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THE METHOD OF LOWER AND UPPER SOLUTIONS FOR *n*th-ORDER PERIODIC BOUNDARY VALUE PROBLEMS[1,2]

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ABSTRACT

In this paper we develop the monotone method in the presence of lower and upper solutions for the problem

$$u^{(n)}(t) = f(t,u(t)); u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R},\ i = 0,\ldots,n-1,$$

where f is a Carathéodory function. We obtain sufficient conditions for f to guarantee the existence and approximation of solutions between a lower solution α and an upper solution β for $n \geq 3$ with either $\alpha \leq \beta$ or $\alpha \geq \beta$.

For this, we study some maximum principles for the operator $Lu \equiv u^{(n)} + Mu$. Furthermore, we obtain a generalization of the method of mixed monotonicity considering f and u as vectorial functions.

Key words: Periodic boundary value problem, lower and upper solutions, monotone method.

AMS (MOS) subject classifications: 34B15, 34C25.

1. INTRODUCTION

In this paper we study the following class of boundary value problems for the ordinary differential equations:

$$u^{(n)}(t) = f(t,u(t)) \text{ for } a.e.\ t \in I = [a,b] \tag{1.1}$$

$$u^{(i)}(a) - u^{(i)}(b) = \lambda_i \in \mathbb{R};\ \ i = 0,1,\backslash,n-1, \tag{1.2}$$

for $n \geq 3$ where f is a Carathéodory function.

Definition 1.1: We say that $f: I \times \mathbb{R}^l \to \mathbb{R}^m$ is a *Carathéodory function*, if $f \equiv (f_1,\backslash,f_m)$ satisfies the following properties:

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1. $f_i(\cdot, x)$ is measurable for all $x \in \mathbb{R}^l$ and $i \in \{1, \ldots, m\}$.
2. $f_i(t, \cdot)$ is continuous for *a.e.* $t \in I$.
3. For every $R > 0$ and $i \in \{1, \ldots, m\}$, there exists $h_{i,R} \in L^1(I)$ such that:

$$| f_i(t,x) | \leq h_{i,R}(t) \text{ for } a.e. \ t \in I,$$

with $\| x \| \leq R$.

To develop the monotone method we use the concept of lower and upper solutions:

Definition 1.2: Let $\alpha \in W^{n,1}(I)$, we say that α is a *lower solution* for the problem (1.1)-(1.2) if α satisfies

$$\alpha^{(n)}(t) \geq f(t, \alpha(t)) \text{ for } a.e. \ t \in I$$

$$\alpha^{(i)}(a) - \alpha^{(i)}(b) = \lambda_i; \quad i = 0, 1, \ldots, n-2$$

$$\alpha^{(n-1)}(a) - \alpha^{(n-1)}(b) \geq \lambda_{n-1}.$$

Definition 1.3: Let $\beta \in W^{n,1}(I)$, we say that β is an *upper solution* for the problem (1.1)-(1.2) if β satisfies

$$\beta^{(n)}(t) \leq f(t, \beta(t)) \text{ for } a.e. \ t \in I$$

$$\beta^{(i)}(a0 - \beta^{(i)}(b) = \lambda_i; \quad i = 0, 1, \ldots, n-2$$

$$\beta^{(n-1)}(a) - \beta^{(n-1)}(b) \leq \lambda_{n-1}.$$

We suppose that f satisfies one of the following conditions, depending on various circumstances:

(H_1) $f(t,x) - f(t,y) \leq M(x-y)$ for *a.e.* $t \in I$ with $\alpha(t) \leq y \leq x \leq \beta(t)$ and $M > 0$.

(H_2) $f(t,x) - f(t,y) \geq M(x-y)$ for *a.e.* $t \in I$ with $\beta(t) \leq y \leq x \leq \alpha(t)$ and $M < 0$.

This problem has been studied by different authors for second order equations when $\alpha \leq \beta$ ([1]-[4], [6], [8], [10], [11]). If $\alpha \geq \beta$ the monotone method is not valid if f satisfies the condition (H_2) for some $M < 0$ ([2], [7], [12], [14]).

For $n \geq 3$ the method of lower and upper solutions has been little studied ([2], [9], [13]). In [2] the author obtains the best value on the constant M for

$n = 2$, $n = 3$ and $n = 4$ (in this last case, if $M < 0$) for which the conditions (H_1) or (H_2) imply that the monotone method is valid.

To prove the validity of the monotone method to more general cases, we present some maximum principles for the operator

$$L_n : F^n_{a,b} \to L^1(I),$$

defined by $L_n u = u^{(n)} + Mu$. Where M is a real constant different from zero, and

$$F^n_{a,b} = \left\{ u \in W^{n,1}(I); u^{(i)}(a) = u^{(i)}(b), i = 0,\ldots,n-2; u^{(n-1)}(a) \geq u^{(n-1)}(b) \right\}.$$

We say that an operator L is **inverse positive** in $F^n_{a,b}$ if $Lu \geq 0$ implies $u \geq 0$ for all $u \in F^n_{a,b}$ and that L is **inverse negative** in $F^n_{a,b}$ if $Lu \geq 0$ implies $u \leq 0$ for all $u \in F^n_{a,b}$.

In Section 2, we obtain a new maximum principle for the operator L_n, using that this operator is given by the composition of the operators of first and second order.

This result is used in Section 3 to extend to more general cases the validity of the monotone method for the problem (1.1)-(1.2) and in Section 4 it is applied to obtain a new generalization of the method of mixed monotony [5] when f and u are vectorial functions.

2. MAXIMUM PRINCIPLES

In this section we improve the following result obtained in [2], which generalizes theorem 4 in [15].

Lemma 2.1: *Let* $A(n) \equiv \frac{n^n n!}{\left[\frac{n}{2}\right]^n (b-a)^n (n-1)^{n-1}}$, *where* $\left[\frac{n}{2}\right]$ *is the integer part of* $\frac{n}{2}$. *Then if* $M \in (0, A(n)]$ $(M \in [-A(n), 0))$, *the operator* L_n *is inverse positive* (*inverse negative*) *on* $F^n_{a,b}$.

Furthermore, if $M \in [-[A(n)]^2, 0)$ *the operator* L_{2n} *is inverse negative on* $F^{2n}_{a,b}$.

For this, we use the following known result.

Lemma 2.2:

1. *L_1 is inverse positive (inverse negative) on $F^1_{a,b}$ for all $M>0$ ($M<0$).*
2. *L_2 is inverse negative on $F^2_{a,b}$ for all $M<0$.*
3. *([13], Lemma 2.1) The operator $N_{A,B}u = u'' - 2Au' + (A^2+B^2)u$ is inverse positive on $F^2_{0,2\pi}$ if and only if $(0<)\ B \le \frac{1}{2}$.*

Now, we prove the following preliminary lemma.

Lemma 2.3: *Let $Lu = u^{(n)} + \sum_{i=0}^{n-1} a_i u^{(i)}$ and $Nu = u^{(m)} + \sum_{i=0}^{m-1} b_i u^{(i)}$. Then if L is inverse positive on $F^n_{a,b}$ and N is inverse positive (inverse negative) on $F^m_{a,b}$ then $L \circ N$ is inverse positive (inverse negative) on $F^{n+m}_{a,b}$.*

Proof: Since $u \in F^{n+m}_{a,b}$ it is clear that

$$(Nu)^{(i)}(a) = (Nu)^{(i)}(b),\ i = 0, \ldots, n-2$$

and

$$(Nu)^{(n-1)}(a) \ge (Nu)^{(n-1)}(b).$$

In consequence, since L is inverse positive on $F^n_{a,b}$, we have that $Nu \ge 0$. Now, using that N is inverse positive (inverse negative) on $F^m_{a,b}$, we obtain that $u \ge 0$ ($u \le 0$). □

Thus, we are in position to prove the following lemma.

Lemma 2.4: *Let $M > 0$. The following properties hold:*

1. *Let $n = 4k$, $k \in \{1, 2, \ldots\}$.*
   If $M \le \left[\frac{\pi}{(b-a)\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n$, then L_n is inverse positive on $F^n_{a,b}$.
2. *Let $n = 2 + 4k$, $k \in \{1, 2, \ldots\}$.*
   If $M \le \left[\frac{\pi}{b-a}\right]^n$, then L_n is inverse positive on $F^n_{a,b}$.
3. *Let n be odd.*
   If $M \le \left[\frac{\pi}{(b-a)\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n$, then L_n is inverse positive on $F^n_{a,b}$.

Proof: Since, if $u \in W^{n,1}(I)$ satisfies

$$L_n u(t) = \sigma(t), u^{(i)}(a) = u^{(i)}(b), i = 0, \ldots, n-2 \text{ and } u^{(n-1)}(a) - u^{(n-1)}(b) = \lambda,$$

then $v(t) = \left(\frac{2\pi}{b-a}\right)^{n-1} u\left(\frac{b-a}{2\pi}t + a\right)$ satisfies

$$v^{(n)}(t)+\left(\frac{b-a}{2\pi}\right)^n Mv(t)=\left(\frac{b-a}{2\pi}\right)\sigma\left(\frac{b-a}{2\pi}t+a\right),$$

with

$$v^{(i)}(0)=v^{(i)}(2\pi),\ i=0,\ldots,n-2 \text{ and } v^{(n-1)}(0)-v^{(n-1)}(2\pi)=\lambda.$$

It is sufficient to study the operator L_n on $F^n_{0,2\pi}$ because to obtain the estimate on the interval $[a,b]$ we multiply by $\left(\frac{2\pi}{b-a}\right)^n$ the estimate obtained on $[0,2\pi]$.

Let $m>0$ such that $m^n=M$.

First, we suppose that n is even.

In this case the polynomial function $p(\lambda)=\lambda^n+m^n=0$ if and only if

$$\lambda=\lambda_l=m\left[\cos\left(\frac{2l+1}{n}\pi\right)\pm i\sin\left(\frac{2l+1}{n}\pi\right)\right]\equiv a_l\pm i\beta_l,$$

$l=0,1,\ldots,\frac{n-2}{2}$.

As consequence we have that

$$\lambda^n+m^n=\prod_{l=0}^{\frac{n-2}{2}}(\lambda^2-2\alpha_l\lambda+m^2),$$

and

$$L_n\equiv T_0\circ T_1\circ\ldots\circ T_{\frac{n-2}{2}}. \tag{2.3}$$

Where $T_l u=u''-2\alpha_l u'+m^2u$.

If $n=4k$ for some $k\in\{1,2,\ldots\}$, then $\beta_l\le\beta_{\frac{n}{4}}=m\sin\left(\frac{n+2}{2n}\pi\right)$ for all $l\in\{0,1,\ldots,\frac{n-2}{2}\}$. Thus, using lemma 2.2, if $m\le\left[2\sin\left(\frac{n+2}{2n}\pi\right)\right]^{-1}$ the operator T_l is inverse positive on $F^2_{0,2\pi}$ for all $l\in\{0,1,\ldots,\frac{n-2}{2}\}$. Therefore lemma 2.3 implies that L_n is inverse positive on $F^n_{0,2\pi}$.

If $n=2+4k$ for some $k\in\{1,2,\ldots\}$, then $\beta_l\le\beta_{\frac{n-2}{4}}=m$ for all $l\in\{0,1,\ldots,\frac{n-2}{2}\}$ and as a consequence, T_l is inverse positive on $F^2_{0,2\pi}$ when $m\le\frac{1}{2}$. By (2.3) and the two previous lemmas, we obtain that L_n is inverse positive on $F^n_{0,2\pi}$.

Now, we suppose that n is odd.

In this case, $p(\lambda)=0$ if and only if $\lambda=-m$ or $\lambda=\lambda_l=\alpha_l\pm i\beta_l$,

$l = 0, \ldots, \frac{n-3}{2}$. Thus

$$\lambda^n + m^n = (\lambda + m)\prod_{l=0}^{\frac{n-3}{2}}(\lambda^2 - 2\alpha_l\lambda + m^2),$$

and

$$L_n \equiv T_0 \circ T_1 \circ \ldots \circ T_{\frac{n-3}{2}} \circ S_1.$$

Where $S_1 u = u' + mu$.

In this case $\beta_l \leq \beta_{\frac{n-1}{4}} = m\, sin\left(\frac{n+1}{2n}\pi\right)$ for all $l \in \{0, 1, \ldots, \frac{n-3}{2}\}$. Thus, if $m \geq [2\, sin\left(\frac{n+1}{2n}\pi\right)]^{-1}$ lemmas 2.2 and 2.3 imply that the operator L_n is inverse positive on $F^n_{0,2\pi}$. □

Analogously we can prove the following result for $M < 0$.

Lemma 2.5: *Let $M < 0$. The following properties hold:*

1. *Let $n = 4k$, $k \in \{1, 2, \ldots\}$.*
   If $M \geq -\left[\frac{\pi}{b-a}\right]^n$ then L_n is inverse negative on $F^n_{a,b}$.
2. *Let $n = 2 + 4k$, $k \in \{0, 1, \ldots\}$.*
   If $M \geq -\left[\frac{\pi}{(b-a)\, sin\left(\frac{n+2}{2n}\pi\right)}\right]^n$ then L_n is inverse negative on $F^n_{a,b}$.
3. *Let n be odd.*
   If $M \geq -\left[\frac{\pi}{(b-a)\, sin\left(\frac{n+1}{2n}\pi\right)}\right]^n$ then L_n is inverse negative on $F^n_{a,b}$.

Remark 2.1: Note that these estimates are not the best possible for all $n \in \mathbb{N}$.

In [2] it is proved that L_3 is inverse positive (inverse negative) on $F^3_{0,2\pi}$ if and only if $M \in (0, M_3^3]$ $(M \in [-M_3^3, 0))$. Where M_3 is the unique solution of the equation

$$arctan\left(\frac{sin\sqrt{3}\, m\pi}{cos\sqrt{3}m\pi - e^{m\pi}}\right) + \pi = \frac{\sqrt{3}}{3}log\left(\frac{e^{3m\pi} - e^{m\pi}}{\sqrt{1 + e^{2m\pi} - 2e^{m\pi}cos\sqrt{3}m\pi}}\right).$$

Furthermore, L_4 is inverse negative on $F^4_{0,2\pi}$ if and only if $M \in [-M_4^4, 0)$, with M_4 given as the unique solution in $(\frac{1}{2}, 1)$ of the equation

$$-tanh\, m\pi = tan\, m\pi.$$

Note that the estimates obtained in lemmas 2.4 and 2.5 are the best possible for $n=1$ and $n=2$.

3. THE MONOTONE METHOD

In this section we study the existence of solutions of the problem (1.1)-(1.2) in the sector $[\alpha,\beta]$ or $[\beta,\alpha]$, where $[v,w]=\{u\in L^1(I): v\le u\le w \text{ on } I\}$. We improve the following result given in [2], which generalizes theorem 5 in [15].

Theorem 3.1: *The following properties hold.*

1. *If there exists $\alpha\le\beta$ ($\alpha\ge\beta$) lower and upper solutions respectively of the problem (1.1)-(1.2), and f satisfies the condition (H_1) ((H_2)) for some $M\in(0,A(n)]$ ($M\in[-A(n),0)$) then there exists a solution of the problem (1.1)-(1.2) in $[\alpha,\beta]$ ($[\beta,\alpha]$). Furthermore, there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0=\alpha$ and $\beta_0=\beta$ which converge uniformly to the extremal solutions in $[\alpha,\beta]$ ($[\beta,\alpha]$) of the problem (1.1)-(1.2).*
2. *The previous property is true when n is even and f satisfies the condition (H_2) for some $M\in(-[A(\frac{n}{2})]^2,0]$.*

Using lemma 2.4 we prove the following result.

Theorem 3.2: *If there exists $\alpha\ge\beta$ lower and upper solutions respectively of the problem (1.1)-(1.2) and if any of the following properties are true:*

1. *Let $n=4k$, $k\in\{1,2,\ldots\}$. Suppose that f satisfies the property (H_2) for some $M\in\left[-\left[\frac{\pi}{(b-a)\sin\left(\frac{n+2}{2n}\pi\right)}\right]^n,0\right)$.*
2. *Let $n=2+4k$, $k\in\{1,2,\ldots\}$. Suppose that f satisfies the property (H_2) for some $M\in\left[-\left[\frac{\pi}{b-a}\right]^n,0\right)$.*
3. *Let n be odd. Suppose that f satisfies the property (H_2) for some $M\in\left[-\left[\frac{\pi}{(b-a)\sin\left(\frac{n+1}{2n}\pi\right)}\right]^n,0\right)$.*

Then there exists u a solution of the problem (1.1)-(1.2) in $[\beta,\alpha]$.

Furthermore, there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0=\alpha$ and $\beta_0=\beta$, which converge uniformly to the extremal solutions in $[\beta,\alpha]$ of *the problem* (1.1)-(1.2).

Proof: We consider the problem:

$$u^{(n)}(t) - Mu(t) = f(t,\eta(t)) - M\eta(t) \text{ for } a.e.\ t \in I \tag{3.1}$$

$$u^{(i)}(a) - u^{(i)}(b) = \lambda_i,\ i = 0,1,\ldots,n-1 \tag{3.2}$$

with $\eta \in L^1(I)$, $\beta(t) \leq \eta(t) \leq \alpha(t)$.

We have:

$$(\alpha - u)^{(n)}(t) - M(\alpha - u)(t) \geq -f(t,\eta(t))$$
$$+ M\eta(t) + f(t,\alpha(t)) - M\alpha(t) \geq 0$$
$$(\alpha - u)^{(i)}(a) - (\alpha - u)^{(i)}(b) = 0;\ \ i = 0,\ldots,n-2$$
$$(\alpha - u)^{(n-1)}(a) - (\alpha - u)^{(n-1)}(b) \geq 0.$$

Lemma 2.4 implies that $u \leq \alpha$.

Analogously we can prove that $u \geq \beta$.

Let $u_i = Q\eta_i$ the unique solution of the problem (3.1)-(3.2) for $\eta = \eta_i \in L^1(I)$. Since for $\beta \leq \eta_1 \leq \eta_2 \leq \alpha$,

$$(u_2 - u_1)^{(n)}(t) - M(u_2 - u_1)(t) = f(t,\eta_2(t))$$
$$- M\eta_2(t) - f(t,\eta_1(t)) + M\eta_1(t) \geq 0$$
$$(u_2 - u_1)^{(i)}(a) - (u_2 - u_1)^{(i)}(b) = 0;\ \ i = 0,\ldots,n-1,$$

the following property holds:

$$\text{If } \beta \leq \eta_1 \leq \eta_2 \leq \alpha \text{ then } u_1 = Q\eta_1 \leq Q\eta_2 = u_2.$$

The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are obtained by recurrence: $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_n = Q\alpha_{n-1}$ and $\beta_n = Q\beta_{n-1}$; $n \geq 1$.

By standard arguments we prove that $\{\alpha_n\}$ and $\{\beta_n\}$ converge to the extremal solutions on $[\beta,\alpha]$ of the problem (1.1)-(1.2). □

Analogously, using lemma 2.5 we can prove the following theorem.

Theorem 3.3: *If there exists* $\alpha \leq \beta$ *lower and upper solutions respectively of the problem* (1.1)-(1.2) *and any of the following properties are* *verified:*

1. *Let* $n = 4k$, $k \in \{1,2,\ldots\}$. *Suppose that* f *satisfies the property* (H_1) *for some* $M \in (0, [\frac{\pi}{b-a}]^n]$.
2. *Let* $n = 2 + 4k$, $k \in \{1,2,\ldots\}$. *Suppose that* f *satisfies the property* (H_1) *for some* $M \in \left(0, \left[\frac{\pi}{(b-a)sin\left(\frac{n+2}{2n}\pi\right)}\right]^n\right]$.
3. *Let* n *be odd. Suppose that* f *satisfies the property* (H_1) *for some* $M \in \left(0, \left[\frac{\pi}{(b-a)sin\left(\frac{n+1}{2n}\pi\right)}\right]^n\right]$.

Then there exists u *a solution of the problem* (1.1)-(1.2) *in* $[\alpha, \beta]$.

Furthermore there exist two monotone sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *with* $\alpha_0 = \alpha$ *and* $\beta_0 = \beta$ *which converge uniformly to the extremal solutions in* $[\alpha, \beta]$ *of the problem* (1.1)-(1.2).

Remark 3.1: Similarly to the remark 2.1, note that the estimates obtained for the function f in theorems 3.2 and 3.3 are not the best possible for all $n \in \mathbb{N}$.

4. THE METHOD OF MIXED MONOTONY

In this section we study the method of mixed monotony, studied by Khavanin and Lakshmikantham in [5], in which they consider the initial and periodic first order problems. In this case, under stronger conditions on the function f it is possible to guarantee the unicity of the solution when we have an nth-order system.

In [5] the following results are obtained.

Theorem 4.1: *Consider the following system*

$$u'(t) = f(t, u(t)); t \in [0, T]$$

with $f \in C([0,T] \times \mathbb{R}^N, \mathbb{R}^N)$.

If there exists $F \in C([0,T] \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $\alpha, \beta \in C^1([0,T], \mathbb{R}^N)$ *which satisfy the following conditions:*

(*i*) $\alpha'(t) \geq F(t, \alpha(t), \beta(t)), \beta'(t) \leq F(t, \beta(t), \alpha(t))$. *With* $\beta \leq \alpha$ *on* $[0, T]$.

(*ii*) $F(t, u, v)$ *is nondecreasing on* u *and nonincreasing on* v.

(*iii*) $F(t, u, u) = f(t, u)$ *and*

$$-B(z_1 - z_2) \le F(t, y_1, z_1) - F(t, y_2, z_2) \le B(y_1 - y_2),$$

with $\beta(t) \le y_2 \le y_1 \le \alpha(t)$, $\beta(t) \le z_2 \le z_1 \le \alpha(t)$ *and* B *an* $N \times N$ *matrix with nonnegative elements.*

Then:

If $\beta(0) \le u_0 \le \alpha(0)$, *then there exist two sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *nonincreasing and nondecreasing respectively which converge uniformly to the unique solution of the problem*

$$u'(t) = f(t, u(t)); u(0) = u_0.$$

Furthermore, if $(T = 2\pi)$ $\beta(0) \le \beta(2\pi)$ *and* $\alpha h(0) \ge \alpha(2\pi)$ *with* $I \ne e^{2B\pi}$ *the same result is valid for the problem*

$$u'(t) = f(t, u(t));\ u(0) = u(2\pi).$$

Theorem 4.2: *If there exists* $\alpha, \beta \in C^1([0,T], \mathbb{R}^N)$, *with* $\beta \le \alpha$ *on* $[0,T]$ *verifying:*

$$\alpha'(t) \ge f(t, \alpha(t)) + B(\alpha(t) - \beta(t)) \text{ and } \beta'(t) \le f(t, \beta(t)) - B(\alpha(t) - \beta(t)),$$

and f *satisfies*

$$-B(x - y) \le f(t, x) - f(t, y) \le B(x - y),$$

with $\beta(t) \le y \le x \le \alpha(t)$, *where* B *is an* $N \times N$ *matrix with nonnegative elements, then the conclusions of theorem* 4.1 *are valid.*

Using lemma 2.5 we prove the following result.

Theorem 4.3: *Let*

$$u^{(n)}(t) = f(t, u(t)) \text{ for a.e. } t \in [a, b] \tag{4.1}$$

$$u_j^{(i)}(a) - u_j^{(i)}(b) = \lambda_{i,j} \in \mathbb{R},\ i = 0, \ldots, n-1;\ j = 1, \ldots, N, \tag{4.2}$$

with $f: I \times \mathbb{R}^N \to \mathbb{R}$ *a Carathéodory function and* $n \ge 2$.

If there exists a Carathéodory function $F: I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ *and* α, $\beta \in W^{n,1}(I, \mathbb{R}^N)$, $\alpha \le \beta$ *on* I, *verifying the following properties:*

(i)

$$\alpha^{(n)}(t) \ge F(t, \alpha(t), \beta(t)) \text{ for a.e. } t \in I$$

$$\alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) = \lambda_{i,j}; i = 0,\ldots,n-2; j = 1,\ldots,N$$
$$\alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) \geq \lambda_{n-1,j}; j = 1,\ldots,N.$$

(*ii*)
$$\beta^{(n)}(t) \leq F(t,\beta(t),\alpha(t)) \; for \; a.e. \; t \in I$$
$$\beta_j^{(i)}(a) - \beta_j^{(i)}(b) = \lambda_{i,j}; i = 0,\ldots,n-2; j = 1,\ldots,N$$
$$\beta_j^{(n-1)}(a) = \beta_j^{(n-1)}(b) \leq \lambda_{n-1,j}; j = 1,\ldots,N.$$

(*iii*) *$F(t,u,v)$ is nonincreasing on u and nondecreasing on v.*

(*iv*) *$F(t,u,u) = f(t,u)$ and*

$$F(t,y,z) - F(t,z,y) = -B(y-z),$$

B being an $N \times N$ matrix with nonnegative elements such that $exp(C(b-a)) \neq I$. Where C is given by the expression

$$C \equiv \left(\begin{array}{c|c} 0 & I_{(n-1)N} \\ \hline -B & 0 \end{array}\right).$$

Here $I_{(n-1)N}$ is the $(n-1)N \times (n-1)N$ identity matrix.

Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the unique solution of the problem (4.1)-(4.2).

Proof: Let $M_1 = -\left[\frac{\pi}{b-a}\right]^n$ and $\eta, \nu \in L^1(I, \mathbb{R}^N)$, $\eta, \nu \in [\alpha, \beta]$.

Consider the following linear problem for each $j = 1,\ldots,N$:

$$u_j^{(n)}(t) + M_1 u_j(t) = F_j(t,\eta(t),\nu(t)) + M_1\eta_j(t) \text{ for } a.e. \; t \in [a,b] \tag{4.3}$$
$$u_j^{(i)}(a) - u_j^{(i)}(b) = \lambda_{i,j} \in \mathbb{R}, \; i = 0,\ldots,n-1; j = 1,\ldots,N. \tag{4.4}$$

Let $u = A[\eta,\nu]$ be the unique solution of the problem (4.3)-(4.4) for each η, ν.

First, we prove that $\alpha \leq A[\alpha,\beta] = \alpha_1$,

$$(\alpha_j^{(n)} - \alpha_{1,j}^{(n)})(t) + M_1(\alpha_j - \alpha_{1,j})(t) \geq 0$$
$$(\alpha_j^{(i)} - \alpha_{1,j}^{(i)})(a) - (\alpha_j^{(i)} - \alpha_{1,j}^{(i)})(b) = 0; i = 0,\ldots,n-2$$
$$(\alpha_j^{(n-1)} - \alpha_{1,j}^{(n-1)})(a) - (\alpha_j^{(n-1)} - \alpha_{1,j}^{(n-1)})(b) \geq 0.$$

Thus, lemma 2.5 implies that $\alpha \le \alpha_1$ on I.

Similarly, we obtain that $\beta \ge \beta_1 = A[\beta, \alpha]$.

Let $\eta_1, \eta_2, \nu \in [\alpha, \beta]$, with $\eta_1 \le \eta_2$. Let $u_1 = A[\eta_1, \nu]$ and $u_2 = A[\eta_2, \nu]$. We have that

$$(u_{1,j} - u_{2,j})^{(n)}(t) + M_1(u_{1,j} - u_{2,j})(t) = F_j(t, \eta_1, \nu) + M_1 \eta_{1,j} - F_j(t, \eta_2, \nu) - M_1 \eta_{2,j} \ge 0$$

which implies that $u_1 \le u_2$.

Analogously, one can prove that $A[\eta, \nu_1] \le A[\eta, \nu_2]$ if $\nu_1 \ge \nu_2$.

It is now easy to define the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$, $\alpha_{n+1} = A[\alpha_n, \beta_n]$ and $\beta_{n+1} = A[\beta_n, \alpha_n]$.

Clearly, $\alpha \le \alpha_1 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_1 \le \beta$ on I.

By standard arguments we can show that $\lim_{n \to \infty} \alpha_n = \phi$ and $\lim_{n \to \infty} \beta_n = \psi$ exist uniformly on I and ϕ and ψ satisfy

$$\phi^{(n)}(t) = F(t, \phi, \psi), \; \psi^{(n)}(t) = F(t, \psi, \phi)$$
$$\phi_j^{(i)}(a) - \phi_j^{(i)}(b) = \psi_j^{(i)}(a) - \psi_j^{(i)}(b) = \lambda_{i,j};$$

$i = 0, \ldots, n-1$; $j = 1, \ldots, N$.

That is

$$(\phi - \psi)^{(n)}(t) = F(t, \phi, \psi) - F(t, \psi, \phi) = -B(\phi - \psi) \tag{4.5}$$
$$(\phi - \psi)^{(i)}(a) = (\phi - \psi)^{(i)}(b); \; i = 0, \ldots, n-1. \tag{4.6}$$

Now, we define $p(t) = ((\phi - \psi)(t), (\phi - \psi)'(t), \ldots, (\phi - \psi)^{(n-1)}(t)) \in \mathbb{R}^{nN}$. Therefore $p' = Cp$, $p(a) = p(b)$. Since $p(b) = exp(C(b-a))p(a)$, we obtain that $p \equiv 0$ and, in consequence, $\phi = \psi$. That is, $\phi^{(n)}(t) = F(t, \phi, \phi) = f(t, \phi)$, which concludes the proof. □

Similarly, using lemma 2.4 we prove the following result.

Theorem 4.4: *The conclusions obtained in theorem* 4.3 *are valid if* $\alpha \ge \beta$ *and the properties* (*iii*) *and* (*iv*) *are changed by*

(*iii*)′ $F(t, u, v)$ *is nondecreasing on* u *and nonincreasing on* v.

(iv)' $F(t,u,u) = f(t,u)$ *and*

$$F(t,y,z) - F(t,z,y) = B(y-z),$$

B being an $N \times N$ *matrix with nonnegative elements as such that* $exp(D(b-a)) \neq I$, *where D is defined as follows:*

$$D \equiv \left(\begin{array}{c|c} 0 & I_{(n-1)N} \\ \hline B & 0 \end{array} \right).$$

Here $I_{(n-1)N}$ *is the* $(n-1)N \times (n-1)N$ *identity matrix.*

As consequence of the two previous lemmas we prove the following result.

Theorem 4.5: *Let* $n \geq 2$. *Suppose that there exist* α *and* $\beta \in W^{n,1}(I, \mathbb{R}^N)$, $\alpha \leq \beta$ $(\alpha \geq \beta)$ *and f a Carathéodory function, satisfying*

$$-B(x-y) \leq f(t,x) - f(t,y) \leq B(x-y),$$

with $y \leq x$ *between* $\alpha(t)$ *and* $\beta(t)$, *where B is an* $N \times N$ *matrix with nonnegative elements.*

If α *and* β *satisfies*

$$\alpha^{(n)}(t) \geq f(t, \alpha(t)) + B \mid \beta(t) - \alpha(t) \mid \text{ for a.e. } t \in I$$
$$\alpha_j^{(i)}(a) - \alpha_j^{(i)}(b) = \lambda_{i,j}; i = 0, 1, \dots, n-2, j = 1, \dots, N$$
$$\alpha_j^{(n-1)}(a) - \alpha_j^{(n-1)}(b) \geq \lambda_{n-1,j}; j = 1, \dots, N$$

and

$$\beta^{(n)}(t) \leq f(t, \beta(t)) - B \mid \beta(t) - \alpha(t) \mid \text{ for a.e. } t \in I$$
$$\beta_j^{(i)}(a) - \beta_j^{(i)}(b) = \lambda_{i,j}; i = 0, 1, \dots, n-2, j = 1, \dots, N$$
$$\beta_j^{(n-1)}(a) - \beta_j^{(n-1)}(b) \leq \lambda_{n-1,j}; j = 1, \dots, N.$$

And $exp(C(b-a)) \neq I$ $(exp(D(b-a)) \neq I)$ *(C and D given in theorems* 4.3 *and* 4.4).

Then there exists a unique solution u between α *and* β *of the problem* (4.1)-(4.2). *Furthermore, there exist two monotone sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$, *with* $\alpha_0 = \alpha$ *and* $\beta_0 = \beta$, *which converge uniformly to the solution u.*

Proof: If $\alpha \leq \beta$ we define F as follows:

$$F(t,u,v)=\frac{1}{2}[f(t,u)+f(t,v)-B(u-v)].$$

It is easy to prove that the function F satisfies the conditions of theorem 4.3. If $\alpha \geq \beta$ the function F is defined as follows:

$$F(t,u,v)=\frac{1}{2}[f(t,u)+f(t,v)+B(u-v)].$$

Clearly, the function F satisfies the conditions of theorem 4.4. □

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