

# INTEGRAL MANIFOLDS OF IMPULSIVE DIFFERENTIAL EQUATIONS<sup>1</sup>

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## ABSTRACT

The present paper is concerned with the existence of integral manifolds of impulsive differential equations as  $t \rightarrow +\infty$ . Under the assumption of exponential trichotomy on the linear part of the right-hand side of the equation, it is proved that if the nonlinear perturbation is small enough, then there exist integral manifolds as  $t \rightarrow +\infty$  for the perturbed equations.

**Key words:** Integral manifold, exponential trichotomy.

**AMS (MOS) subject classification:** Primary 34C35.

## 1. INTRODUCTION

Impulsive differential equations have found many applications in physics, chemistry, control theory, etc. In recent years, many interesting results on qualitative theory of these equations have been obtained [1], [2], [5], [10]. The existence of stable and unstable integral manifolds of such equations has been investigated in [10] under the assumption that the linear part of the right-hand sides of these equations is hyperbolic. In [2], the authors have investigated the existence of integral manifolds, some components of which remain bounded as  $t \rightarrow \pm \infty$ .

In this paper we are concerned with the existence of integral manifolds, some components of which remain bounded as  $t \rightarrow +\infty$ . Introducing the notion of trichotomy of the linear part of the right-hand sides of these equations, we have proved that under a small

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nonlinear perturbation there exist integral manifolds with properties mentioned above for differential equations with impulse effect.

## 2. PRELIMINARIES

In the present paper, we consider equations of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + F(t, x) \text{ if } t \neq t_n, t \in \mathbb{R}^+ \\ \Delta x |_{t=t_n} &= B_n x(t_n - 0) + I_n(x(t_n - 0)), n \in \mathbb{N} \end{aligned} \quad (1)$$

where  $x \in X$ ,  $X$  is a complex Banach space,  $\Delta x |_{t=\tau} = x(\tau + 0) - x(\tau - 0)$ ,  $F: \mathbb{R}^+ \times X \rightarrow X$  is a piecewise continuous function with discontinuities of the first kind at the points  $t = t_n$ ,  $n = 1, 2, \dots$  with respect to  $t$  and continuous with respect to  $x$ ,  $A(t)$  is a continuous and bounded operator with respect to  $t$ ,  $B_n$  are bounded linear operators,  $I_n: X \rightarrow X$  are impulsive operators. Furthermore,

$$0 < t_1 < t_2 < \dots, \lim_{n \rightarrow \infty} t_n = +\infty.$$

We assume that there exists uniformly in  $t \in \mathbb{R}^+$

$$\lim_{T \rightarrow +\infty} \frac{i(t, t+T)}{T} = p < +\infty \quad (2)$$

where  $i(a, b)$  is defined as the number of points  $t_n$  belonging to  $(a, b)$ .

**Definition 1:** By *solution* of (1), we understand a piecewise continuous function  $x(t)$ ,  $t \in \mathbb{R}^+$  with discontinuities of the first kind at the points  $t_n$ ,  $n \in \mathbb{N}$  such that for every  $t \neq t_n$  we have

$$\frac{dx}{dt} = A(t)x(t) + F(t, x(t))$$

and  $x(t)$  satisfies (2) for every  $t = t_n$ .

**Definition 2:** The linear part of (1) is said to have an *exponential trichotomy* if there exists a splitting  $X = X_1 \oplus X_2 \oplus X_3$  and positive constants  $\kappa$ ,  $\alpha$ ,  $\beta$ ,  $\alpha > \beta$  with the following properties:

- i)  $A(t)$ ,  $B_n$  leave  $X_i$ ,  $i = 1, 2, 3$  invariant for every  $t \in \mathbb{R}^+$  and  $n \in \mathbb{N}$ ,
- ii)  $\|X_2(t)X_2^{-1}(s)\| \leq \kappa \exp(-\alpha(t-s))$ , for  $t \geq s$   
 $\|X_3(t)X_3^{-1}(s)\| \leq \kappa \exp(-\alpha(s-t))$ , for  $s \geq t$   
 $\|X_1(t)X_1^{-1}(s)\| \leq \kappa \exp(\beta|t-s|)$ , for all  $t, s$

where  $X(t) = \text{diag}(X_1(t), X_2(t), X_3(t))$  is a fundamental solution operator of the linear part of

(1).

**Definition 3:** We call an *integral manifold* of (1) any subset  $M$  of the extended phase space consisting of whole integral curves.

From now on we shall deal with integral manifolds which can be represented in the form of a graph of a mapping with certain properties.

In what follows  $L(\rho, \eta)$  denotes the set of mappings  $\varphi: \mathbb{R}^+ \times X_3 \rightarrow X_1 \oplus X_2$  enjoying the following properties:

$$\begin{aligned} \|\varphi(t, u)\| &\leq \rho \quad \text{for } (t, u) \in \mathbb{R}^+ \times X_3 \\ \|\varphi(t, u) - \varphi(t, v)\| &\leq \eta \|u - v\| \quad \text{for all } t \in \mathbb{R}^+, u, v \in X_3. \end{aligned}$$

We shall consider only nonlinear perturbations  $F, I_n$  in (1) satisfying the Lipschitz condition with respect to  $x$ , i.e.

$$\begin{aligned} \|F(t, x) - F(t, y)\| &\leq \delta \|x - y\| \quad \text{for all } t \in \mathbb{R}^+, x, y \in X \\ \|I_n(x) - I_n(y)\| &\leq \delta \|x - y\| \quad \text{for all } n \in \mathbb{N}, x, y \in X. \end{aligned}$$

Furthermore, we assume that

$$\|F(t, x)\| \leq Q, \|I_n(x)\| \leq Q \quad \text{for all } t, n, x.$$

### 3. PREPARATORY LEMMAS

The proof of our main result needs some technical preparations which we carry out in this section.

**Lemma 1:** *Suppose that the nonlinear perturbations  $F, I_n$  in (1) are independent of  $x$  and  $X_3 = \{0\}$ , (i.e., the linear part of (1) has an exponential dichotomy). then (1) has at least one bounded solution. In addition, all bounded solutions of (1) satisfy the following equation*

$$x(t) = X(t)y + \int_0^\infty G(t, \tau)F(\tau)d\tau + \sum_{i=1}^\infty G(t, t_i)I_i \quad (3)$$

where  $X(t)$  is the principal fundamental solution operator of the homogeneous equation (i.e.,  $X(0) = Id$ ),  $G(t, \tau)$  is Green's function of the homogeneous equation,  $y \in X_2$ .

**Proof:** The proof of the lemma can be carried out in a well known way (see e.g. [1]). Thus we omit it.

**Lemma 2:** Assume that  $X_3 = \{0\}$ . Then (1) has at least one bounded solution (as  $t \rightarrow +\infty$ ) and all bounded solutions of (1) satisfy the equation

$$x(t) = X(t)y + \int_0^{\infty} G(t,s)F(t,x(s))ds + \sum_{i=1}^{\infty} G(t,t_i)I_i(x(t_i)) \quad (4)$$

where the notation is the same as in Lemma 1.

It may be noted that under the assumptions stated above equation (1) satisfies the Existence and Uniqueness Theorem.

Now we rewrite equation (1) in another form

$$\begin{aligned} \frac{du}{dt} &= A_1(t)u + F_1(t,u,v), u \in X_1, v \in X_2 \oplus X_3 \\ \frac{dv}{dt} &= A_2(t)v + F_2(t,u,v), \text{ if } t \neq t_n, t \in \mathbb{R}^+ \\ \Delta u |_{t=t_n} &= B_n^1 u(t_n - 0) + I_n^1(u(t_n - 0), v(t_n - 0)) \\ \Delta v |_{t=t_n} &= B_n^2 v(t_n - 0) + I_n^2(u(t_n - 0), v(t_n - 0)). \end{aligned} \quad (5)$$

Suppose that  $M$  is any integral manifold of (5) which is represented by the function  $\varphi: \mathbb{R}^+ \times X_1 \rightarrow X_2 \oplus X_3$ ,  $\varphi \in L(\rho, \eta)$ . Now we are going to find the equation defining  $\varphi$ . Let  $x(t)$  be any solution of (1) such that  $(t, x(t)) \in M$  for all  $t \in \mathbb{R}^+$ . Then  $x(t) = (u(t), \varphi(t, u(t)))$ , where  $u(t) = P_1 \times (t)$ ,  $P_2, P_2, P_3$  denote the projections:  $X \rightarrow X_1, X_2, X_3$  respectively. It is clear that  $u(t)$  is a solution of the following equation

$$\begin{aligned} \frac{du}{dt} &= A_1(t)u + F_1(t, u, \varphi(t, u)) \text{ if } t \neq t_n \\ \Delta u |_{t=t_n} &= B_n^1 u + I_n^1(u, \varphi(t_n, u)). \end{aligned} \quad (6)$$

On the other hand,  $\varphi(t, u(t))$  is a solution of the equation

$$\begin{aligned} \frac{d\varphi(t, u(t))}{dt} &= A_2(t)\varphi(t, u(t)) + F_2(t, u(t), \varphi(t, u(t))) \text{ if } t \neq t_n, t \in \mathbb{R}^+ \\ \Delta \varphi |_{t=t_n} &= I_n^2 \varphi(t_n, u(t_n - 0)) + I_n^2(u(t_n - 0), \varphi(t_n, u(t_n - 0))). \end{aligned} \quad (7)$$

From Lemma 2, it follows that there exists  $y \in X_2$  such that

$$\begin{aligned} \varphi(t, u(t)) &= X(t, t_0)y + \int_{t_0}^{\infty} G(t,s)F_2(s, u(s), \varphi(s, u(s)))ds + \\ &+ \sum_{i=1}^{\infty} G(t, t_i)I_i^2(u(t_i), \varphi(t_i, u(t_i))). \end{aligned} \quad (8)$$

**Lemma 3:** Suppose that  $u(t)$  is piecewise continuous with discontinuities of the first kind at the points  $t_n, n = 1, 2, \dots$  on the interval  $[t_0, +\infty)$  and satisfies

$$u(t) \geq \int_{t_0}^t u(s)v(s)ds + F(t) + \sum_{t_0 < t_i < t} \beta_i u(t_i) + \sum_{t_0 < t_i < t} \alpha_i(t) \tag{9}$$

where  $v \geq 0$  is piecewise continuous,  $\alpha_i, F$  are increasing,  $\beta_i \geq 0$ . Then we have

$$u(t) \leq (F(t) + \sum_{t_0 < t_i < t} \alpha_i(t)) \prod_{t_0 < t_i < t} (1 + \beta_i) e^{\int_{t_0}^t v(s)ds}.$$

*Proof:* See [2].

Now we assume that  $\varphi(t, u, y)$  satisfy the inequality

$$\|\varphi(t, u_1, y_1) - \varphi(t, u_2, y_2)\| \leq \eta \|u_1 - u_2\| + Le^{-\nu(t-t_0)} \|y_1 - y_2\| \tag{10}$$

where  $u_1, u_2 \in X_1; y_1, y_2 \in X_2, 0 < \eta, 0 < \nu$ .

Let  $U(t, s, u | \varphi)$  be the solution of (6) corresponding to  $\varphi$ , starting at the moment  $s$  from  $u$ . We denote

$$\psi_1(t) = U(t, s, u_1 | \varphi_1), \psi_2(t) = U(t, s, u_2 | \varphi_2).$$

It is easy to verify that for equation (6), we have the variation of parameters formula

$$\begin{aligned} \psi_1(t) = & X_1(t, s)u_1 + \int_s^t X_1(t, r)F_1(r, u_1, \varphi(r, u_1, y_1))dr + \\ & + \sum_{s < t_i < t} X_1(t, t_i)I_i^1(\psi_1(t_i) + \varphi(t_i, \psi_1(t_i))) \end{aligned} \tag{11}$$

where  $X_1(t, s)$  is the Cauchy operator of the linear part of (6). So, we have the following estimate for  $t \geq s$

$$\begin{aligned} \|\psi_1 - \psi_2\| \leq & \|X_1(t, s)\| \|u_1 - u_2\| + \int_s^t \|X_1(t, r)\| \|F_1(r, \psi_1(r) + \varphi_1(r, \psi_1(r))) \\ & - F_1(r, \psi_2(r) + \varphi_2(r, \psi_2(r)))\| dr + \\ & + \sum_{s < t_i < t} \|X_2(t, t_i)\| \|I_i^1(\psi_1(t_i) + \varphi_1(t_i, \psi_1(t_i))) - I_i^1(\psi_2(t_i) + \\ & + \varphi_2(t_i, \psi_2(t_i)))\| \leq \end{aligned}$$

$$\begin{aligned} &\leq \kappa e^{\beta(t-s)} \|u_1 - u_2\| + \int_s^t \kappa e^{\beta(t-r)} \delta (\|\psi_1 - \psi_2\| + \eta \|\psi_1 - \psi_2\| + \\ &\quad + L e^{-\nu(r-t_0)} \|y_1 - y_2\|) dr + \sum_{s < t_i < t} \kappa e^{\beta(t-t_i)} \times \\ &\quad \delta (\|\psi_1(t_i) - \psi_2(t_i)\| + \eta \|\psi_1(t_i) - \psi_2(t_i)\| + L e^{-\nu(t_i-t_0)} \|y_1 - y_2\|) \end{aligned}$$

where  $\varphi_1, \varphi_2$  denotes  $\varphi(t, \cdot, y_1), \varphi(t, \cdot, y_2)$  respectively.

But

$$\begin{aligned} u(t) &= e^{-\beta t} \|\psi_1(t) - \psi_2(t)\|, v(t) = \delta \kappa (1 + \eta) \\ F(t) &= \kappa e^{-\beta s} \|u_1 - u_2\| + \delta \kappa L \|y_1 - y_2\| \int_s^t e^{-\nu(r-t_0) - \beta r} dr \\ \beta_i &= (1 + \eta) \kappa \delta, \alpha_i(t) = \delta \kappa L e^{-(\beta + \nu)t_i + \nu t_0} \|y_1 - y_2\|. \end{aligned}$$

Thus, applying Lemma 3, we obtain for  $t \geq s$

$$\begin{aligned} \|\psi_1 - \psi_2\| &\leq [\kappa e^{\beta(t-s)} \|u_1 - u_2\| + \\ &\quad + \frac{\delta \kappa L}{\beta + \nu} e^{\nu t_0} \|y_1 - y_2\| (e^{\beta(t-s) - \nu s} - e^{-\nu t}) + \\ &\quad + \sum_{s < t_i < t} \delta \kappa L e^{\beta t - (\beta + \nu)t_i + \nu t_0} \|y_1 - y_2\| ] (1 + (1 + \eta) \kappa \delta)^{i(s,t)} e^{\delta \kappa (1 + \eta)(t-s)}. \end{aligned}$$

Now for  $t < s$ , we have

$$\begin{aligned} \|\psi_1 - \psi_2\| &\leq \kappa e^{\beta(s-t)} \|u_1 - u_2\| + \int_t^s \kappa e^{\beta(r-t)} \delta ((1 + \eta) \|\psi_1 - \psi_2\| + \\ &\quad + L e^{-\nu(r-t_0)} \|y_1 - y_2\|) dr + \sum_{t < t_i < s} \kappa e^{\beta(t-t_i)} \delta ((1 + \eta) \|\psi_1(t_i) - \psi_2(t_i)\| + \\ &\quad + L e^{-\nu(t_i-t_0)} \|y_1 - y_2\|), \end{aligned} \tag{12}$$

where  $\psi_1(t) = \psi(t, s, u_1 | \varphi_1), \psi_2(t) = \psi(t, s, u_2 | \varphi_2)$ . Thus we get

$$\begin{aligned} e^{\beta t} \|\psi_1(t) - \psi_2(t)\| &\leq \kappa e^{\beta s} \|u_1 - u_2\| + \int_t^s \delta \kappa (1 + \eta) e^{\beta r} \|\psi_1(r) - \psi_2(r)\| dr + \\ &\quad + \delta \kappa \int_t^s L \|y_1 - y_2\| e^{-\nu(r-t_0) + \beta r} dr + \sum_{t < t_i < s} \delta \kappa (1 + \eta) e^{\beta t_i} \|\psi_1(t_i) - \psi_2(t_i)\| + \\ &\quad + \sum_{t < t_i < s} \delta \kappa L e^{-\nu(t_i-t_0) + \beta t_i} \|y_1 - y_2\|. \end{aligned}$$

Put

$$u(t) = e^{\beta t} \|\psi_1(t) - \psi_2(t)\|, v(t) = \delta\kappa(1 + \eta)$$

$$F(t) = \kappa e^{\beta s} \|u_1 - u_2\| + \delta\kappa L \|\gamma_1 - \gamma_2\| \int_t^s e^{-\nu(r-t_0) + \beta r} dr$$

$$\beta_i = (1 + \eta)\kappa\delta, \alpha_i(t) = \delta\kappa L e^{(\beta - \nu)t_i + \nu t_0} \|\gamma_1 - \gamma_2\|.$$

We have the inequality

$$u(t) \leq F(t) + \int_t^s u(r)v(r)dr + \sum_{t < t_i < s} \beta_i U(t_i) + \sum_{t < t_i < s} \alpha_i(t).$$

Note that for  $t < s$ ,  $F$  is nonincreasing. So we have Gronwall's inequality of the same type as for the case  $t > s$ . Finally we get the estimate for each  $t < s$

$$\|\psi(t, s, u_1 | \varphi_1) - \psi(t, s, u_2 | \varphi_2)\| \leq (\kappa e^{\beta(s-t)} \|u_1 - u_2\| +$$

$$+ \delta\kappa L \|\gamma_1 - \gamma_2\| \frac{e^{\nu t_0}}{\beta - \nu} (e^{\beta(s-t) - \nu s} - e^{-\nu t}) + \sum_{t < t_i < s} \delta\kappa L e^{-\beta t + (\beta - \nu)t_i + \nu t_0} \|\gamma_1 - \gamma_2\|) \times$$

$$\times (1 + (1 + \eta)\kappa\delta)^{i(t,s)} e^{\delta\kappa(1 + \eta)(s-t)} \tag{13}$$

We define  $L(\rho, R, \eta, L, \nu, \bar{t})$  as the space of functions  $\varphi(t, u, y)(t \in [t_0, +\infty), u \in X_1, y \in X_2, \|y\| \leq R)$  satisfying condition (10) and  $\|\varphi(t, u, y)\| \leq \rho$  for  $t \geq \bar{t} \geq t_0$ ,

$$\|\varphi\| = \sup_{t \geq \bar{t}; u, \|y\| \leq R} \|\varphi(t, u, y)\|.$$

In this space we shall investigate the operator  $S$  defined by the formula

$$(S\varphi)(t, u, y) = X(t, t_0)y + \int_{t_0}^{\infty} G(t, s)F_2(s, \psi(s, t, u | \varphi) +$$

$$+ \varphi(s, \psi(s, t, u | \varphi), y)ds + \sum_{i=1}^{\infty} G(t, t_i)I_i^2(\psi(t_i, t, u | \varphi)) +$$

$$+ \varphi(t_i, \psi(t_i, t, u | \varphi)) \tag{14}$$

**Lemma 4:** Let  $R, \rho, \eta$  be positive constants,  $L > \kappa, 0 < \nu < \alpha$ . Then there exist  $\delta_0, Q_0$  and  $t_1 > t_0$  such that for every  $0 < \delta < \delta_0, 0 < Q < Q_0, \bar{t} > t_1$  the operator  $S$  acts in  $L(\rho, R, \eta, L, \nu, \bar{t})$ .

**Proof:** First of all, we shall prove that for  $\delta, Q$  small enough,  $\bar{t}$  large enough, we have  $\|S\varphi\| \leq \rho$ . In fact,

$$\begin{aligned} \|(S\varphi)(t, u, y)\| &\leq \kappa e^{-\alpha(t-t_0)} \|y\| + \int_{t_0}^{\infty} \kappa e^{-\alpha|t-s|} Q ds + \\ &\quad + \sum_{i=1}^{\infty} \kappa e^{-\alpha|t-t_i|} Q. \end{aligned} \quad (15)$$

Since  $\{t_n\}$  satisfies condition (2), it is easy to show that

$$\sum_{i=1}^{\infty} e^{-\alpha|t-t_i|} < C_\alpha < +\infty$$

where  $C_\alpha$  depends only on  $\alpha$  and  $\{t_n\}$ . Thus if we choose  $Q$  small enough,  $\bar{t}$  large enough, we get  $\|(S\varphi)(t, u, y)\| \leq \rho$  for  $\|y\| \leq R, t \in [\bar{t}, +\infty), u \in X_1$ . Now we are going to choose  $Q, \delta$  so small that  $\tilde{\varphi} = S\varphi$  satisfies condition (10). We have

$$\begin{aligned} \|\tilde{\varphi}(t, u_1, y_1) - \tilde{\varphi}(t, u_2, y_2)\| &\leq \kappa e^{-\alpha(t-t_0)} \|y_1 - y_2\| + \\ &\quad + \int_{t_0}^{\infty} \delta \kappa e^{-\alpha|t-s|} ((1+\eta) \|\psi(s, t, u_1 | \varphi_1) - \psi(s, t, u_2 | \varphi_2)\| + \\ &\quad \quad + L e^{-\nu(s-t_0)} \|y_1 - y_2\|) ds + \\ &\quad + \sum_{i=1}^{\infty} \delta \kappa e^{-\alpha|t-t_i|} ((1+\eta) \|\psi(t_i, t, u_1 | \varphi_1) - \psi(t_i, t, u_2 | \varphi_2)\| + \\ &\quad \quad + L e^{-\nu(t_i-t_0)} \|y_1 - y_2\|). \end{aligned}$$

Based on the estimates (12), (13), we obtain

$$\begin{aligned} \|\tilde{\varphi}(t, u_1, y_1) - \tilde{\varphi}(t, u_2, y_2)\| &\leq \kappa e^{-\alpha(t-t_0)} \|y_1 - y_2\| + \\ &\quad + \int_{t_0}^{\infty} \delta \kappa L e^{-\alpha|t-s| - \nu(s-t_0)} \|y_1 - y_2\| ds + \sum_{i=1}^{\infty} \delta \kappa L e^{-\alpha|t-t_i| - \nu(t_i-t_0)} \|y_1 - y_2\| + \\ &\quad + \int_{t_0}^{\infty} \delta \kappa (1+\eta) e^{-\alpha|t-s|} \|\psi(s, t, u_1 | \varphi_1) - \psi(s, t, u_2 | \varphi_2)\| ds + \\ &\quad + \sum_{i=1}^{\infty} \delta \kappa (1+\eta) e^{-\alpha|t-t_i|} \|\psi(t_i, t, u_1 | \varphi_1) - \psi(t_i, t, u_2 | \varphi_2)\| \leq \\ &\leq N(\delta) \|y_1 - y_2\| + \int_{t_0}^t \delta \kappa (1+\eta) e^{-\alpha(t-s)} (\kappa e^{\beta(t-s)} \|u_1 - u_2\| + \\ &\quad + \|y_1 - y_2\| D(e^{\beta(t-s) - \nu t} - e^{-\nu s}) + \sum_{s < t_i < t} \delta \kappa L e^{-\beta s + (\beta - \nu)t_i + \nu t_0} \|y_1 - y_2\|) \times \end{aligned}$$



$$\begin{aligned}
& \times (1 + (1 + \eta)\kappa\delta)^{i(s,t)} e^{\delta\kappa(1 + \eta)(t-s)} ds + \\
& + \int_t^\infty \delta\kappa(1 + \eta)e^{-\alpha(s-t)} (\kappa e^{\beta(s-t)} \|u_1 - u_2\| + D' \|y_1 - y_2\| (e^{\beta(s-t) - \nu t} - \\
& - e^{-\nu s}) + \sum_{t < t_i < s} \delta\kappa L e^{\beta s - (\beta + \nu)t_i + \nu t_0} \|y_1 - y_2\|) (1 + (1 + \eta)\kappa\delta)^{i(t,s)} e^{\delta\kappa(1 + \eta)(s-t)} + \\
& + \sum_{t_i < t} \delta\kappa e^{-\alpha(t-t_i)} (\kappa e^{\beta(t-t_i)} \|u_1 - u_2\| + D \|y_1 - y_2\| \times \\
& \times (e^{\beta(t-t_i) - \nu t} - e^{-\nu t_i}) + \sum_{t_i < t_j < t} \delta\kappa L e^{-\beta t_j + (\beta - \nu)t_i + \nu t_0} \|y_1 - y_2\|) \times \\
& \times (1 + (1 + \eta)\kappa\delta)^{i(t_i,t)} e^{\delta\kappa(1 + \eta)(t-t_i)} + \sum_{t < t_i} \delta\kappa(1 + \eta) e^{-\alpha(t_i-t)} \times \\
& \times (\kappa e^{\beta(t_i-t)} \|u_1 - u_2\| + D' \|y_1 - y_2\| (e^{\beta(t_i-t) - \nu t} - e^{-\nu t_i}) + \\
& + \sum_{t < t_j < t_i} \delta\kappa L e^{\beta t_i - (\beta + \nu)t_j + \nu t_0} (1 + (1 + \eta)\kappa\delta)^{i(t,t_i)} \times \\
& \times e^{\delta\kappa(1 + \eta)(t_i-t)}
\end{aligned} \tag{16}$$

where  $N(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  and  $t_1$  is large enough,  $D, D'$  are positive constants. Put  $\delta\kappa(1 + \eta) = \gamma$ . Then, if  $\delta$  is small enough and (2) is satisfied, we have

$$0 < (1 + \gamma)^{i(s,t)} e^{\gamma(s-t)} < e^{-\epsilon(s-t)}, 0 < \epsilon < \alpha - \beta.$$

Finally, we get the estimate

$$\|\tilde{\varphi}(t, u_1, y_1) - \tilde{\varphi}(t, u_2, y_2)\| \leq M(\delta) \|u_1 - u_2\| + \kappa(\delta) e^{-\nu(t-t_0)} \|y_1 - y_2\|$$

where  $M(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ ,  $\kappa(\delta) < L$  if  $\delta$  is small enough and  $\kappa < L$ . This completes the proof of Lemma 4.

#### 4. MAIN RESULT

**Theorem 1:** Assume that the linear part of (1) has an exponential trichotomy with the above notation. Then for every  $\eta, \rho, R, 0 < \nu < \alpha$  and  $L$  large enough, there exist positive constants  $Q_0, \delta_0, \bar{t}$  with the following properties:

Let  $\delta, Q$  belong to  $(0, \delta_0), (0, Q_0)$  respectively. Then for every  $y \in X_2, \|y\| \leq R$  equation (1) has an integral manifold represented by a function  $\varphi \in L(\rho, \eta)$ . In addition,  $\varphi$  satisfies the conditions:

- 1)  $P_2(u + \varphi(t, u, y)) = y$ ,  $P_2$  is the projection:  $X \rightarrow X_2$
- 2)  $\|\varphi(t, u, y)\| \leq \rho$  for  $t - t_0 \geq \bar{t}$ ,  $\|y\| \leq R$
- 3)  $\|\varphi(t, u_1, y_1) - \varphi(t, u_2, y_2)\| \leq \eta \|u_1 - u_2\| + l e^{-\nu(t-t_0)} \|y_1 - y_2\|$ .

**Proof:** From Lemma 4, it follows that there exist positive constants  $Q_0, \delta_0, t_1$  such that  $S$  acts in  $L(\rho, R, \eta, L, \nu, \bar{t})$ . Now we are going to show that if  $Q_0, \delta_0$  are chose small enough, then  $S$  is contracting. In fact, we have

$$\begin{aligned} \|(S\varphi)(t, u, y) - (S\varphi')(t, u, y)\| &\leq \int_{t_0}^{\infty} \|G(t, s)\| \delta(\|\psi(s, t, u | \varphi) - \\ &\quad - \psi(s, t, u | \varphi')\| + \|\varphi(s, \psi, t, u | \varphi) - \varphi'(s, \psi(s, t, u | \varphi'))\|) ds \\ &\quad + \sum_{i=1}^{\infty} \|G(t, t_i)\| \delta(\|\psi(t_i, t, u | \varphi) - \psi(t_i, t, u | \varphi')\| + \\ &\quad \|\varphi(t_i, \psi'(t_i, t, u | \varphi) - \varphi'(t_i, \psi(t_i, t, u | \varphi'))\|), \end{aligned}$$

where  $\psi(t, s, u | \varphi)$  is the solution of (6).

On the other hand, we have

$$\begin{aligned} \|\psi(s, t, u | \varphi) - \psi(s, t, u | \varphi')\| &\leq \int_t^s \kappa \delta e^{\beta(s-\xi)} (\|\psi_1 - \psi_2\| + \|\varphi - \varphi'\|) d\xi \\ &\quad + \sum_{t < t_i < s} \delta \kappa e^{\beta(s-t_i)} (\|\psi_1(t_i) - \psi_2(t_i)\| + \|\varphi - \varphi'\|). \end{aligned}$$

Setting  $u(s) = e^{-\beta s} \|\psi_1(s) - \psi_2(s)\|$  and applying Gronwall's inequality, we have

$$\begin{aligned} u(s) &\leq \left( \int_t^s \kappa \delta e^{-\beta \xi} d\xi + \sum_{t < t_i < s} \kappa \delta e^{-\beta t_i} \right) \|\varphi - \varphi'\| \times \\ &\quad (1 + \delta \kappa)^{i(t,s)} e^{\delta \kappa(s-t)}. \end{aligned} \tag{18}$$

So we have

$$\|\psi(s, t, u | \varphi) - \psi'(s, t, u | \varphi')\| \leq D(\delta) e^{\beta(s-t)} \|\varphi - \varphi'\| \tag{19}$$

where  $D(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

Substituting (19) into (17), we have

$$\begin{aligned} \|(S\varphi)(t, u, y) - (S\varphi')(t, u, y)\| &\leq \left[ \delta \int_{t_0}^{\infty} \kappa e^{-\alpha|t-s|} (D(\delta) e^{\beta|s-t|} + 1) ds \right. \\ &\quad \left. + \sum_{i=1}^{\infty} \kappa \delta e^{-\alpha|t-t_i|} (D(\delta) e^{\beta|t_i-t|} + 1) \right] \|\varphi - \varphi'\|. \end{aligned} \tag{20}$$

From (2), (20) it follows that if  $\delta$  is sufficiently small, then  $S$  is contracting. This completes the proof of the theorem.

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