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QUASILINEARIZATION FOR SOME NONLOCAL PROBLEMS[1]

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ABSTRACT

The method of generalized quasilinearization [4] is applied to study semilinear parabolic equation $u_t - Lu = f(t,x,u)$ with nonlocal boundary conditions $u(t,x) = \int_\Omega \phi(x,y)u(t,y)dy$ in this paper. The convexity of f in u is relaxed by requiring $f(t,x,u) + Mu^2$ to be convex for some $M > 0$. The quadratic convergence of monotone sequence is obtained.

Key words: Quasilinearization, nonlocal problem, semilinear parabolic equation, upper and lower solutions, monotone iterative technique, quadratic convergence.

AMS (MOS) subject classifications: 35K20, 35K60, 35K99.

1. INTRODUCTION

In this paper, we consider the nonlocal boundary value problem (NBVP for short):

$$u_t - Lu = f(t,x,u) \text{ in } D_T = (0,T] \times \Omega, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \forall x \in \Omega, \tag{1.2}$$

$$u(t,x) = \int_\Omega \phi(x,y)u(t,y)dy \quad \forall (t,x) \in \Gamma_T, \tag{1.3}$$

where Ω is a bounded domain in R^n, $\partial\Omega \in C^2$, $\Gamma_T = (0,T) \times \partial\Omega$,

$$L = \sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2}{\partial x^2} + \sum_{i=1}^n b_i(t,x)\frac{\partial}{\partial x}, \quad a_{ij}, b_i \in C^{0,\alpha},\ 0 < \alpha < 1.$$

$u_0(x) \in C(\bar\Omega)$, $f \in C[\bar D_T \times R, R]$ and satisfies Lipschitz condition in u. $\phi \in C[\partial\Omega \times \bar\Omega, R]$ satisfies $\phi(x,y) \geq 0$, $\int_\Omega \phi(x,y)dy \neq 0$, $\forall\ x \in \partial\Omega$ and $\int_\Omega \phi(x,y)dy \leq \rho < 1$, $\forall x \in \partial\Omega$.

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It is shown in [5] that, if v, w are lower and upper solutions of NBVP such that $v \leq w$, then there exist monotone sequences which converge uniformly to the unique solution of NBVP. The method of upper and lower solutions and monotone iterative techniques played most important roles in constructing monotone sequences.

Consider the following ordinary differential equation with initial condition :

$$u' = g(t,u),\ u(0) = u_0,\ t \in J = [0,T]. \tag{1.4}$$

It is well known [1, 2] that the method of quasilinearization provides not only the monotone sequence but also quadratic convergence of the sequence for (1.4) if requiring more on $g(t,u)$, namely, $g(t,u)$ to be convex in u for $t \in J$. The result is of computational interest. In [4], this method has been extended to more general cases, namely, without demanding the convexity of $g(t,u)$, by requiring $g(t,u) + Mu^2$ to be convex for some $M > 0$ and the same results have been obtained [4].

Our purpose, in this paper, is to apply the method used in [4] to study nonlocal problem (1.1) – (1.3).

2. PRELIMINARIES

Consider NBVP (1.1)-(1.3). A function $v_0 \in C^{1,2}(D_T) \cap C(\bar{D}_T)$ is called a lower solution of NBVP if

$$(v_0)_t - Lv_0 \leq f(t,x,v_0) \quad \text{in } D_T,$$

$$v_0(0,x) \leq u_0(x) \quad \forall x \in \Omega,$$

$$v_0(t,x) \leq \int_\Omega \phi(x,y)v_0(t,y)dy \quad \text{on } \Gamma_T.$$

An upper solution w_0 is defined analogously by reversing the above inequalities with respect to w_0. It is shown[5] that $v_0 \leq w_0$ on $\bar{\Omega}$.

We first state two lemmas which we need in the proof of the main result. Define $Q_T = \{(t,x,u): v_0 \leq u \leq w_0, (t,x) \in \bar{D}_T\}$. If the existence of upper and lower solutions of NBVP is known, then we can prove the existence of solutions of NBVP in the closed set Q_T which is the statement of Lemma 2.1. For proof, see [5].

Lemma 2.1: *Consider*

$$u_t - Lu = G(t,x,\eta(t,x),u(t,x)) \quad in\ D_T, \tag{2.1}$$

with (1.2) *and* (1.3), *where* $\eta(t,x)$ *is a given function on* $\bar{D}_T$. *Suppose that*

(*i*) v_0, w_0 *are lower solutions of* (2.1), (1.2) *and* (1.3),

(*ii*) $G(t,x,\eta,u)$ *satisfies Lipschitz condition in* u, *i.e.*, $\exists N > 0$ *such that*

$$-N(u_1 - u_2) \le G(t,x,\eta,u_1) - G(t,x,\eta,u_2) \le N(u_1 - u_2), \tag{2.2}$$

whenever $u_1 \ge u_2$, *for* (t,x,u_1), $(t,x,u_2) \in Q_T$. *Then there exists a unique solution* u *of NBVP such that* $u \in C^{1,2}(D_T) \cap C(\bar{D}_T)$ *and* $v_0 \le u \le w_0$ *on* $\bar{D}_T$.

Lemma 2.2: *Assume that* $p \in C^{1,2}(D_T) \cap C(\bar{D}_T)$ *satisfies*

$$p_t(Lp + Kp + C) \le 0 \quad \text{in } D_T,$$

$$p(0,x) \le 0 \quad \forall x \in \Omega,$$

$$p(t,x) \le \int_\Omega \phi(x,y)p(t,y)dy \quad \forall (t,x) \in \Gamma_T,$$

where $K > 0$, $C > 0$ *are constants. Then*, $p(t,x) \le Ce^{(K+1)t}$ *on* $\bar{D}_T$.

Proof: Let $\Psi = Ce^{(K+1)t}$, then Ψ verifies

$$\Psi_t - (L\Psi + K\Psi + C) = C(e^{(K+1)t} - 1) > 0 \quad \text{in } D_T,$$

$$\Psi(0,x) > 0 \quad \forall x \in \Omega,$$

$$\Psi(t,x) > \int_\Omega \phi(x,y)\Psi(t,y)dy \quad \text{on } \Gamma_T.$$

By Theorem 2.1 of [5], $p(t,x) \le \Psi$ on $\bar{D}_T$. This completes the proof of the lemma.

3. MAIN RESULTS

In this section, we shall apply the method of quasilinearization generalized in [3] to NBVP in order to obtain the monotone sequence and quadratic convergence of the sequence. Suppose now, $f \in C^2$ in u and $f + Mu^2$ is uniformly convex for $(t,x,u) \in Q_T$. Define $F(t,x,u) \equiv f(t,x,u) + Mu^2$, then

$$F_{uu} = f_{uu} + 2M > 0. \tag{3.1}$$

If $u_1 \ge u_2$, it follows

$$F_u(t,x,u_1) = f_u(t,x,u_1) + 2Mu_1 \ge F_u(t,x,u_2) + 2Mu_2;$$

$$F(t,x,u_1) - F_u(t,x,u_2)u_1 \ge F(t,x,u_2) - F_u(t,x,u_2)u_2. \tag{3.2}$$

We also assume that there exist constants $K_1 > 0$, $C_1 > 0$ such that

$$| F_u | \leq K_1 \text{ and } F_{uu} \leq C_1, \text{ for } (t,x,u) \in Q_T.$$

Now, we are in the position to prove the main results.

Theorem 3.1: *Suppose that*

(*i*) *v_0, w_0 are lower and upper solutions of NBVP such that $v_0 \leq w_0$ on $\bar{D}_T$;*

(*ii*) *$F(t,x,u) \equiv f(t,x,u) + Mu^2$ is uniformly convex in Q_T.*

Then, there exists a monotone sequence $\{v_n\}$ which converges uniformly to the unique solution of NBVP and moreover, the convergence is quadratic.

Proof: Consider the modified nonlocal problem (P^*):

$$\begin{aligned} u_t - Lu &= f(t,x,\eta) + (f_u(t,x,\eta) + 2M\eta)(u-\eta) - M(u^2 - \eta^2) \\ &\equiv F(t,x,\eta) + F_u(t,x,\eta)(u-\eta) - Mu^2, \end{aligned} \tag{3.3}$$

with (1.2) and (1.3), where $\eta \in C^{1,2}(D_T) \cap (\bar{D}_T)$ is such that

$$v_0(t,x) \leq \eta(t,x) \leq w_0(t,x) \text{ on } D_T.$$

Define

$$G(t,x,\eta,u) \equiv F(t,x,\eta) + F_u(t,x,\eta)(u-\eta) - Mu^2. \tag{3.4}$$

Since v_0 is a lower solution of NBVP and if $\eta = v_0$, because of (3.4), we have

$$(v_0)_t - Lv_0 \leq f(t,x,v_0) \equiv G(t,x,v_0,v_0) \text{ in } D_T, \tag{3.5}$$

From (3.3) and (3.4),

$$u_t - Lu = G(t,x,v_0,u) \text{ in } D_T. \tag{3.6}$$

Similarly,

$$\begin{aligned} &(w_0)_t - Lw_0 \geq f(t,x,w_0) \\ &\geq F(t,x,v_0) + F_u(t,x,v_0)(w_0 - v_0) - Mw_0^2 \equiv G(t,x,v_0,w_0) \text{ in } D_T. \end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7) together with the initial and boundary conditions, we conclude that $v_0(t,x)$ and $w_0(t,x)$ are lower and upper solutions of (P^*) with $\eta = v_0$. It is easy to see that $G(t,x,v_0,u)$ satisfies Lipschitz condition (2.2) for $(t,x,u) \in Q_T$ since $F(t,x,u)$ is convex in u and $v_0(t,x)$ is a given function. Therefore, by Lemma 2.1, there exists a unique solution u of

(P^*) with $\eta = v_0$ such that $u \in C^{1,2}(D_T) \cap C(\bar{D}_T)$ and $v_0 \le u \le w_0$ on $\bar{D}_T$.

Now, we construct the sequence $\{v_n\}$ by

$$\begin{aligned}(v_{n+1})_t - Lv_{n+1} &= f(t,x,v_n) + (f_u(t,x,v_n) + 2Mv_n)(v_{n+1} - v_n) - M(v_{n+1}^2 - v_n^2)\\ &\equiv F(t,x,v_n) + F_u(t,x,v_n)(v_{n+1} - v_n) - Mv_{n+1}^2 \ \text{ in } D_T,\\ v_{n+1}(0,x) &= u_0(x) \quad \forall x \in \Omega,\\ v_{n+1}(t,x) &= \int_\Omega \phi(x,y)v_{n+1}(t,y)dy \text{ on } \Gamma_T,\end{aligned} \tag{3.8}$$

for $n = 0, 1, 2, \ldots$.

Clearly, from the above discussion, we obtain that (P^*) has a unique solution v_1 with $\eta = v_0$ and $v_0 \le v_1 \le w_0$ on $\bar{D}_T$.

Assume that for some n, (P^*) has a unique solution v_n with $\eta = v_{n-1}$ and $v_{n-1} \le v_n \le w_0$ on $\bar{D}_T$, by (3.2) and (3.8)

$$\begin{aligned}(v_n)_t - Lv_n &= F(t,x,v_{n-1}) + F_u(t,x,v_{n-1})(v_n - v_{n-1}) - Mv_n^2\\ &\le F(t,x,v_n) - Mv_n^2 = f(t,x,v_n) \ \text{ in } D_T.\end{aligned} \tag{3.9}$$

On the other hand,

$$\begin{aligned}(v_{n+1})_t - Lv_{n+1} &= F(t,x,v_n) + F_u(t,x,v_n)(v_{n+1} - v_n) - Mv_{n+1}^2\\ &\equiv G(t,x,v_n,v_{n+1}) \ \text{ in } D_T.\end{aligned} \tag{3.10}$$

Now, (3.9) becomes

$$(v_n)_t - Lv_n \le f(t,x,v_n) \equiv G(t,x,v_n,v_n) \ \text{ in } D_T. \tag{3.11}$$

Similarly,

$$(w_0)_t - Lw_0 \ge f(t,x,w_0) \equiv G(t,x,v_n,w_0) \ \text{ in } D_T. \tag{3.12}$$

Applying the same arguments as above, we claim that (3.10), (3.11) and (3.12) together with the initial and boundary conditions yield the unique solution v_{n+1} of (P^*) with $\eta = v_n$ and $v_n \le v_{n+1} \le w_0$ on $\bar{D}_T$. Thus, by induction, we conclude that for all n,

$$v_n \le v_1 \le \cdots \le v_n \le \cdots \quad \text{on } \bar{D}_T,$$

and $\{v_n\}$ is bounded above by w_0 on $\bar{D}_T$. Therefore, we obtain a sequence $\{v_n\}$ which is monotone increasing and uniformly bounded in $C^{1,2}(D_T) \cap C(\bar{D}_T)$. By standard arguments [3, 5], $\{v_n\}$ converges uniformly to the unique solution of NBVP.

It remains to show that convergence of $\{v_n\}$ to the solution u of NBVP is quadratic. Observe that, for all $n \geq 0$, $v_n \leq u$. Set $p_n = u(t,x) - v_n(t,x)$ on $\bar{D}_T$. Define $\| p_n \| = \sup_{(t,x) \in \bar{D}_T} | u(t,x) - v_n(t,x) |$. Then,

$$(p_n)_t - Lp_n = f(t,x,u) - [F(t,x,v_{n-1}) + F_u(t,x,v_{n-1})(v_n - v_{n-1}) - Mv_n^2]$$

$$= F_u(t,x,\theta)(u - v_{n-1}) - F_u(t,x,v_{n-1})(u - v_{n-1}) + F_u(t,x,v_{n-1})(u - v_n) + M(v_n^2 - u^2)$$

$$= F_{uu}(t,x,\xi)(u - v_{n-1})(\theta - v_{n-1}) + F_u(t,x,v_{n-1})(u - v_n) + M(v_n^2 - u^2) \text{ in } D_T,$$

where $v_{n-1} \leq \xi \leq \theta \leq u$ for $(t,x) \in D_T$.

Since $0 < F_{uu} \leq C_1$, $| F_u | \leq K_1$ on Q_T and $v_0 \leq v_n$, $u \leq w_0$, we have

$$(p_n)_t - Lp_n \leq C_1(u - v_{n-1})^2 + Kp_n - M(v_n + u)p_n$$

$$\leq (K_1 - 2Mv_0)p_n + C_1 \| p_{n-1} \|^2 \leq Kp_n + C \text{ in } D_T,$$

where $K = K_1 + 2MV$ and $| v_0 | \leq V$ on $\bar{D}_T$, $C = C_1 \| p_{n-1} \|^2$. Therefore, by Lemma 2.2, we get

$$p_n(t,x) \leq Ce^{(K+1)t} \leq C_1 e^{(K+1)T} \| p_{n-1} \|^2 = C_1 e^{(K+1)T} \sup_{(t,x) \in \bar{D}_T} | u - v_{n-1} |^2 \text{ on } \bar{D}_T,$$

which yields

$$\sup_{(t,x) \in \bar{D}_T} | u - v_n | \leq \alpha \sup_{(t,x) \in \bar{D}_T} | u - v_{n-1} |^2, \quad \alpha = C_1 e^{(K+1)T}.$$

The proof of Theorem 3.1 is therefore complete.

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