (x_j - x_j^s)^2/\sigma_s^2\Big), \tag{6}$$

$i = 1, \ldots, N$, and

$$V(x) = (1/2)\left\{M - \sum_{s=1}^{M} exp\Big(-\sum_{j=1}^{N} (x_j - x_j^s)^2/\sigma_s^2\Big)\right\}, \tag{7}$$

where x_i is the ith component of X, x_i^s is the ith component of the sth stored vector, and M is the number of stored vectors. It is claimed that all solutions converge to the minima of V. Questions of delay are not considered, but (4) is very well adapted to delays, as we will see.

The third model on which we wish to focus is that of Marcus and Westervelt ([11], [12]) who start with a streamlined version of (1) into which they introduce a delay and write

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{N} J_{ij} f(u_j(t-\tau)) \tag{8}$$

where f has a maximum slope of β at zero, f is sigmoidal, $|f(x)| < 1$, $\tau > 0$ is a positive constant. The authors give a linear stability analysis of (8) both for $\tau > 0$ and $\tau = 0$, concluding that there are sustained oscillations in some cases. A nonlinear stability analysis is also given which yields a critical value of τ at which oscillations cease.

But in actual neural networks of both biological and electrical type, the response tends to be based on an accumulation of charges (Hopfield's "short term average"), say through a capacitor, and the result is a delay term in the form of an integral, not a pointwise delay. Indeed, if a Stieltjes integral is used, then the integral can represent various pointwise delays, as is noted by Langenhop [9], for example. Our work here concentrates on integral delays.

Remarks on literature. We have focused on the two papers of Hopfield, the paper by Han, Sayeh, and Zhang [4], and the papers of Marcus and Westervelt ([11], [12]) because they provide central motivation for work. But the literature concerning the Hopfield model is enormous. Miller [13] has written a two volume loose-leaf survey work on neural networks with exhaustive bibliography and survey of investigators.

2. BOUNDEDNESS, STABILITY, AND DELAYS

Let us return now to the derivation of (1). Elementary circuit theory states that when $I(t)$ is the current, then the charge on the capacitor is given by $\int_0^t I(s)ds$. Haag [3; p. 169] discusses this process with the capacitor discharging when the charge reaches a certain level. For the neural network (1) the effect of the capacitor can be modeled by replacing $T_{ij}g_j(u_j)$ by

$$\int_{-\infty}^{t} a_{ij}(t-s)g_j(u_j(s))ds \tag{9}$$

(which can also be written as $\int_0^\infty a_{ij}(s)g_j(u_j(t-s))ds$) where $\int_0^\infty a_{ij}(s)ds = T_{ij}/C_i$. A typical form for $a_{ij}(t)$ would be

$$a_{ij}(t) = \begin{cases} 3T_{ij}(t-h)^2/h^3C_i & \text{for } 0 \le t \le h, \\ 0 \text{ for } t > h. \end{cases}$$

Thus, $a_{ij}(0) = 3T_{ij}/hC_i$, $a_{ij}(h) = 0$, and $\int_0^\infty a_{ij}(s)ds = T_{ij}/C_i$. Here, h is the τ of the Marcus and Westervelt problem, or the "short term" of the short term average mentioned by Hopfield [6; p. 2555]. If $h = \infty$, then an appropriate example is $a_{ij}(t) = T_{ij}e^{-t}/C_i$.

Thus, our model is

$$du_i/dt = -\lambda_i u_i + \sum_j \int_{-\infty}^{t} a_{ij}(t-s) g_j(u_j(s))ds + I_i/C_i \tag{10}$$

where

$$\begin{cases} \lambda_i = 1/R_iC_i, \ I_i \text{ is constant, } \int_0^\infty |\, a_{ij}(t)\,|\, dt = M_{ij} \le M, \\ \text{for some } M > 0, \ a_{ij}(t) \text{ is piecewise continuous.} \end{cases} \tag{11}$$

Obviously, C_i is not taken to be the capacitance in this system. It should be noted that for proper choice of $a_{ij}(t)$, (9) can represent terms $\int_{t-h}^{t} g_j(u_j(s))ds$ and $g_j(u_j(t-h))$ at the same time (cf. Langenhop [9]). Moreover, if $a_{ij}(t) = T_{ij}e^{-\alpha_i t}/C_i$, α_i constant, then (10) can be reduced to a higher dimensional system of ordinary differential equations. This idea is developed in Section 4.

It is readily proved that for each set of bounded and piecewise continuous initial functions $u_i(s)$ on $-\infty < s \le 0$, there is a solution $u_i(t)$ on some interval $0 \le t < \alpha$; and if the solution remains bounded, then $\alpha = \infty$ (see [1] for methods of proof).

It is to be noted that if $u^0 = (u_1^0, \ldots, u_n^0)$ is an equilibrium point for (1), then it is also for (10). Hopfield [6; p. 3089] has made a careful study of those equilibrium points. Our long term goal is to show that solutions of (10) approach the equilibrium points of (1). To that end, we follow the lead of Hopfield [6; p. 3090] where he constructs the Lyapunov function E given in (2). We will try to extend that Lyapunov function to (10).

Before doing so we first focus on Hopfield's argument [6; p. 3090]. He states that E is bounded, that $dE/dt \le 0$, and that $dE/dt = 0$ implies that $dV_i/dt = 0$ for all i so that all solutions approach points where $dE/dt = 0$. His conclusion is most certainly correct, but he needs to first show that solutions are bounded; this is an easy matter, as we shall see. Basically, Hopfield is invoking an old result of Yoshizawa ([14] or (1; p. 232]) or, as the system is autonomous, a result of Krasovskii [8; p. 67] which may be stated as follows.

***Theorem* (Yoshizawa):** *Let $F:[0,\infty) \times R^n \to R^n$ be continuous and bounded for x bounded and suppose that all solutions of $x' = F(t,x)$ are bounded. If there is a continuous function $E:[0,\infty) \times R^n \to (-\infty,\infty)$ which is locally Lipschitz in x and bounded below for x*

bounded, if there is a continuous function $W: R^n \to [0,\infty)$ *which is positive definite with respect to a closed set* Ω*, and if* $E' \leq -W(x)$*, then every solution approaches* Ω *as* $t \to \infty$*.*

The crucial requirement is that solutions be bounded, not that E be bounded (except for x bounded), as the following example shows. Let $x' = xe^{-x^2}$ and $E = e^{-x^2}$ so that $E' = -2x^2e^{-2x^2}$; E is bounded, but all nontrivial solutions tend to $\pm\infty$. Of course, this does not happen in the Hopfield case.

The following proof applies equally well to Hopfield's equation and to that of Marcus and Westervelt, but it does not apply to the Marcus and Westervelt linearized system. The type of boundedness proved here is commonly called uniform boundedness and uniform ultimate boundedness for bound B at $t = 0$ (cf. [1; p.248]).

Lemma 1: *There is a* $B > 0$*, and for each* $B_1 > 0$ *there is a* $B_2 > 0$ *and a* $T > 0$ *such that if the initial functions all satisfy* $|u_i(s)| \leq B_1$ *on* $(-\infty, 0]$*, then the solutions of* (10) *will satisfy* $|u_i(t)| \leq B_2$ *for all* $t \geq 0$*, while* $|u_i(t)| \leq B$ *if* $t \geq T$*.*

Proof: Since the $|g_i(u_i)| \leq 1$, the I_i are constants, and $\int_0^\infty |a_{ij}(t)|\,dt \leq M$, the solution $u_i(t)$ satisfies

$$u_i'(t) = -\lambda_i u_i + h(t)$$

where $|h(t)| \leq M + I$ and $I = max|I_i/C_i|$. Certainly, $h(t)$ depends on the initial function, but M does not. Thus, by the variation of parameters formula,

$$|u_i(t)| \leq |u_i(0)|e^{-\lambda_i t} + M\int_0^t e^{-\lambda_i(t-s)}ds,$$

from which the result follows.

System (1) seems to us to be precisely the one which describes the Hopfield problem and is worthy of careful study. It is, however, quite nontrivial and may be the focus of stability analysis for some time to come. We begin by showing that a study is feasible by giving a basic result patterned after a one dimensional theorem of Levin [10] concerning an unrelated question. In this result, our initial functions are points in R^n at $t = 0$, but are zero for $t < 0$. Such are also Hopfield's initial conditions. The initial functions have the effect of changing (10) to

$$u_i' = -\lambda_i u_i + \sum_j \int_0^t a_{ij}(t-s)g_j(u_j(s))ds + I_i/C_i.$$

While we stated earlier that (10) can include the Marcus and Westervelt system, that is not true under the conditions of the following result.

Theorem 1: *Let $I_i = 0$ for all i, $u_i(s) = 0$ for $s < 0$, $u_i(0) \in R^n$, (11) hold, and $u_i(t)$ satisfy (10) on $[0, \infty)$. Suppose also that $a_{ij}(t) = a_{ji}(t)$ and that the matrices $(-a_{ij}(t))$, $(a'_{ij}(t))$ and $(-a''_{ij}(t))$ are positive semi-definite. Then for*

$$V(t) = \sum_i \int_0^{u_i} g_i(s)ds \tag{12}$$

$$-(\tfrac{1}{2})\sum_i \sum_j a_{ij}(t) \int_0^t g_i(u_i(s))ds \int_0^t g_j(u_j(s))ds$$

$$+\tfrac{1}{2}\int_0^t a'_{ij}(t-s)\int_s^t g_j(u_j(v))dv \int_s^t g_i(u_i(v))dv\, ds$$

we have

$$V'(t) = \sum_i -u_i g_i(u_i)\lambda_i \tag{13}$$

$$-(\tfrac{1}{2})\sum_i \sum_j a'_{ij}(t)\int_0^t g_i(u_i(s))ds \int_0^t g_j(u_j(s))ds$$

$$+\sum_i \sum_j (\tfrac{1}{2})\int_0^t a''_{ij}(t-s)\int_s^t g_j(u_j(v))dv \int_s^t g_i(u_i(v))dvds \le 0.$$

Moreover, $u(t)$ approaches an equilibrium point of (1).

Proof: We have

$$V' = \sum_i -\lambda_i u_i g_i(u_i) + \sum_i \sum_j \{g_i(u_i)\int_0^t a_{ij}(t-s)g_j(u_j(s))ds\}$$

$$-(\tfrac{1}{2})\sum_i \sum_j \{a'_{ij}(t)\int_0^t g_i(u_i(s))ds \int_0^t g_j(u_j(s))ds$$

$$+(\tfrac{1}{2})\int_0^t a''_{ij}(t-s)\int_s^t g_j(u_j(v))dv \int_s^t g_i(u_i(v))dv\, ds\}$$

$$-(\tfrac{1}{2})\sum_i \sum_j \{a_{ij}(t)[g_i(u_i(t))\int_0^t g_j(u_j(s))ds$$

$$+ g_j(u_j(t)) \int_0^t g_i(u_i(s))ds]$$

$$+ \int_0^t a'_{ij}(t-s)[g_j(u_j(t)) \int_s^t g_i(u_i(v))dv$$

$$+ g_i(u_i(t)) \int_s^t g_j(u_j(v))dv]\}.$$

Integration by parts of the last term yields.

$$-\tfrac{1}{2}a_{ij}(t-s)g_j(u_j(t)) \int_s^t g_i(u_i(v))dv|_{s=0}^{s=t}$$

$$-\frac{1}{2}\int_0^t a_{ij}(t-s)g_i(u_i(s))ds g_j(u_j)$$

$$-\tfrac{1}{2}a_{ij}(t-s)g_i(u_i(t)) \int_s^t g_j(u_j(v))dv \,|_{s=0}^{s=t}$$

$$-\frac{1}{2}\int_0^t a_{ij}(t-s)g_j(u_j(s))ds g_i(u_i).$$

Note that if $A = (a_{ij})$ and if G is the vector with components $g_i(u_i)$, then $A^T = A$ implies that

$$\sum_i \sum_j \int_0^t g_i(u_i(t))a_{ij}g_j(u_j(s))ds = \int_0^t G^T(t)AG(s)ds$$

$$= \int_0^t G^T(s)AG(t)ds = \sum_i \sum_j \int_0^t g_i(u_i(s))a_{ij}g_j(u_j(t))ds.$$

This yields (13). Each term of (13) is nonpositive, all solutions are bounded so du_i/dt is bounded. Thus, by Yoshizawa's argument [14], $u_i \to 0$ and so $du_i/dt \to 0$. Hence, $u(t)$ approaches an equilibrium point of (10) and these are the same as those of (1). This completes the proof.

Remark: Theorem 1 is viewed as a first result. Nevertheless, the definiteness conditions on $(a_{ij}(t))$, $(a'_{ij}(t))$ and $(a''_{ij}(t))$ may not be as severe as they first seem. These require self connections. Since $-u_i/R_i$ appears in (1) we can think of each neuron as being self connected. To see this, in (1) determine T^*_{ii} such that $[u_i/R_i] - T^*_{ii}g_i(u_i)$ has the sign of u_i, so that (1) can be written as

$$C_i(du_i/dt) = \sum_j \widetilde{T}_{ij} V_j - \Big([u_i/R_i] - T^*_{ii} g_i(u_i)\Big) + I_i \qquad (1)^{**}$$

where $\widetilde{T}_{ij} = T_{ij}$ if $i \neq j$, and $\widetilde{T}_{ii} = T_{ii} - T^*_{ii}$. Then design the delay system so that (10) becomes

$$du_i/dt = -\Big([u_i/R_i] - T^*_{ii} g_i(u_i)\Big)/C_i + \sum_j \int_{-\infty}^{t} a_{ij}(t-s) g_j(u_j(s))ds + I_i/C_i$$

and (13) becomes

$$V' = \sum_i -\Big([u_i/R_i] - T^*_{ii} g_i(u_i)\Big)/C_i - \tfrac{1}{2}\sum_i \sum_j a'_{ij}(t) \int_0^t g_i(u_i(s))ds \int_0^t g_j(u_j(s))ds + \sum_i \sum_j \tfrac{1}{2} \int_0^t a''_{ij}(t-s) \int_s^t g_j(u_j(v))dv \int_s^t g_i(u_i(v))\, ds \leq 0 \qquad (13)^*$$

since $[u_i/R_i] - T^*_{ii} g_i(u_i)$ and $g_i(u_i)$ both have the sign of u_i. The matrices $(a_{ij}(t))$, $(a'_{ij}(t))$ and $(a''_{ij}(t))$ will have nonzero diagonal elements.

In Section 4 we will simplify (10) and obtain results independent of the definiteness of these matrices.

3. ROBUSTNESS AND DELAYS

Equations $(1)^*$ and (5) show that (1) and (5) are very robust in the sense that comparatively large perturbations can be added and solutions will still converge to the equilibrium points of the unperturbed equation. Recently, Kosko [7] has discussed robustness of this type for a variety of neural networks when the perturbations are stochastic.

Lyapunov's direct method is well suited to proving robustness under real perturbations. Intuitively we have the following situation. Given a positive definite Lyapunov function $V(u)$ for a differential equation $u' = F(u)$, the derivative of V along a solution is

$$V' = grad\ V \cdot F = |\, grad\ V \,|\ |\, F \,|\, cos\ \theta$$

where θ is the angle between the tangent vector $F(u)$ to the solution and $grad\ V$ which is the outward normal to the surface $V = constant$. A gradient system has $cos\ \theta = -1$, the optimal value. This means that the solution $u(t)$ enters the region $V(u) \leq constant$ along the inward normal. Hence, if we perturb the differential equation to $u' = F(u) + G(u)$, so long as $G(u)$ is not too large relative to $F(u)$, the vector $F(u) + G(u)$ will still point inside the region $V(u) \leq constant$.

Now (5) is actually a gradient system so the perturbation result for it is better than the one for (1)* which is merely almost a gradient system. Perturbation results are crucial for any real system since the mathematical equation will seldom represent the physical reality exactly.

Let β and h be positive constants and A an $n \times n$ matrix of piecewise continuous functions with $\| A \| \leq 1$ where $\| A \| = \max_{\substack{1 \leq \; < n \\ 0 \leq t \leq h}} \sum_j | a_{ij}(t) |$, and consider

$$X' = -grad\ V(X) + \beta \int_{t-h}^{t} A(t-s)\ grad\ V(X(s))ds. \tag{14}$$

Several other forms could be chosen, but this will demonstrate the strong stability. Note that (4) and (14) have the same equilibrium points (under our subsequent assumption (16)). To solve (14) it is required that there be given a piecewise continuous initial function $\varphi: [-h, 0] \rightarrow R^n$. There is then a continuous solution $X(t, \varphi)$ on some interval $0 \leq t < \alpha$ with $X(t, \varphi) = \varphi(t)$ for $-h \leq t \leq 0$; $X(t, \varphi)$ satisfies (14) on $(0, \alpha)$. See methods of [1] for existence details.

Theorem 2: *Let $V(X) \geq 0$. Then there is a $\beta > 0$ (see (16)) such that for each piecewise continuous $\varphi: [-h, 0] \rightarrow R^n$ any solution $X(t, \varphi)$ of (14) is defined on $[0, \infty)$; if it is bounded, then it converges to an equilibrium point of (4).*

Proof: Define a Lyapunov functional along a solution of (14) by

$$W(t) = V(X(t)) + \gamma \int_{-h}^{0} \int_{t+v}^{t} | grad\ V(X(s)) |^2 ds\ dv \tag{15}$$

so that

$$W'(t) \leq -| grad\ V(X) |^2 + \beta | grad\ V(X) | \int_{t-h}^{t} \| A(t-s) \| \ | grad\ V(X(s)) | ds$$
$$+\gamma \int_{-h}^{0} [\ | grad\ V(X(t)) |^2 - | grad\ V(X(t+v) |^2]dv$$
$$\leq [-1 + \frac{\beta h}{2} + \gamma h] | grad\ V(X) |^2$$

$$-(\gamma-\frac{\beta}{2})\int_{t-h}^{t} |\, grad\ V(X(s))\,|^2 ds.$$

Let

$$\gamma > \frac{\beta}{2} \text{ and } 1 > \gamma h + \frac{\beta h}{2}. \tag{16}$$

Then there is a $\mu > 0$ with

$$W'(t) \leq -\mu(\,|\, grad\ V(X)\,|^2 + \int_{t-h}^{t} |\, A(t-s) grad\ V(X(s))\,|^2 ds).$$

But $\int_{t-h}^{t} |\, r(s)\,|\, ds \leq h^{1/2}\left(\int_{t-h}^{t} |\, r(s)\,|^2 ds\right)^{1/2}$ or $\left(\int_{t-h}^{t} |\, r(s)\,|^2 ds\right)^2 \leq h\int_{t-h}^{t} |\, r(s)\,|^2 ds$ so we have

$$W'(t) \leq -k(\,|\, X'(t)\,|\,)^2, \text{ for some } k > 0. \tag{17}$$

It is known that the only way in which a solution $X(t)$ of (14) can fail to be defined for all $t \geq 0$ is for there to exist a $T > 0$ such that $\limsup_{t\to T^-} |\, X(t)\,| = +\infty$. Thus, if $T-1 < t < T$ then from (17) we have

$$0 \leq W(t) \leq W(T-1) - k\int_{T-1}^{t} (\,|\, X'(s)\,|\,)^2 ds \leq W(T-1) - k\left(\int_{T-1}^{t} |\, X'(s)\,|\, ds\right)^2$$

$$\leq -k(\,|\, X(t) - X(T-1)\,|\,)^2,$$

a contradiction to $\limsup_{t\to T^-} |\, X(t)\,| = +\infty$. Thus, each solution $X(t,\varphi)$ can be defined for all $t \geq 0$.

Suppose that $X(t,\varphi)$ is bounded. Then *grad* $V(X)$ is continuous and $A(t)$ is bounded so $X'(t,\varphi)$ is bounded. The argument of Yoshizawa [14] is fully applicable and $X(t,\varphi)$ approaches the set in which $|\, grad\ V(X)\,| = 0$, the equilibrium points of (4). This completes the proof.

There are several simple conditions which will ensure that solutions of (14) are bounded. Certainly, (17) with $W \geq 0$ will not do it as may be seen from the scalar equation $x' = 2xe^{-x^2}$ with $V = e^{-x^2}$. We have *grad* $V = -2xe^{-x^2}$ and $V' = -4x^2e^{-2x^2} = -(grad\ V(x))^2$; but all solutions except $x = 0$ are unbounded.

We could ensure boundedness by asking one of the following:

(a) Since $W'(t) \le 0$, if for each continuous $\varphi:[-h,0]\to R^n$ and for $C = \{\psi:[-h,0]\to R^n\}$, we have $\liminf_{\substack{\|\psi\|\to\infty \\ \psi\in C}} W(\psi(t)) > W(\varphi)(0)$, then all solutions of (14) are bounded.

(b) If there is a continuous function $G:[0,\infty)\to[0,\infty)$ with $G(r) = \inf_{|X|=r} |\mathrm{grad}\, V(X)|$ and $\int_0^\infty G(r)dr = \infty$, then all solutions of (14) are bounded.

The validity of (a) should be clear. To prove (b) we note that there is a $\bar{k} > 0$ with

$$W'(t) \le -\bar{k}\, |X'(t)|\, |\mathrm{grad}\, V(X(t))| \le -\bar{k}\, |X'(t)|\, G(|X(t)|)$$

so that

$$\begin{aligned} 0 \le W(t) &\le W(0) - \bar{k}\int_0^t G(|X(s)|)\,|X'(s)|\,ds \\ &\le W(0) - \bar{k}\int_0^t G(|X(s)|)\,|\,|X(s)|'\,|\,ds \\ &\le W(0) - \bar{k}\,\left|\int_0^t G(|X(s)|)\,|X(s)|'ds\right| \\ &= W(0) - \bar{k}\,\left|\int_{|X(0)|}^{|X(t)|} G(s)ds\right| \end{aligned}$$

so that $|X(t)|$ is bounded.

Remark: The conclusion of this theorem can not be strengthened to stating that bounded solutions approach the minima of $V(X)$, as was desired in [4] where maxima and saddle points were to be avoided. In the scalar equation

$$x' = -x^2(1-x)$$

with

$$V(x) = (\tfrac{x^4}{4}) - (\tfrac{x^3}{3}) + (\tfrac{1}{2}),$$

the minimum is at $x = 1$, but if $x_0 < 0$, then $x(t)\to 0$; gradient systems of the same type are easily constructed.

We turn now to the model of Hopfield which is more challenging when introducing a delay because (3) is slightly more complicated than (5). Moreover, since (1) is not quite a gradient system, the perturbation will be not quite as large as in (14).

To verify (3), use the symmetry of (T_{ij}) in (2) to obtain

$$\begin{aligned}\partial E/\partial u_i &= -\sum_j T_{ij} g_i'(u_i) V_j + (1/R_i) u_i g_i'(u_i) - I_i g_i'(u_i) \\ &= \Big(-\sum_j T_{ij} V_j + (1/R_i) u_i - I_i\Big) g_i'(u_i) \\ &= -C_i u_i' g_i'(u_i)\end{aligned}$$

so

$$C_i u_i' = -(\partial E/\partial u_i)/g_i'(u_i).$$

To obtain a delay system for (1) we let $A(t)$ be an $n \times 1$ matrix of piecewise continuous functions and let A_i be the ith component of A with $|A_i(t)| \leq 1$ for $0 \leq t \leq h$ and all i. Consider the system

$$C_i u_i' = -[(\partial E/\partial u_i)/g_i'(u_i)] + \alpha_i \int_{t-h}^{t} A_i(t-s)(\partial E(u(s))/\partial u_i)/\sqrt{g_i'(u_i(s))}\,ds, \tag{18}$$

where the α_i are constants. Note that equilibrium points of (1) are preserved.

We now prove a simple lemma parallel to that of Lemma 1. While (18) is the preferred form for showing limit sets, for boundedness we write (18) as

$$C_i du_i/dt = \sum_j T_{ij} V_j - u_i/R_i + I_i + \alpha_i \int_{t-h}^{t} A_i(t-s)\Big\{\sum_j T_{ij} V_j(s) - u_i(s)/R_i + I_i\Big\} ds$$

which we can represent by

$$du_i/dt = -\lambda_i u_i + (\alpha_i/R_i C_i) \int_{t-h}^{t} A_i(t-s) u_i(s)\,ds + f(t)$$

or in vector notation as

$$u' = -\Lambda u + \int_{t-h}^{t} D(t-s) u(s)\,ds + F(t) \tag{19}$$

where Λ is a diagonal matrix of constants $\lambda_i = 1/R_i C_i$, D is a diagonal matrix of elements

$\alpha_i A_i(t-s)/R_iC_i$, and there is a constant P which is independent of the initial function with $|F(t)| \le P$. Let

$$\lambda = min\ \lambda_i,\ \overline{\alpha} = max\,|\alpha_i/R_iC_i|. \tag{20}$$

Lemma 2: *There is an $\alpha > 0$ (defined by (20) and (21)) such that if $|\alpha_i| \le \alpha$, then all solutions of (18) are bounded in the same sense as in Lemma 1.*

Proof: Define a functional

$$W(t) = |u(t)| + \int_{-h}^{0}\int_{t+s}^{t} \gamma\,|u(v)|\,dv\,ds,\ \gamma > 0.$$

Then

$$W'(t) \le -\lambda\,|u(t)| + \int_{t-h}^{t} |D(t-s)|\,|u(s)|\,ds + P$$

$$+\gamma h\,|u| - \int_{t-h}^{t} \gamma\,|u(s)|\,ds$$

$$\le -(\lambda - \gamma h)\,|u| - (\gamma - \overline{\alpha})\int_{t-h}^{t} |u(s)|\,ds + P$$

$$\le -\eta V + P, \text{ for some } \eta > 0,$$

provided that

$$\gamma > \overline{\alpha} \text{ and } \lambda > \gamma h. \tag{21}$$

The conclusion now follows from the differential inequality.

Theorem 3: *There is an $\alpha > 0$ (see (22)) such that if all $|\alpha_i| \le \alpha$ then every solution of (18) is bounded and converges to the set of equilibrium points of (1).*

Proof: Define a Lyapunov functional along a solution of (18) by

$$V(t) = E + \delta\int_{-h}^{0}\int_{t+s}^{t}\Big\{\sum_i |\partial E(u(v))/\partial u_i|^2/g_i'(u_i(v))C_i\Big\}dv\,ds$$

so that

$$V' = \sum_i -(\partial E/\partial u_i)^2/g_i'(u_i)C_i$$

$$+(\alpha_i/C_i)(\partial E/\partial u_i)\int_{t-h}^{t}[A_i(t-s)(\partial E(u(s))/\partial u_i)/\sqrt{g_i'(u_i(s))}]ds$$

$$+\delta h[\,|\partial E/\partial u_i|^2/g_i'(u_i)C_i] - \delta\int_{t-h}^{t}[\,|\partial E(u(s))/\partial u_i|^2/g_i'(s))C_i]ds$$

$$\le \sum_i\{-[(\partial E/\partial u_i)^2/g_i'(u_i)] + \alpha_i h(\partial e/\partial u_i)^2 + [\delta h(\partial E/\partial u_i)^2/g_i'(u_i)$$

$$-(\delta-\alpha_i)\int_{t-h}^{t}[(\partial E(u(s))/\partial u_i)^2/g_i'(u_i(s))]ds\}/C_i$$

$$= \sum_i\{-1+\alpha_i h g_i'(u_i) + \delta h](\partial E/\partial u_i)^2/g_i'(u_i)$$

$$-(\delta-\alpha_i)\int_{t-h}^{t}[(\partial E(u(s))/\partial u_i)^2/g_i'(u_i(s))]ds\}/C_i.$$

Thus, we require that

$$\delta > \alpha_i \text{ and } 1 > \alpha_i h\xi + \delta h \tag{22}$$

where $\xi = \max_{\substack{-\infty < u_i < \infty \\ 1 \le i \le n}} g_i'(u_i)$. It then follows readily from Yoshizawa's argument [14] that these (bounded) solutions converge to equilibrium points of (1) when $(20)-(22)$ hold. This completes the proof.

4. A FULLY DELAYED GRADIENT SYSTEM

In the Hopfield model, if the train of action potentials is also dependent on the average potential of the neuron itself, a simplified form of (10) would be

$$u_i' = -\int_{-\infty}^{t}[a_i(t-s)(\partial E(u(s))/\partial u_i)/g_i'(u_i(s))]ds \tag{23}$$

and the analog of (4) is

$$x_i'(t) = -\int_{-\infty}^{t} a_i(t-s)(\partial V(x(s))/\partial x_i)ds. \tag{24}$$

Thus, in (10) we are taking $a_{ij}(t) = a_i(t)$ for $1 \le j \le n$. Numerous papers in mathematical biology have dealt with such intractable memory systems and have noted that if

$$a_i(t) = \gamma_i e^{-\alpha_i t} \tag{25}$$

with

$$\gamma_i \text{ and } \alpha_i \text{ positive constants,} \tag{26}$$

then the memory can be eliminated at the expense of doubling the order of the system. In implementing the case in which (25) holds, in effect, we add a neuron to the Hopfield model, as is indicated by the increased dimension.

Equation (25) does yield a very reasonable memory system. Theorem 6 will reduce (25), but will restrict initial functions, as did Theorem 1.

Theorem 4: *Let* (25), (26) *hold and let* V *be bounded below. Then every solution of* (24) *is bounded and approaches the set of equilibrium points of* (4).

Proof: We have

$$x_i'e^{\alpha_i t} = -\int_{-\infty}^{t} \gamma_i e^{\alpha_i s}(\partial V(x(s))/\partial x_i)ds,$$

so a differentiation yields

$$x_i'' + \alpha_i x_i' = -\gamma_i \partial V(x)/\partial x_i$$

and

$$x_i'x_i'' + \alpha_i(x_i')^2 = -\gamma_i(\partial V/\partial x_i)x_i'$$

or

$$\sum_i (1/\gamma_i)[x_i'^2(t) - x_i'^2(0) + 2\alpha_i \int_0^t x_i'(s)^2 ds] = -2[V(x) - V(x(0))].$$

As V is bounded from below, all solutions are bounded and the argument of Yoshizawa will show that $x_i'(t) \to 0$ so that solutions approach the equilibrium points of (4).

Theorem 5: *Let* (25), (26) *hold and suppose that* $|g_i''(u_i)| < 2\alpha_i g_i'(u_i)$ *for all* i *and all* $u_i \in R$. *Then all solutions of* (23) *approach equilibrium points of* (1).

Proof: We have

$$u_i'' + \alpha_i u_i' = -(\partial E/\partial u_i)/g_i'(u_i)$$

so

$$u_i'u_i''g_i'(u_i) + \alpha_i g_i'(u_i)(u_i')^2 = (\partial E/\partial u_i)u_i'\gamma_i.$$

Now

$$\int_0^t u_i'u_i''g_i'(u_i) = (u_i')^2 g_i'(u_i)/2 \,\Big|_0^t - \int_0^t [u_i'(s)^2 g_i''(u_i)/2]ds$$

and so

$$\sum_i (1/\gamma_i)[u_i'(t)^2 g_i'(u_i(t)) - u_i'(0)^2 g_i'(u_i(0))$$

$$+ \int_0^t [2\alpha_i g_i'(u_i) - g_i''(u_i)]u_i'(s)^2 ds] = 2[E(t) - E(0)].$$

If solutions are bounded, then $2\alpha_i g_i'(u_i) - g_i''(u_i) \geq \delta > 0$ and so

$$\sum_i (1/\gamma_i)u_i'(t)^2 g_i'(u_i(t)) - E(t) \leq K - \sum_i (\delta/\gamma_i) \int_0^t u_i'(s)^2 ds$$

and Yoshizawa's argument implies that $u_i'(t) \to 0$ as $t \to \infty$.

To see that solutions are bounded, write (23) as

$$u_i' = \int_{-\infty}^t e^{-\alpha_i(t-s)} \Big(\sum_j T_{ij} V_j - u_i/R_i + I_i \Big) ds$$

$$= h(t) - (1/R_i) \int_{-\infty}^t e^{-\alpha_i(t-s)} u_i(s) ds$$

where $h(t)$ and $h'(t)$ are bounded and the bound depends on the initial function. Then

$$u_i'' + \alpha_i u_i' = h'(t) + \alpha_i h(t) - u_i/R_i$$

or

$$u_i'' + \alpha_i u_i' + u_i/R_i = h'(t) + \alpha_i h(t),$$

all of whose solutions are bounded. This completes the proof.

Now if one is interested in linear analysis of (23), such as was given by Marcus and Westervelt [12] for the pointwise delay with a view to obtaining local information, say near $u_i = 0$, then (23) is written as

$$u_i' = - \int_{-\infty}^t [a_i(t-s)(\partial E(u(s))/\partial u_i)/\xi_i] ds \tag{27}$$

where $\xi_i = g_i'(0) > 0$. That is, we have linearized the denominator.

Theorem 6: *Let* (25), (26) *hold. Then every bounded solution of* (27) *converges to the set of equilibrium points of* (1).

The proof is, of course, an exact repetition of that of Theorem 4.

We return now to (24) with initial conditions of Theorem 1.

Theorem 7: *In (24) suppose that $x_i(t) = 0$ if $t < 0$ and $x_i(0) \in R$. Suppose also that $a_i'(t) \le 0$ and $a_i''(t) \ge 0$ for all $t > 0$. Then for*

$$W = V + \sum_i \left\{ \tfrac{1}{2} a_i(t) \left\{ \int_0^t (\partial V(x(s))/\partial x_i) ds \right\}^2 \right. \tag{28}$$

$$\left. - \tfrac{1}{2} \int_0^t a_i'(t-s) \left(\int_s^t (\partial V(x(v))/\partial x_i) dv \right)^2 ds \right\}$$

we have

$$W' = \sum_i \left\{ (\tfrac{1}{2}) a_i'(t) \left\{ \int_0^t (\partial V(x(s))/\partial x_i) ds \right\}^2 \right. \tag{29}$$

$$\left. - (\tfrac{1}{2}) \int_0^t a_i''(t-s) \left(\int_s^t (\partial V(x(v))/\partial x_i) dv \right)^2 ds \right\} \le 0.$$

If, in addition, $a'''(t) \le 0$ and $a(t) \not\equiv a(0)$, then for each bounded solution $x(t)$ of (24) with these initial conditions, $x(t)$ approaches the equilibrium points of (4).

Proof: We have

$$W' = \sum_i - (\partial V/\partial x_i) \int_0^t a_i(t-s)(\partial V(x(s))/\partial x_i) ds$$

$$+ (\tfrac{1}{2}) a_i'(t) \left\{ \int_0^t (\partial V(x(s))/\partial x_i) ds \right\}^2$$

$$+ a_i(t) \int_0^t (\partial V(x(s))/\partial x_i) ds (\partial V(x)/\partial x_i)$$

$$- (\tfrac{1}{2}) \int_0^t a_i''(t-s) \left\{ \int_s^t (\partial V(x(v))/\partial x_i) dv \right\}^2 ds$$

$$- \int_0^t a_i'(t-s) \int_s^t (\partial V(x(v))/\partial x_i) ds (\partial V(s)/\partial x_i).$$

An integration of the last term by parts yields

$$\left\{ - a_i(t) \int_0^t (\partial V(x(v))/\partial x_i) ds + \int_0^t a_i(t-s)(\partial V(x(s))/\partial x_i) ds \right\} (\partial V(x)/\partial x_i)$$

which will now verify (29). The final conclusion follows from an argument of Levin [10; p.

540]. That will complete the proof.

5. DISCUSSION

System (10) seems to be a proper formulation for the general problem described by Hopfield and a more justifiable delay system than that of Marcus and Westervelt. It seems to be very difficult to evaluate its stability properties in its full generality, but it is significant that Lemma 1 yields boundedness of solutions. Analysis in the full generality is expected to be a long-term project, but the results of Section 4 indicate that (10) should be very stable. Noting that (1) is almost a gradient system should significantly enhance the stability analysis.

6. REMARKS ON MEMORY

The object of the memory is to enable the T_{ij} in (1) to reflect the time lag. System (10) has a memory in every sense of the word. For a general $a_{ij}(t)$, (10) can not be reduced to an ordinary differential equation without memory. When (25) holds, then systems (23) and (24) have limited memory in that $a_i(t)$ can be removed at the expense of doubling the order. Any ordinary differential equation can be expressed as an integral equation and sometimes it appears to have a memory. For example, Hopfield's system can be written as

$$u_i' = -\gamma_i u_i + h_i(t,u), \tag{30}$$

so that using the integrating factor $e^{\gamma_i t}$, we obtain

$$u_i(t) = u_i(0)e^{-\gamma_i t} + \int_0^t e^{-\gamma_i(t-s)} h_i(s,u(s))ds.$$

But since solutions of (30) are uniquely determined by (t_0, u_0) alone, equation (31) does not have a memory.

Section 4 has focused on $a(t) = e^{-t}$ and this can be generalized to $a(t) = \sum_i f_i(t)$ where each $f_i(t)$ is the solution of a linear homogeneous ordinary differential equation of degree n with constant coefficients. (See [1; p. 84]).

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