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ON (h_0, h)-STABILITY OF AUTONOMOUS SYSTEMS[1]

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ABSTRACT

In this paper, we discuss the qualitative behavior of a map h along solutions of an autonomous system whose initial values are measured by a second map h_0. By doing this, we may deal with, in a unified way, several concepts and associated problems, which are usually considered separately. Five theorems on asymptotic stability are given and two examples are worked out.

Key words: Stability in terms of two measures, autonomous systems, invariance principle, Lyapunov function.

AMS (MOS) subject classifications: 34D20, 93D05.

1. INTRODUCTION

It is well known that LaSalle's invariance principle [3] is one of the most useful results in applications since it allows using the total energy as a Lyapunov function to obtain asymptotic stability in mechanical problems with dissipation. The key idea of the invariance principle is to use Lyapunov's method to locate an attractive set and then to refine the result by using invariance properties of its subsets. Recently, Hatvani [1] successfully extended the invariance principle to the study of partial stability which improves LaSalle's result significantly.

Due to the needs of applications, there are several different concepts of stability studied in the literature, such as orbital stability, partial stability, conditional stability, just to name a few. To unify these varieties of stability notions and to offer a general framework for investigation, the stability concepts in terms of two different measures have been proven very useful. See Lakshmikantham and Liu [2] for a detailed discussion of this point.

We shall discuss, in this paper, stability properties in terms of two measures for autonomous differential systems and extend the invariance principle to the study of (h_0, h)-stability.

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2. PRELIMINARIES

Let us list the following definitions and classes of functions for convenience.

$K = [\sigma \in C[R_+, R_+]; \sigma(u)$ is strictly increasing and $\sigma(0) = 0]$.

$\Gamma = [h \in C[R^n, R_+; \inf_{x \in R^n} h(x) = 0]$.

Consider the differential system

$$x' = f(x) \tag{2.1}$$

where $x \in R^n$, $f \in C[R^n, R^n]$. For $\rho > 0$, we define $s(h,\rho) = [x \in R^n; h(x) < \rho]$. Let $x(t) = x(t, x_0)$ denote any solution of (2.1) with $x(0) = x_0$. We shall assume that if $x(t)$ is a solution of (2.1) so that $h(x(t)) < \rho$ for $t \in [0, \alpha)$, then $x(t)$ can be continued to the closed interval $[0, \alpha]$. For $y \in S(h, \rho)$, we denote by $\gamma(y)$ the positive trajectory of (2.1) passing through y i.e. $\gamma(y) = \{x(t, y); t \geq 0\}$, and by $\Omega_\sigma(x_0)$ all the positive trajectories of (2.1) passing through x_0 with $h_0(x_0) < \sigma$, i.e. $\Omega_\sigma(x_0) = [x \in R^n; x = x(t, x_0), t \geq 0$, and $h_0(x_0) < \sigma]$.

Definition 2.1: Let $h_0, h \in \Gamma$. Then we say that h_0 is finer than h if there exists a constant $\rho > 0$ and a function $\phi \in K$ such that

$$h_0(x) < \rho \text{ implies } h(x) < \phi(h_0(x)).$$

Definition 2.2: Let $V \in C[R^n, R_+]$. Then the generalized derivative of V along solutions of (2.1) is defined by

$$D^+V(x) = \lim_{\delta \to 0^+} sup \frac{1}{\delta}[V(x + \delta f(x)) - V(x)].$$

We define the set E by

$$E = \{x \in \overline{s(h,\rho)},\ D^+V(x) = 0\}.$$

Definition 2.3: Let $h_0, h \in \Gamma$ and $V \in C[R^n, R_+]$. Then V is said to be

(i) h-positive definite if for some $\rho > 0$ and $b \in K$, $h(x) < \rho$ implies $V(x) \geq b(h(x))$;

(ii) h_0-decrescent if for some $\rho > 0$ and $a \in K$, $h_0(x) < \rho$ implies $V(x) \leq a(h_0(x))$.

Definition 2.4: Let $h_0, h \in \Gamma$. The system (2.1) is said to be

(i) (h_0, h)-stable if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $h_0(x_0) < \delta$ implies $h(x(t)) < \epsilon$, $t \geq 0$, where $x(t) = x(t, x_0)$ is any solution of (2.1);

(ii) (h_0, h)-attractive if there exists a $\sigma > 0$ such that $h_0(x_0) < \sigma$ implies $h(x(t)) \to 0$ as $t \to \infty$;

(iii) (h_0, h)-asymptotically stable if (i) and (ii) hold;

(*iv*) (h_0, h)-unstable if (*i*) fails to hold.

Definition 2.4 describes the qualitative behavior of a map $h \in \Gamma$ along solutions $x(t)$ of system (2.1) whose initial values are measured by a second map $h_0 \in \Gamma$. By using this definition, we can deal with, in a unified way, several concepts and associated problems, which are usually considered separately. It is easy to see that Definition 2.4 reduces to

(1) the well-known stability of the trivial solution of (2.1) if $h(x) = h_0(x) = \| x \|$, where $\| \cdot \|$ denote the Euclidian norm in R^n;

(2) orbital stability if $h(x) = h_0(x) = d(x, \gamma)$, where γ is a given periodic orbit of (2.1), and d is the distance function;

(3) partial stability of the trivial solution of (2.1) if $h(x) = \| x \|_s$, $1 \leq s < n$, and $h_0(x) = \| x \|$;

(4) the stability of an invariant set $A \subset R^n$ if $h(x) = h_0(x) = d(x, A)$;

(5) the stability of conditionally invariant set B with respect to A, where $A \subset B \subset R^n$, if $h(x) = d(x, B)$ and $h_0(x) = d(x, A)$.

3. MAIN RESULTS

We state and prove our main results in this section. Let us begin by proving a result on (h_0,h)-asymptotic stability under weaker assumptions.

Theorem 3.1: *Assume that*

(*i*) *$h_0, h \in \Gamma$ and h_0 is uniformly finer than h;*

(*ii*) *$V \in C[R^n, R_+], V(x)$ is locally Lipschitzian in x, h-positive definite, h_0-decrescent and*

$$D^+ V(x) \leq 0 \text{ on } S(h, \rho),$$

(*iii*) *the set $s(h, \rho) \cap \Omega_\sigma(x_0)$ is precompact;*

(*iv*) *for any $c > 0$, the set $E \cap V^{-1}(c)$ contains no complete positive trajectory of* (2.1), *where $V^{-1}(x) = \{x \in \overline{s(h, \rho)}; V(x) = c\}$.*

Then the system (2.1) *is (h_0, h)-asymptotically stable.*

Proof: Assumptions (*i*)-(*ii*) imply that the system (2.1) is (h_0, h)-stable. Thus for $\rho > 0$, there exists a $\delta_0 = \delta_0(\rho) > 0$ such that

$$h_0(x) < \delta_0 \text{ implies } h(x(t, x_0)) < \rho, \; t \geq 0. \tag{3.1}$$

Choose $\delta = min\{\delta_0, \sigma\}$. Then by assumption (iii) and (3.1) we see that $h_0(x_0) < \delta$ implies that $x(t, x_0)$ is bounded and $h(x(t, x_0)) < \rho$, $t \geq 0$. Since $V(x(t, x_0))$ is nonincreasing and bounded from below, it follows that $\lim_{t \to \infty} V(x(t, x_0)) = c \geq 0$. Suppose, for the sake of contradiction, that $c > 0$. Since $x(t, x_0)$ is bounded, it follows that $\Omega(x_0)$ is nonempty and invariant. Then for $y \in \Omega(x_0), x(t, y) \in \Omega(x_0), t \in [0, \infty)$. Thus $V(x(t, y)) \equiv c$ and $D^+V(x(t, y)) \equiv 0$ for $t \in [0, \infty)$. Hence $\gamma(y) \subset E \cap V^{-1}(c)$, which contradicts assumption (iv). So, we must have $c = 0$. Since $V(x)$ is h-positive definite, this shows $\lim_{t \to \infty} h(x(t, x_0)) = 0$. Thus the system (2.1) is (h_0, h)-asymptotically stable and the proof is complete.

Remark: If $h_0(x) = h(x) = \| x \|$, then condition (iii) is a consequence of conditions (i) and (ii) and Theorem 3.1 reduces to LaSalle's result [3]. In case $h_0(x) = \| x \|$ and $h(x) = | x |_s$, $a \leq s < n$, then Theorem 3.1 includes Oziraner's result [4] on partial stability.

If we remove the condition (iii) in Theorem 3.1, i.e. without demanding the boundedness of solutions of (2.1), then we have the following result.

Theorem 3.2: *Assume that*

(i) *$h, h^*, h_0 \in \Gamma$, and $h(x) + h^*(x) = \phi(h_0(x))$, where $\phi \in K$ and $\phi(h_0(x)) \to \infty$ as $\| x \| \to \infty$;*

(ii) *assumptions (ii) and (iv) of Theorem 3.1 hold.*

Then there exists a constant $\delta > 0$ such that $h_0(x_0) < \delta_0$ implies that either $V(x(t, x_0)) \to 0$ or $h^*(x(t, x_0)) \to \infty$ as $t \to \infty$.

Proof: By condition (i), $h(x) \leq \phi(h_0(x))$, which means that h_0 is uniformly finer than h. This, together with condition (iii) of Theorem 3.1 implies that the system (2.1) is (h_0, h)-stable. Thus for $\epsilon = \rho > 0$, there exists a $\delta_0 = \delta_0(\rho) > 0$ such that

$$h_0(x_0) < \delta_0 \text{ implies } h(x(t, x_0)) < \rho, \ t \geq 0. \tag{3.2}$$

Let $x(t, x_0)$ be a solution of (2.1) such that $h^*(x(t, x_0)) \not\to \infty$ as $t \to \infty$. Then there exists a sequence $t_n \in R_+$, $t_n \to \infty$ as $n \to \infty$ such that $\{h^*(x(t_n, x_0))\}$ and $\{h(x(t_n, x_0))\}$ are bounded which implies, by condition (i), that $\{x(t_n, x_0)\}$ is bounded. Thus $\Omega(x_0)$ is nonempty. Since $V(x(t, x_0)) \to c \geq 0$ as $t \to \infty$, then there exists a $y \in \Omega(x_0)$ such that $V(y) = c$. But the set $\Omega(x_0)$ is invariant, consequently, the set $E \cap V^{-1}(c)$ contains a complete positive trajectory. This, together with assumption (iv) of Theorem 3.1, implies $c = 0$. Thus the proof is complete.

Employing Theorem 3.2, one can get (h_0, h)-asymptotic stability as follows.

Theorem 3.3: *Let the assumptions of Theorem 3.2 hold. Suppose further that*

(A) $V(x) \to 0$ *as* $D^+V(x) \to 0$ *and* $h^*(x) \to \infty$.

Then the system (2.1) *is* (h_0, h)*-asymptotically stable.*

Proof: (h_0, h)-stability of (2.1) is immediate. Thus there exists $\delta_0 = \delta_0(\rho) > 0$ such that $h_0(x_0) < \delta_0$ implies $h(x(t, x_0)) < \rho$, $t \geq 0$. Let $x(t, x_0)$ be a solution of (2.1) with $h_0(x_0) < \delta_0$. Then by Theorem 3.2, $V(x(t, x_0)) \to 0$ as $t \to \infty$ or $h^*(x(t, x_0)) \to \infty$ as $t \to \infty$. Since $V(x)$ is nonnegative and $D^+V(x(t, x_0)) \leq 0$, $t \geq 0$, it follows that $\lim_{t \to \infty} \sup D^+V(x(t, x_0)) = 0$, which implies that there exists a sequence $t_n \in R^+$ such that $D^+V(x(t_n, x_0)) \to 0$ as $n \to \infty$. Suppose that $\lim_{t \to \infty} V(x(t, x_0)) = c \neq 0$. Then $h^*(x(t_n, x_0)) \to \infty$ as $n \to \infty$, which implies, by assumption (A), $V(x(t_n, x_0)) \to 0$ as $n \to \infty$. This is impossible. Thus we must have $V(x(t, x_0)) \to 0$ as $t \to \infty$. Since $V(x)$ is h-positive definite, this in turn implies that $h(x(t, x_0)) \to 0$ as $t \to \infty$ and hence the system (2.1) is (h_0, h)-asymptotically stable, completing the proof.

Example 3.1: Consider the nonlinear system

$$\begin{cases} x_1' = -x_1(1 + x_3^2), \\ x_2' = -x_2, \\ x_3' = x_3 - x_3^3. \end{cases} \tag{3.3}$$

Choose $V(x) = x_1^2 + x_2^2 + x_2^2 x_3^2$, $h(x) = x_1^2 + x_2^2$, $h^*(x) = x_3^2$ and $h_0(x) = h(x) + h^*(x)$. Then $V(x)$ is h-positive definite, h_0-decrescent and continuously differentiable. The derivative of $V(x)$ along solutions of (3.3) is

$$D^+V(x) = -2[x_1^2(1 + x_3^2) + x_2^2 + x_2^2 x_3^4] \leq 0.$$

It is easy to see that $D^+V(x) \to 0$ *iff* $x_1 \to 0$, $x_2 \to 0$ and $x_2^2 x_3^4 \to 0$. If $h^*(x) \to \infty$, then we must have $x_2^2 x_3^2 \to 0$, which implies

$$V(x) \to 0 \text{ as } D^+V(x) \to 0 \text{ and } h^*(x) \to \infty.$$

Thus condition (A) of Theorem 3.3 is satisfied. It is easy to verify that all other conditions of Theorem 3.3 are met. Thus the system (3.3) is (h_0, h)-asymptotically stable.

Theorem 3.4: *Assume that conditions* (i) *and* (ii) *of Theorem 3.1 hold. Suppose further that*

(iii^*) *for any* $c > 0$, *any complete positive trajectory of* (2.1) *contained in* $V^{-1}(c) \cap E$ *is also contained in the set* $N = \{x \in R^n, h(x) = 0\}$;

(iv^*) $h(x) \to 0$ *as* $\| x \| \to \infty$ *if* $h(x)$ *is bounded.*

Then the system (2.1) *is* (h_0, h)*-asymptotically stable.*

Proof: Evidently, system (2.1) is (h_0, h)-stable. Thus there exists $\delta_0 = \delta_0(\rho) > 0$ such that

$$h_0(x_0) < \delta_0 \text{ implies } h(x(t, x_0)) < \rho, \; t \geq 0. \tag{3.4}$$

Let $x(t, x_0)$ be a solution of (2.1) satisfying (3.4) and define $m(t) = V(x(t, x_0))$, $t \in R_+$. Then $m(t)$ is nonincreasing and bounded from below, so $\lim_{t \to \infty} m(t) = c$ exist. Let us assume $c > 0$ for otherwise $c = 0$ and the theorem is proved. Since $h(x(t, x_0))$ is bounded for any $t \geq 0$, it follows from condition (iv^*) that we only need consider the case when $\| x(t, x_0) \| \not\to \infty$ as $t \to \infty$. If $\| x(t, x_0) \| \not\to \infty$ as $t \to \infty$, then $w(x_0) \neq \phi$ and $w(x_0) \subset V^{-1}(c) \cap E$. Let $y \in w(x_0)$. Then $\gamma(y) \subset w(x_0)$ since $w(x_0)$ is invariant. By condition (iii^*), $\gamma(y) \subset N$ which implies $w(x_0) \subset N$ and $h(y) = 0$ for any $y \in w(x_0)$. If there exists a sequence $\{x(t_k, x_0)\}$ such that $\| x(t_k, x_0) \| \to \infty$ as $k \to \infty$, then by conditions (iv^*) $\lim_{k \to \infty} h(x(t_k, x_0)) = 0$. Thus we conclude that $\lim_{t \to \infty} h(x(t, x_0)) = 0$ for any solution $x(t, x_0)$ of (2.1) satisfying (3.4) and hence the system (2.1) is (h_0, h)-asymptotically stable.

The following result is a direct consequence of Theorem 3.4 and its proof is omitted.

Theorem 3.5: *Let conditions* (*i*) *and* (*ii*) *of Theorem 3.1 and condition* (iii^*) *of Theorem 3.4 hold. Suppose further that*

$$\lim_{\substack{|x| \to \infty \\ 0 < h(x) < \rho}} V(x) = \infty. \tag{3.5}$$

Then the system (2.1) *is* (h_0, h)*-asymptotically stable.*

Example 3.2: Consider the nonlinear system

$$\begin{cases} x_1' = x_3, \\ x_2' = x_4, \\ x_3' = \dfrac{1}{1 + x_1^2}(x_1 x_4^2 - x_1 x_3^2 - 2x_1^2 x_3), \\ x_4' = \dfrac{1}{2 + x_1^2 + x_2^2}(-2x_2 - x_2 x_4^2 - 2x_1 x_3 x_4 - 3x_2 x_4). \end{cases} \tag{3.6}$$

Let $V(x) = \frac{1}{2}(x_1^2 x_4^2 + x_2^2 x_4^2 + x_1^2 x_3^2 + x_3^2 + 2x_4^2) + x_2^2$, $h(x) = x_3^2 + x_4^2$ and $h_0(x) = \sum_{i=1}^{4} x_i^2$. Then $V(x)$ is h-positive definite, h_0-decrescent and continuously differentiable. The derivative of $V(x)$ along solutions of (3.6) is

$$D^+ V(x) = -2x_1^2 x_3^2 - 3x_2^2 x_4^2 \leq 0.$$

$E = \{x \in R^4; x_1x_3 = 0 \text{ and } x_2x_4 = 0\}$. The largest invariant set contained in E is the (x_1, x_2)-plane where $h(x)$ vanishes. Since $\lim_{x_2 \to \infty} V(x) = \infty$, it follows that condition (3.5) is satisfied. Thus by Theorem 3.5 the system (3.6) is (h_0, h)-asymptotically stable.

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