Journal of Applied Mathematics and Stochastic Analysis
7, Number 3, 1994, 373-396.

ON MARKOVIAN TRAFFIC WITH APPLICATIONS TO TES PROCESSES

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(Received November, 1993; revised July, 1994)

ABSTRACT

Markov processes are an important ingredient in a variety of stochastic applications. Notable instances include queueing systems and traffic processes offered to them. This paper is concerned with Markovian traffic, i.e., traffic processes whose inter-arrival times (separating the time points of discrete arrivals) form a real-valued Markov chain. As such this paper aims to extend the classical results of renewal traffic, where interarrival times are assumed to be independent, identically distributed. Following traditional renewal theory, three functions are addressed: the probability of the number of arrivals in a given interval, the corresponding mean number, and the probability of the times of future arrivals. The paper derives integral equations for these functions in the transform domain. These are then specialized to a subclass, TES^+, of a versatile class of random sequences, called TES (*Transform-Expand-Sample*), consisting of marginally uniform autoregressive schemes with modulo-1 reduction, followed by various transformations. TES models are designed to simultaneously capture both first-order and second-order statistics of empirical records, and consequently can produce high-fidelity models. Two theoretical solutions for TES^+ traffic functions are derived: an operator-based solution and a matric solution, both in the transform domain. A special case, permitting the conversion of the integral equations to differential equations, is illustrated and solved. Finally, the results are applied to obtain instructive closed-form representations for two measures of traffic burstiness: peakedness and index of dispersion, elucidating the relationship between them.

Key words: Traffic Processes, Markov Processes, Markovian Traffic, TES Processes, Stochastic Process, Peakedness Functional, Peakedness Function, Index of Dispersion for Intervals.

AMS (MOS) subject classifications: 60G07, 60G10, 60G55, 60J27, 65C05, 90B15, 90B22.

1. Introduction

Let $\{X_n\}_{n=0}^{\infty}$ be a stationary non-negative Markovian stochastic process, interpreted as inter-arrival times in a traffic process. We shall refer to $\{X_n\}$ as a traffic process, or interchangeably, as an arrival process. The purpose of this paper is twofold: to extend the classic theory of renewal traffic (Cox [1]) (where $\{X_n\}$ is a sequence of independent identically distributed random variables) to the case where $\{X_n\}$ is a Markov sequence, and to specialize the results to the class of TES$^+$ processes [8, 9, 10], to be overviewed below.

The following notation and assumptions relating to $\{X_n\}$ are adopted. The common mean and variance of the X_n are assumed finite and positive. The marginal density of the X_n is denoted by $f_X(x)$, and the n-step transition density of $\{X_n\}$ is denoted by $f_n(y \mid x)$. It will be assumed that $f_1(y \mid x)$ is positive in y on a set of positive Lebesgue measure for almost every x. A subscript is used to resolve ambiguities as to the random variable involved. The autocorrelation function of $\{X_n\}$ is

$$\rho_X(\tau) = \frac{E[X_n X_{n+\tau}] - E^2[X_n]}{Var[X_n]}, \quad \tau \geq 0, \tag{1.1}$$

and the corresponding spectral density has the representation

$$s_X(\omega) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{\tau=1}^{\infty} \rho_X(\tau)\, cos(\omega\tau), \quad 0 \leq \omega \leq \pi. \tag{1.2}$$

Assume that $-X_0$ and 0 are arrival points, and that X_1 is considered as the first arrival point. For fairly general Markovian traffic, we shall be interested in the following functions:

1. The probability functions $q_n(t) = P\{N(t) = n\}$, $n \geq 0$, of the number of arrivals, $N(t)$, in the interval $(0, t]$.
2. The mean number $M(t) = E[N(t)]$ of arrivals in the interval $(0, t]$.
3. The density function $p_n(t)$ of the time $S_n = \sum_{j=1}^{n} X_j$ from 0 to the n^{th} arrival point.

We then proceed to specialize the discussion to TES traffic – TES processes being essentially transformed versions of autoregressive schemes with modulo-1 reduction (see Jagerman and Melamed [8, 9, 10]). The modulo-1 (fractional part) operator, $\langle \cdot \rangle$, is defined for any real x by $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor = max\{n \text{ integer: } n \geq x\}$. We shall be primarily interested in the class of TES$^+$ processes, $\{X_n^+\}_{n=0}^{\infty}$, where the plus superscript is used to distinguish this process from other flavors of TES processes (see *ibid.*) A TES$^+$ process, $\{X_n^+\}$, has the form

$$X_n^+ = D(U_n^+), \quad n \geq 0, \tag{1.3}$$

where D is a measurable real function called a *distortion*, and $\{U_n^+\}_{n=0}^{\infty}$ is a stochastic sequence of the form

$$U_n^+ = \begin{cases} U_0, & n = 0 \\ \langle U_{n-1}^+ + V_n \rangle, & n > 0 \end{cases} \tag{1.4}$$

where U_0 is distributed uniformly on the interval $[0, 1)$, and $\{V_n\}_{n=1}^{\infty}$ is an arbitrary sequence of independent identically distributed random variables with common density function f_V; the V_n form a sequence of *innovations*, i.e., for each $n \geq 1$, V_n is independent of $\{U_0^+, U_1^+, \ldots, U_{n-1}^+\}$, namely, the history of $\{U_n^+\}$ to date. The auxiliary TES$^+$ processes of the form (1.4) are called the *background* processes, whereas the target TES$^+$ processes of the form (1.3) are called the *foreground* processes.

Throughout this paper, the following notational conventions are used. A plus superscript is consistently appended to mathematical objects associated with TES$^+$ sequences. To improve typographical clarity, we use the notation f_X^+ instead of f_{X^+}, and so on. However, to economize on notation, the subscript will often be omitted; in that case, it is understood that the object is associated with the foreground sequence, $\{X_n^+\}$ – the focus of this paper. An exception is the transition density of the background process, $\{X_n^+\}$, which will be denoted by $g_U^+(v \mid u)$, in conformance with previous notation (Jagerman and Melamed [8, 9]). Real functions are implicitly extended to vanish on the complement of their domains. The indicator function of a set A is denoted by 1_A. A vertical bar in the argument list always denotes conditioning. The Laplace Transform of a function f is denoted by $\tilde{f}(s) = \int_{-\infty}^{\infty} e^{-sy} f(y)dy$; unless otherwise specified, all Laplace transforms are evaluated for a real argument s.

Since this paper is concerned with traffic modeling, we shall make throughout the reasonable assumption that D is strictly positive almost everywhere on $[0,1)$, thereby guaranteeing that interarrival times modeled by TES$^+$ processes give rise to simple traffic. Furthermore, for a marginal distribution, F, we shall take $D = F^{-1}$, where $F^{-1}(y) = inf\{u: F(u) = y\}$ is always single-valued, even if F is not one-one (F is always monotone increasing, but not necessarily strictly monotone). Distortions of the form $D = F^{-1}$ ensure that $\{X_n^+\}$ has marginal distribution F, allowing us to match any empirical distribution; in practice, F is usually obtained from an empirical histogram of data measurements. Jagerman and Melamed [8] showed that the background TES$^+$ processes, (1.4), are Markovian with transition densities

$$g_U^+(v \mid u) = \sum_{\nu = -\infty}^{\infty} \tilde{f}_V(i2\pi\nu) e^{i2\pi\nu(v-u)}, \tag{1.5}$$

and the corresponding foreground processes, (1.3), have autocorrelation functions (see Equation (1.1))

$$\rho_X^+(\tau) = \frac{1}{Var[X_n^+]} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu) \mid \tilde{D}(i2\pi\nu) \mid^2. \tag{1.6}$$

The rest of this paper is organized as follows. Section 2 presents some technical preliminaries. Section 3 develops functional equations in the transform domain for the traffic functions of interest, valid for fairly general Markovian traffic. Section 4 specializes the integral equations to TES$^+$ traffic and presents operator-based solutions, while Section 5 presents a matric solution, both in the transform domain. Section 6 shows how to solve for the traffic functions of a TES$^+$ process with exponential innovations by converting the integral equations to differential equations. Finally, Section 7 computes the peakedness measure for the burstiness of TES$^+$ processes. The resulting formula is shown to be related to, but more general than, another measure of traffic burstiness, called IDI (index of dispersion for intervals).

2. Preliminaries

This section presents some preliminary technical material concerning Laplace transforms of certain integrals and a theory for solving a class of Fredholm-type integral equations.

The following lemma extends the standard Laplace transform of convolution. In the lemma, the (one-one) correspondence between a function, f, and its Laplace transform, $\tilde{f}$, is denoted by $f \doteqdot \tilde{f}$ (see Van Der Pol and Bremmer [15]); further, for a function $g(t,x)$ of two variables, the Laplace transform, with respect to either variable, will be denoted by $\tilde{g}(s,x)$ and $\tilde{g}(t,s)$ as the case may be.

Lemma 1: *Suppose that for some* s_0,

$$\int_0^{\infty} e^{-s_0 t} \mid g(t,x) \mid dt < \infty \quad \forall x, \qquad \int_0^{\infty} e^{-s_0 x} \mid f(x) \mid ds < \infty.$$

Then, for $s \geq s_0$,

$$\int_0^t g(t-x,x)f(x)dx \div \int_0^\infty e^{-sx}\tilde{g}(s,x)f(x)dx.$$

Proof: From the absolute convergence of the Laplace integrals,

$$\int_0^\infty e^{-s_0 x}f(x)dx \int_0^\infty e^{-s_0 u}g(u,x)du = \int_0^\infty \int_0^\infty e^{-s_0(u+x)}g(u,x)f(x)dudx,$$

in which the double integral on the right is absolutely convergent. Setting $t = u + x$ in the double integral yields by Fubini's theorem

$$\int_0^\infty e^{-s_0 x}\tilde{g}(s_0,x)f(x)dx = \int_0^\infty f(x)dx \int_x^\infty e^{-s_0 t}g(t-x,x)dt$$

$$= \int_0^\infty e^{-s_0 t}dt \int_0^t g(t-x,x)f(x)dx.$$

The result follows immediately, since a Laplace integral always converges for points $s \geq s_0$, where s_0 is a convergence point. □

Note that if $g(t,x) = g(t)$ is independent of x, then $\int_0^t g(t-x,x)f(x)dx$ becomes the familiar convolution, and Lemma 1 yields the known result $\tilde{g}(s)\tilde{f}(s)$ on the right-hand side.

We next present a theory of Fredholm-type integral equations, to be used in the sequel to compute statistics of Markovian traffic, namely, the functions, $q_n(t)$, $M(t)$ and $p_n(t)$ defined in Section 1.

Consider the Fredholm-type integral equation

$$\varphi_{s,z}(x) = h(x) + z\int_0^\infty \varphi_{s,z}(y)K_s(x,y)dy, \tag{2.1}$$

in which $h(x)$ is the forcing function and the kernel $K_s(x,y)$ is given by

$$K_s(x,y) = e^{-sy}f_1(y \mid x), \quad s > 0. \tag{2.2}$$

It will be assumed throughout that $K_s(x,y) \in L^2([0,\infty) \times [0,\infty))$, for all $s > 0$, namely,

$$\int_0^\infty \int_0^\infty e^{-2sy}f_1^2(y \mid x)dx\,dy < \infty, \quad s > 0, \tag{2.3}$$

and that $h(x) \in L^2([0,\infty))$, that is

$$\int_0^\infty h^2(x)dx < \infty.$$

Thus, (2.1) is a Fredholm-type integral equation (Zabreyko et al. [16]), and in terms of the Fredholm operator $\mathcal{K}_s$, defined by

$$\mathcal{K}_s[f(x)] = \int_0^\infty f(y)K_s(x,y)dy, \qquad f \in L^2([0,\infty)), \tag{2.4}$$

the integral equation (2.1) takes the form

$$\varphi_{s,z}(x) = h(x) + z\mathcal{K}_s[\varphi_{s,z}(x)]. \tag{2.5}$$

Furthermore, since $K_s(x,y) \in L[0,\infty)$ as a function of y for every $x \geq 0$, the domain of $\mathcal{K}_s$ may be enlarged, for each x, to include functions which are bounded on $[0,\infty)$ and summable over every compact interval.

The $L^2([0,\infty))$ norm is defined by $\| f \|_2 = [\int_0^\infty f^2(x)dx]^{1/2}$, and the $L^2([0,\infty)\times[0,\infty))$ norm by $\| g \|_2 = [\int_0^\infty \int_0^\infty g^2(x,y)dx\,dy]^{1/2}$. If M is an operator mapping $L^2([0,\infty))$ onto itself, then $\| M \| = inf\{C > 0\colon \| M[f] \|_2 \leq C \| f \|_2,\ f \in L^2([0,\infty))\}$ denotes the induced operator norm of M. Since $\| M \| \leq \| M \|_2$, follows that the operators $\mathcal{K}_s$, defined in (2.4), are bounded as a consequence of (2.3).

We now return to the solution of the integral equation (2.5). The iterated kernels $K_{s,n}(x,y)$, given by

$$K_{s,n}(x,y) = \begin{cases} K_s(x,y), & n = 1 \\ \displaystyle\int_0^\infty e^{-sw} f_1(w \mid x) K_{s,n-1}(w,y)dw, & n > 1 \end{cases}$$

provide an integral representation for the n^{th} iterate, $\mathcal{K}_s^n$, of the operator $\mathcal{K}_s$, namely $\mathcal{K}_s^n[h(x)] = \int_0^\infty h(y)K_{s,n}(x,y)dy$. Formally, the integral equation (2.5) has the solution

$$\varphi_{s,z}(x) = (I - z\mathcal{K}_s)^{-1}[h(x)], \tag{2.6}$$

with the corresponding Neumann series representation (see Tricomi [14])

$$\varphi_{s,z}(x) = h(x) + \sum_{n=1}^{\infty} z^n \mathcal{K}_s^n[h(x)]. \tag{2.7}$$

The resolvent operator, $R_s(z)$ of $\mathcal{K}_s$, satisfies by definition the equation

$$[I - z\mathcal{K}_s]^{-1} = I + zR_s(z), \tag{2.8}$$

whence, it can be represented as

$$R_s(z) = \sum_{n=1}^{\infty} z^{n-1}\mathcal{K}_s^n. \tag{2.9}$$

Substituting the right-hand side of (2.8) in (2.6) yields a solution for $\varphi_{s,z}(x)$ in terms of the resolvent $R_s(z)$, namely,

$$\varphi_{s,z}(x) = h(x) + zR_s(z)[h(x)]. \tag{2.10}$$

The resolvent is also an integral operator with kernel $Q_{s,z}(x,y) = \sum_{n=1}^{\infty} K_{s,n}(x,y)z^{n-1}$, whence

$$\varphi_{s,z}(x) = h(x) + z\int_0^{\infty} h(y)Q_{s,z}(x,y)dy. \tag{2.11}$$

To investigate the radius of convergence of the Neumann expansion (2.7) or, equivalently, the expansion for the resolvent (2.9), we seek the lowest characteristic value λ_s and the corresponding eigenfunction $\psi_s(x)$ satisfying

$$\psi_s(x) = \lambda_s \int_0^{\infty} \psi_s(y)e^{-sy}f_1(y \mid x)dy. \tag{2.12}$$

It is known that the Neumann series converges for $|z| < |\lambda_s|$ (see, e.g., Tricomi [14]). Since the kernel (2.2) satisfies $K_s(x,y) \geq 0$, it is also known that $\lambda_s > 0$ and that $\psi_s(x)$ may be chosen to satisfy $\psi_s(x) \geq 0$. It will now be shown that for $s > 0$, the circle of convergence of (2.7) and (2.10) includes the circle $|z| = 1$.

Theorem 1: *For $s > 0$, the radius of convergence of the Neumann series* (2.7) *and the resolvent series* (2.10) *is greater than one.*

Proof: Since λ_s is the radius of convergence, it suffices to show that $\lambda_s > 1$. To this end, write

$$\sup_{x \geq 0}\{\psi_s(x)\} = \lambda_s \sup_{x \geq 0}\left\{\int_0^{\infty} \psi_s(y)e^{-sy}f_1(y \mid x)dy\right\}$$

$$\leq \lambda_s \sup_{x \geq 0}\left\{\int_0^{\infty} \sup_{z \geq 0}\{\psi_s(z)\}e^{-sy}f_1(y \mid x)dy\right\},$$

where the first equality follows from (2.12), and the succeeding inequality is a consequence of the positivity of the kernel and the positivity of $f_1(y \mid x)$ on a set of positive Lebesgue measure. Dividing throughout by $\sup_{z \geq 0}\{\psi_s(z)\} > 0$, we can write

$$1 \leq \lambda_s \sup_{x \geq 0}\left\{\int_0^{\infty} e^{-sy}f_1(y \mid x)dy\right\} < \lambda_s \sup_{x \geq 0}\left\{\int_0^{\infty} f_1(y \mid x)dy\right\} = \lambda_s,$$

where the inequalities again follow from the positivity of the kernel and the positivity of $f_1(y \mid x)$ on a Borel set of positive Lebesgue measure. □

We conclude that $\| \mathcal{K}_s \| = \lambda_s^{-1} < 1$, whence $\mathcal{K}_s$ is a contraction. By the Banach-Cacciopoli theorem (Jerri [11]) the (equivalent) solutions, (2.7) and (2.10), are unique.

3. Integral Equations for Markovian Traffic

Recall that by assumption, $t = 0$ is an arrival point, $-X_0$ is the previous arrival point and X_1 the next arrival point. Denoting $q_n(t \mid x) = P\{N(t) = n \mid X_0 = x\}$, define the generating function $G(z,t \mid x) = E[z^{N(t)} \mid X_0 = x] = \sum_{n=0}^{\infty} q_n(t \mid x)z^n$. We now proceed to derive an integral equation for $G(z,t \mid x)$. Noting the sample path relation

$$z^{N(t)} = \begin{cases} 1, & \text{on } \{X_1 > t\} \\ z\, z^{N(t - X_1)}, & \text{on } \{X_1 \le t\} \end{cases} \tag{3.1}$$

and recalling that $f_1(y \mid x)$ is the 1-step transition density of $\{X_n\}$, it follows from (3.1) that

$$E[1_{\{X_1 > t\}} \mid X_0 = x] = \int_t^{\infty} f_1(y \mid x)dy$$

$$E[z^{1 + N(t - X_1)} 1_{\{X_1 \le t\}} \mid X_0 = x] = z \int_0^t G(z, t - y \mid y) f_1(y \mid x)dy.$$

Hence, the required integral equation for $G(z,t \mid y)$ is

$$G(z,t \mid x) = \int_t^{\infty} f_1(y \mid x)dy + z \int_0^t G(z, t - y \mid y) f_1(y \mid x)dy. \tag{3.2}$$

Unfortunately, (3.2) is neither of Volterra nor Fredholm type. For later applications it will be more convenient to transform it into a Fredholm-type integral equation. To do that, observe that the conditions of Lemma 1 are satisfied for $f_1(y \mid x)$ and $G(z,t \mid y)$ in the set $\{z\colon |z| \le 1\}$. Thus, taking Laplace transforms in (3.2) yields

$$\widetilde{G}(z,s \mid x) = \frac{1 - \widetilde{f}_1(s \mid x)}{s} + z \int_0^{\infty} e^{-sy} \widetilde{G}(z,s \mid y) f_1(y \mid x)dy, \quad s > 0. \tag{3.3}$$

The integral equation for the Laplace transform $\widetilde{M}(s \mid x)$ of the conditional expectation $M(t \mid x) = E[N(t) \mid x]$ is obtained from the integral equation (3.3) using

$$\widetilde{M}(s \mid x) = \frac{\partial}{\partial z} \widetilde{G}(z,s \mid x) \big|_{z = 1} = \frac{1}{s} \widetilde{f}_1(s \mid x) + \int_0^{\infty} e^{-sy} \widetilde{M}(s \mid y) f_1(y \mid x)dy, \tag{3.4}$$

justified by the uniform convergence of $\frac{\partial}{\partial z}\widetilde{G}(z,s \mid x)$ in the set $\{z\colon |z| \le 1\}$, by appeal to Theorem 1.

Finally, we obtain an integral equation for the Laplace transform of the generating function $L(z,t \mid x) = \sum_{n=1}^{\infty} p_n(t \mid x) z^{n-1}$, where $p_n(t \mid x) = \frac{\partial}{\partial t} P\{S_n \le t \mid X_0 = x\}$. Since

$$P\{S_n > t \mid X_0 = x\} = P\{N(t) < n \mid X_0 = x\} = \sum_{k=0}^{n} q_k(t \mid x),$$

we can write $p_n(t \mid x) = -\frac{\partial}{\partial t} \sum_{k=0}^{n-1} q_k(t \mid x)$, whence

$$L(z,t \mid x) = -\frac{\partial}{\partial t} \sum_{n=1}^{\infty} z^{n-1} \sum_{k=0}^{n-1} q_k(t \mid x) = -\frac{\partial}{\partial t} \frac{G(z,t \mid x)}{1 - z}. \tag{3.5}$$

Applying Laplace transforms to (3.5) yields the relation

$$\widetilde{L}(z,s \mid x) = \frac{1 - s\widetilde{G}(z,s \mid x)}{1-z},$$

and applying (3.3) to the relation above results in the integral equation

$$\widetilde{L}(z,s \mid x) = \widetilde{f}_1(s \mid x) + z\int_0^\infty e^{-sy}\widetilde{L}(z,s \mid y) f_1(y \mid x)dy \tag{3.6}$$

for $\widetilde{L}(z,s \mid x)$. It is interesting to note that

$$\widetilde{M}(s \mid x) = \tfrac{1}{s}\widetilde{L}(1,s \mid x), \tag{3.7}$$

and this relation extends the known formula for the transform of the renewal density in renewal theory (Cox [1]).

The three equations (3.3), (3.4) and (3.6), all have the generic form (2.1). We are now in a position to use the Fredholm-type integral equation theory, developed in Section 2, to solve the integral equations (3.3), (3.4) and (3.6).

For the integral equation (3.3), the forcing term, $h_G(x) = \frac{1-\widetilde{f}_1(s \mid x)}{s}$, is assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.3) is

$$\widetilde{G}(z,s \mid x) = \frac{1-\widetilde{f}_1(s \mid x)}{s} + \sum_{n=1}^{\infty} z^n \mathcal{K}_s^n\left[\frac{1-\widetilde{f}_1(s \mid x)}{s}\right]. \tag{3.8}$$

Since the circle of convergence of the Neumann series (2.7) includes $\mid z \mid = 1$, the integral equation (3.4) for $\widetilde{M}(s \mid x)$ is established, as noted earlier, as well as its existence. For the integral equation (3.4), the forcing term, $h_M(x) = \frac{\widetilde{f}_1(s \mid x)}{s}$, is also assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.4) is

$$\widetilde{M}(s \mid x) = \tfrac{1}{s}\widetilde{f}_1(s \mid x) + \tfrac{1}{s}\sum_{n=1}^{\infty}\mathcal{K}_s^n[\widetilde{f}_1(s \mid x)] = \tfrac{1}{s}\widetilde{f}_1(s \mid x) + \tfrac{1}{s}R_s(1)\widetilde{f}_1(s \mid x). \tag{3.9}$$

For the integral equation (3.6), the forcing term, $h_L(x) = \widetilde{f}_1(s \mid x)$, is again assumed to belong to $L^2([0,\infty))$, or to be bounded on $[0,\infty]$ and summable over every finite interval. The solution of (3.6) is

$$\widetilde{L}(z,s \mid x) = \widetilde{f}_1(s \mid x) + \sum_{n=1}^{\infty} z^n \mathcal{K}_s^n[\widetilde{f}_1(s \mid x)]. \tag{3.10}$$

The coefficient of z^n in (3.8) is the transform $\widetilde{q}_n(s \mid x)$, whence

$$\widetilde{q}_n(s \mid x) = \mathcal{K}_s^n\left[\frac{1-\widetilde{f}_1(s \mid x)}{s}\right] = \int_0^\infty \frac{1-\widetilde{f}_1(s \mid y)}{s} K_{s,n}(x,y)dy, \quad n \geq 1. \tag{3.11}$$

Finally, the coefficient of z^n in (3.10) is the transform $\widetilde{p}_{n+1}(s \mid x)$, whence

$$\widetilde{p}_n(s \mid x) = \mathcal{K}_s^{n-1}[\widetilde{f}_1(s \mid x)] = \int_0^\infty \widetilde{f}_1(s \mid y) K_{s,n-1}(x,y)dy, \quad n \geq 1. \tag{3.12}$$

4. Specialization of TES$^+$ Traffic Processes

We now proceed to specialize the discussion to a TES$^+$ arrival process $\{X_n^+\}$, with innovation density f_V. From (1.5), the transition density, $g_U^+(v \mid u)$ of the uniform background TES$^+$ process, $\{U_n^+\}$, has the representation

$$g_U^+(v \mid u) = g_U(v-u),$$

in which the function g_U on the right-hand side is given by

$$g_U(w) = \sum_{n=-\infty}^{\infty} f_V(w+n) = \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)e^{i2\pi\nu w}, \tag{4.1}$$

where the second equality follows from the Poisson summation formula (Lighthill [12]).

Next, we rewrite the integral equations from Section 3 for their $\{X_n^+\}$ counterparts $\tilde{G}_X^+(z,s \mid x)$, $\tilde{M}_X^+(s \mid x)$ and $\tilde{L}_X^+(z,s \mid x)$ in terms of the function $g_U(w)$. This has the important advantage of transforming the infinite integration range to the compact set $[0,1]$. To this end, observe that the probability element $f_1^+(y \mid x)dy$, with $x = D(u)$ and $y = D(v)$, is transformed to

$$f_1^+(y \mid x)dy = f_1^+(D(v) \mid D(u))D'(v)dv = g_U(v-u)dv. \tag{4.2}$$

Consequently, the integral equation (3.3) is transformed to

$$\tilde{G}_X^+(z,s \mid D(u)) = \frac{1-\tilde{f}_1^+(s \mid D(u))}{s} + z\int_0^1 \tilde{G}_X^+(z,s \mid D(v))e^{-sD(v)}g_U(v-u)dv, \tag{4.3}$$

the integral equation (3.4) is transformed to

$$\tilde{M}_X^+(s \mid D(u)) = \frac{1}{2}\tilde{f}_1^+(s \mid D(u)) + \int_0^1 \tilde{M}_X^+(s \mid D(v))e^{-sD(v)}g_U(v-u)dv, \tag{4.4}$$

and the integral equation (3.6) is transformed to

$$\tilde{L}_X^+(z,s \mid D(u)) = \tilde{r}_1^+(s \mid D(u)) + z\int_0^1 \tilde{L}_X^+(z,s \mid D(v))e^{-sD(v)}g_U(v-u)dv. \tag{4.5}$$

For each u, the function $\tilde{M}_X^+(s \mid D(u))$ is analytic in the plane $\{s: Re[s] > 0\}$ with a pole at $s = 0$. The contribution of this pole (to be used in Section 7) is given in the next theorem.

Theorem 2: *For any TES$^+$ process $\{X_n^+\}$ of the form (1.3), the asymptotic expansion of $\tilde{M}_X^+(s \mid D(u))$ at 0 is given, for each u, by*

$$\widetilde{M}_X^+(s \mid D(u)) \sim \frac{\lambda_X^+}{s^2} + \frac{b_X^+(u)}{s}, \quad s \to 0+, \tag{4.6}$$

where $\lambda_X^+ = 1/E[X_n]$, *and*

$$b_X^+(u) = b_0^+ - \lambda_X^+ \sum_{\nu = -\infty}^{\epsilon} \frac{\widetilde{f}_V(-i2\pi\nu)}{1 - \widetilde{f}_V(-i2\pi\nu)} \widetilde{D}(i2\pi\nu)e^{i2\pi\nu u}, \tag{4.7}$$

for some constant b_0^+ *to be determined in Section* 7.

Proof: The asymptotic analysis will be carried out, postulating the expansion (4.6) and substituting it into (4.4). Accordingly,

$$\frac{\lambda_X^+}{s^2} + \frac{b_X^+(u)}{s} = \frac{1}{s} + \int_0^1 \left(\frac{\lambda_X^+}{s^2} + \frac{b_X^+(v)}{s}\right)[1 - sD(v)]g(v-u)dv, \tag{4.8}$$

in which only the relevant powers of s have been retained, and the first term on the right-hand side, $1/s$, arises from $\widetilde{f}_1^+(0 \mid D(u)) \equiv 1$ by expanding it in powers of s around 0. Equating the coefficients of $1/s^2$ in (4.8) yields

$$\lambda_X^+ = \lambda_X^+ \int_0^1 g(v-u)dv,$$

which is clearly satisfied since $\int_0^1 g(v-u)dv \equiv 1$. Equating the coefficients of $1/s$ in (4.8) yields the following integral equation for $b_X^+(u)$,

$$b_X^+(u) = 1 - \lambda_X^+ \int_0^1 D(v)g(v-u)dv + \int_0^1 b_X^+(v)g(v-u)dv.$$

To solve the integral equation above for b_X^+, substitute the Fourier series representation of $g(v-u)$ from (4.1), which gives

$$b_X^+(u) = 1 - \lambda_X^+ \sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(i2\pi\nu)\widetilde{D}(-i2\pi\nu)e^{-i2\pi\nu u}$$

$$+ \sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(i2pi\nu)\widetilde{b}_X^+(-i2\pi\nu)e^{-i2\pi\nu u}.$$

To put the Fourier series above in standard form, we use complex conjugates to obtain

$$b_X^+(u) = 1 - \lambda_X^+ \sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(-i2\pi\nu)\widetilde{D}(i2\pi\nu)e^{i2\pi\nu u}$$

$$+ \sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(-i2pi\nu)\widetilde{b}_X^+(i2\pi\nu)e^{i2\pi\nu u}. \tag{4.9}$$

On the other hand, since

$$b_X^+(u) = \sum_{\nu=-\infty}^{\infty} \widetilde{b}_X^+(i2\pi\nu)e^{i2\pi\nu u}, \tag{4.10}$$

one may equation coefficients in the representations (4.9) and (4.10) and deduce that

$\widetilde{b}_X^+(i2\pi\nu) =$

$$\begin{cases} 1-\lambda_X^+\widetilde{D}(0)+\widetilde{b}_X^+(0), & \nu=0 \\ -\lambda_X^+\widetilde{f}_V(-i2\pi\nu)\widetilde{D}(i2\pi\nu)+\widetilde{f}_V(-i2\pi\nu)\widetilde{b}_X^+(i2\pi\nu), & \nu\neq 0. \end{cases}$$

Since $\widetilde{D}(0)=1/\lambda_X^+$, the equation for $\widetilde{b}_X^+(0)$ is consistent but does not determine $\widetilde{b}_X^+(0)$. This constant will be determined later on in Section 7. But for $\nu\neq 0$,

$$\widetilde{b}_X^+(i2\pi\nu)=-\lambda_X^+\frac{\widetilde{f}_V(-i2\pi\nu)}{1-\widetilde{f}_V(-i2\pi\nu)}\widetilde{D}(i2\pi\nu).$$

The theorem follows, since the postulated asymptotic expansion is consistent. □

Since $M(t\mid D(u))$ is monotone increasing in t (for fixed $D(u)$), a real Tauberian theorem [15] may be used to obtain the following asymptotic expansion at infinity,

$$M(t\mid D(u))\sim\lambda t,\quad t\to\infty.$$

One may also expect

$$M(t\mid D(u))\sim\lambda t+b_X^+(u),$$

although this does not directly follow from the Tauberian theorem.

The three equations (4.3)-(4.5) all have the generic form

$$\varphi(u)=h(u)+z\int_0^1\varphi(v)T_s(u,v)dv, \tag{4.11}$$

where the kernel $T_s(u,v)$ is given by

$$T_s(u,v)=e^{-sD(v)}g_U(v-u). \tag{4.12}$$

Thus, the Fredholm-type integral equation (4.11) is a special case of the integral equation (2.1) and the kernel $T_s(u,v)$ in (4.12) is a special case of the kernel $K_s(x,y)$ of (2.2). In conformance with the notational conventions of Section 2, the associated operator $\mathcal{T}_s$, $s>0$, on $L^2([0,\infty))$ is

$$\mathcal{T}_s[f(u)]=\int_0^1 e^{-sD(v)}g_U(v-u)f(v)dv,\quad f(u)\in L^2([0,1)).$$

The iterated kernels $T_{s,n}(u,v)$ take the form

$$T_{s,n}(u,v)=\begin{cases} T_s(u,v), & n=1 \\ \int_0^1 e^{-sD(v)}g_U(v-w)T_{s,n-1}(w,u)dw, & n>1. \end{cases}$$

Since the theory outline in Section 2 for Fredholm-type integral equations holds for $K_s=T_s$ as a special case, we may apply the solutions as well to the special case of TES$^+$ processes.

From (3.8), the solution of the integral equation (4.3) is

$$\tilde{G}_X^+(z,s \mid D(u)) = \frac{1-\tilde{f}_1^+(s \mid D(u))}{s} + \sum_{n=1}^{\infty} z^n \mathcal{T}_s^n\left[\frac{1-\tilde{f}_1^+(s \mid D(u))}{s}\right], \tag{4.13}$$

and from (3.11),

$$\tilde{q}_n^+(s \mid D(u) = \mathcal{T}_s^n\left[\frac{1-\tilde{f}_1^+(s \mid D(u))}{s}\right]$$

$$= \frac{1}{s}\int_0^1 [1-\tilde{f}_1^+(s \mid D(v))]T_{s,n}(u,v)dv, \quad n \geq 1. \tag{4.14}$$

From (3.9), the solution of the integral equation (4.4) is

$$\tilde{M}_X^+(s \mid D(u)) = \frac{1}{s}\tilde{f}_1^+(s \mid D(u)) + \frac{1}{s}\sum_{n=1}^{\infty} \mathcal{T}_s^n[\tilde{f}_1^+(s \mid D(u))]$$

$$= \frac{1}{s}\tilde{f}_1^+(s \mid D(u)) + \frac{1}{s}R_s^+(1)\tilde{f}_1^+(s \mid D(u)), \tag{4.15}$$

where R_s^+ is the resolvent (2.8), corresponding to the TES kernel $T_s(u,v)$.

From (3.10), the solution of the integral equation (4.5) is

$$\tilde{L}_X^+(z,s \mid D(u)) = \tilde{f}_1^+(s \mid D(u)) + \sum_{n=1}^{\infty} z^n \mathcal{T}_s^n[\tilde{f}_1^+(s \mid D(u))], \tag{4.16}$$

and from (3.12),

$$\tilde{p}_n^+(s \mid D(u)) = \mathcal{T}_s^{n-1}[\tilde{f}_1^+(s \mid D(u))]$$

$$= \int_0^1 \tilde{f}_1^+(s \mid D(u))T_{s,n-1}(u,v)dv, \quad n \geq 1. \tag{4.17}$$

It is interesting to compare the structure of the integral equations (4.3), (4.4) and (4.5) and their respective solutions (4.13), (4.15) and (4.16) above (when the interarrival process is a Markovian TES$^+$ sequence) with the renewal case (when the inter-arrival process consists of independent indentically distributed random variables), implying $\overset{1}{f}_1^+(s \mid x) \equiv \tilde{f}_X^+(s)$). The renewal case is just a special TES$^+$ process with $g_U^+(v \mid u) = g_U(v-u) \equiv 1$. from standard renewal theory or from the respective integral equations (4.3), (4.4) and (4.5), the corresponding transforms are

$$\tilde{G}_r(z,s) = \frac{1}{s}\,\frac{1-\tilde{f}_X^+(s)}{1-z\tilde{f}_X^+(s)}$$

$$\tilde{M}_r(s) = \frac{1}{s}\,\frac{\tilde{f}_X^+(s)}{1-\tilde{f}_X^+(s)}$$

$$\tilde{L}_r(z,s) = \frac{\tilde{f}_X^+(s)}{1-z\tilde{f}_X^+(s)}.$$

For any $g_U(w)$, let $\widetilde{G}_d(z,s,u) = \widetilde{G}_X^+(z,r \mid D(u)) - \widetilde{G}_r(z,s)$, $\widetilde{M}_d(s,u) = \widetilde{M}_X^+ \ (s \mid D(u)) - \widetilde{M}_r(s)$ and $\widetilde{L}_d(z,s,u) = \widetilde{L}_X^+(z,s \mid D(u)) - \widetilde{L}_r(z,s)$ be the respective deviation functions from the renewal case. Each of the deviation functions satisfies an integral equation as follows: $\widetilde{G}_d(z,s,u)$ satisfies

$$\widetilde{G}_d(z,s,u) = \frac{1 - \widetilde{f}_1^+(s \mid D(u))}{s} - \widetilde{G}_r(z,s)[1 - z\widetilde{f}_1^+(s \mid D(u))] + z\mathcal{T}_s[\widetilde{G}_d(z,s,u)],$$

$\widetilde{M}_d(s,u)$ satisfies

$$\widetilde{M}_d(s,u) = \tfrac{1}{s}\widetilde{f}_1^+(s \mid D(u)) - \widetilde{M}_r(s)[1 - \widetilde{f}_1^+(s \mid D(u))] + \mathcal{T}_s[\widetilde{M}_d(s,u)],$$

and $\widetilde{L}_d(z,s,u)$ satisfies

$$\widetilde{L}_d(z,s,u) = \widetilde{f}_1^+(s \mid D(u)) - \widetilde{L}_r(z,s)[1 - z\widetilde{f}_1^+(s \mid D(u))] + z\mathcal{T}_s[\widetilde{L}_d(z,s,u)].$$

For TES^+ processes which are approximately renewal, one would expect the corresponding Neumann expansions (2.7) for $\widetilde{G}_d$, $\widetilde{M}_d$ and $\widetilde{L}_d$ to provide good approximations. This aspect of the integral equations may be used to construct analytical approximations for the solutions.

5. A Matric Form of Solutions for TES^+ Traffic Processes

The Fourier series representation of $g_U(w)$ in (4.1) may be used to construct a matric solution for the integral equation (4.11). Following Tricomi [14] to this end, substitute (4.1) for $g_U(w)$ in (4.11), yielding

$$\varphi(u) = h(u) + z\int_0^1 \varphi(v)e^{-sD(v)}\sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(i2\pi\nu)e^{i2\pi\nu(v-u)}dv.$$

Next, we write

$$\varphi(u) = h(u) + z\sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(i2\pi\nu)e^{-i2\pi\nu u}\int_0^1 \varphi(v)e^{-sD(v)+i2\pi\nu v}dv$$

$$= h(u) + z\sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(-i2\pi\nu)e^{i2\pi\nu u}\int_0^1 \varphi(v)e^{-sD(v)-i2\pi\nu v}dv, \tag{5.1}$$

where Parseval's equality (Hardy and Rogosinski [6]) justifies the interchange of integration and summation in the first equality, and complex conjugation justifies the second equality. Denoting for integer μ,

$$c_\mu(s) = \int_0^1 \varphi(v)e^{-sD(v)-i2\pi\mu v}dv \tag{5.2}$$

to be the Fourier coefficients of $\varphi(v)e^{-sD(v)}$, we can rewrite (5.1) as

$$\varphi(u) = h(u) + z\sum_{\nu=-\infty}^{\infty} \widetilde{f}_V(-i2\pi\nu)c_\nu(s)e^{i2\pi\nu u}. \tag{5.3}$$

Equation (5.3) is a solution for $\varphi(u)$ in terms of $(\ldots, c_{-1}(s), c_0(s), c_1(s), \ldots)$, which is an unknown vector with components $c_\mu(s)$. In order to determine this unknown vector, substitute (5.3) into

(5.2). This yields

$$
\begin{aligned}
c_{\mu}(s) &= \int_0^1 h(v)e^{-sD(v) - i2\pi\mu v}dv \\
&\quad + z\int_0^1 e^{-sD(v) - i2\pi\mu v}\sum_{\nu = -\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)c_{\nu}(s)e^{i2\pi\nu v}dv \\
&= \int_0^1 h(v)e^{-sD(v) - i2\pi\mu v}dv \\
&\quad + z\sum_{\nu = -\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)c_{\nu}(s)\int_0^1 e^{-sD(v) + i2\pi(\nu - \mu)v}dv,
\end{aligned}
\tag{5.4}
$$

where Parseval's formula again justifies the interchange of integration and summation. Let $\boldsymbol{c}(s)$ be the vector with components $c_{\mu}(s)$, and let $\boldsymbol{h}(s)$ be the vector with components $h_{\mu}(s) = \int_0^1 h(v)e^{-sD(v) - i2\pi\mu v}dv$. Finally, let $\boldsymbol{M}(s)$ be the matrix with components $M_{\mu,\nu}(s) = \tilde{f}_V(-i2\pi\nu)\int_0^1 e^{-sD(v) + i2\pi(\nu - \mu)v}dv$. Equation (5.4) now takes the form

$$c_{\mu}(s) = h_{\mu}(s) + z\sum_{\nu = -\infty}^{\infty} M_{\mu,\nu}(s)c_{\nu}(s),$$

which can be written in matric form as

$$\boldsymbol{c}(s) = \boldsymbol{h}(s) + z\boldsymbol{M}(s)\boldsymbol{c}(s).$$

The solution of the matric equation above is

$$\boldsymbol{c}(s) = [\boldsymbol{I} - z\boldsymbol{M}(s)]^{-1}\boldsymbol{h}(s), \tag{5.5}$$

where **I** is the (infinite-dimensional) identity matrix. The solution (5.5) may be effectuated, in practice, by using an $n \times n$ submatrix extracted from $\boldsymbol{M}(s)$ by symmetrical truncation. This is the same as symmetrically truncating the Fourier series (4.1) for $g_U(w)$.

6. Example: TES$^+$ Processes with Exponential Innovations

When the transform $\tilde{f}_V(s)$ of the innovation density is rational then, in principle, it is possible to obtain the exact solution for the integral equation (4.11). In that case, $g_U(w)$ has the form of an exponential polynomial, so that a differential operator may be found to eliminate the integration; this will replace the integral equation by a differential equation. Difficulties still remain, however, since the differential equation will have variable coefficients.

In this section, we illustrate this procedure for the exponential innovation density $f_V(x) = \lambda e^{-\lambda x}$, $x > 0$, with its Laplace transform $\tilde{f}_V(s) = \frac{\lambda}{\lambda + s}$. From (4.1), the corresponding density $g_U(w)$ is given by

$$g_U(w) = \sum_{\nu = -\infty}^{\infty} \frac{\lambda}{\lambda + i2\pi\nu}e^{i2\pi\nu w} = \frac{\lambda}{1 - e^{-\lambda}}e^{-\lambda\langle w\rangle}, \tag{6.1}$$

and the corresponding transformed transition density is

$$\tilde{f}_1^+(s \mid D(u)) = \frac{\lambda}{1-e^{-\lambda}} \int_0^1 e^{-sD(v) - \lambda\langle v-u \rangle} dv. \tag{6.2}$$

Putting (6.1) into the integral equation (4.11) yields

$$\begin{aligned}
\varphi_{s,z}(u) &= h(u) + z\frac{\lambda}{1-e^{-\lambda}} \int_0^1 \varphi_{s,z}(v) e^{-sD(v) - \lambda\langle v-u \rangle} dv \\
&= h(u) + z\frac{\lambda}{1-e^{-\lambda}} \int_0^u \varphi_{s,z}(v) e^{-\lambda - sD(v) - \lambda v + \lambda u} dv \\
&\quad + z\frac{\lambda}{1-e^{-\lambda}} \int_u^1 \varphi_{s,z}(v) e^{-sD(v) - \lambda v + \lambda u} dv.
\end{aligned} \tag{6.3}$$

After differentiating (6.3) with respect to u, we get

$$\begin{aligned}
\varphi'_{s,z}(u) &= h'(u) + z\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}} \varphi_{s,z}(u) e^{-sD(u)} - z\frac{\lambda}{1-e^{-\lambda}} \varphi_{s,z}(u) e^{-sD(u)} \\
&\quad + \lambda z \int_0^u \varphi_{s,z}(v) e^{-sD(v)} \frac{\lambda e^{-\lambda - \lambda v + \lambda u}}{1-e^{-\lambda}} dv \\
&\quad + \lambda z \int_u^1 \varphi_{s,z}(v) e^{-sD(v)} \frac{\lambda e^{-\lambda v + \lambda u}}{1-e^{-\lambda}} dv \\
&= h'(u) - \lambda z \varphi_{s,z}(u) e^{-sD(u)} + \lambda z \int_0^1 \varphi_{s,z}(v) e^{-sD(v)} \frac{\lambda e^{-\lambda\langle v-u \rangle}}{1-e^{-\lambda}} dv.
\end{aligned}$$

Replacing the integral term above, with the aid of (6.3), yields the differential equation in $\varphi_{s,z}(u)$

$$\varphi'_{s,z}(u) - \lambda\left[1 - ze^{-sD(u)}\right] \varphi_{s,z}(u) = h'(u) - \lambda h(u), \quad u \in [0,1]. \tag{6.4}$$

The right-hand side, $\eta(u) = h'(u) - \lambda h(u)$, of (6.4) is readily computed for each of the integral equations (4.3), (4.4) and (4.5), using $\tilde{f}_1^+(s \mid D(u))$ from (6.2). Specifically, for (4.3),

$$h_G(u) = \tfrac{1}{s} - \tfrac{1}{s}\tilde{f}_1^+(s \mid D(u)) \Rightarrow \eta_G(u) = -\tfrac{\lambda}{s} + \tfrac{\lambda}{s} e^{-sD(u)};$$

for (4.4),

$$h_M(u) = \tfrac{1}{s}\tilde{f}_1^+(s \mid D(u)) \Rightarrow \eta_M(u) = -\tfrac{\lambda}{s} e^{-sD(u)};$$

and for (4.5),

$$h_L(u) = \tilde{f}_1^+(s \mid D(u)) \Rightarrow \eta_L(u) = -\lambda e^{-sD(u)}.$$

To simplify the exposition we assume a distortion-free TES$^+$ process ($D(u) \equiv u$). To solve for $\tilde{L}_X^+(z, s \mid u)$ in (4.5), the differential equation (6.4) now simplifies to

$$\varphi'_{s,z}(u) - \lambda[1 - ze^{-su}]\varphi_{s,z}(u) = -\lambda e^{-su}, \quad u \in [0,1], \tag{6.5}$$

with the complementary solution (see Ritger and Rose [13])

$$\varphi^{(c)}_{s,z}(u) = ce^{\lambda u + (\lambda z/s)e^{-su}}.$$

To determine a particular solution, define the operator

$$S_z[\psi(u)] = \psi'(u) - \lambda[1 - ze^{-\lambda u}]\psi(u), \quad \psi \in L^2([0,1)),$$

and postulate a solution $\varphi_{s,z}(u)$ of the form

$$\varphi_{s,z}(u) = \sum_{n=1}^{\infty} a_n e^{-nsu},$$

for some coefficients $a_n = a_n(s,z)$. Applying the operator S_z to the postulated solution above then results in

$$S_z[\varphi_{s,z}(u)] = -(\lambda + s)a_1 e^{-su} + \sum_{n=2}^{\infty}[-(\lambda + ns)a_n + \lambda z a_{n-1}]e^{-nsu}.$$

But from (6.3), $S_z[\varphi_{s,z}(u)] = -\lambda e^{-su}$, implying the recursive relations $a_1 = \frac{\lambda}{\lambda + s}$ and $a_n = \frac{\lambda z}{\lambda + ns}a_{n-1}$, for $n \geq 2$. Hence $a_n = \frac{1}{z} \prod_{k=1}^{n} \frac{\lambda z}{\lambda + ks}$, and the solution of (6.5) can now be written explicitly in terms of the constant c as

$$\tilde{L}(z, s \mid u) = ce^{\lambda u + (\lambda z/s)e^{-su}} + \frac{1}{z}\sum_{n=1}^{\infty} e^{-nsu}\prod_{k=1}^{n}\frac{\lambda z}{\lambda + ks}. \tag{6.6}$$

The constant $c = c(z, s, \lambda)$ is independent of u and will be determined by substituting (6.6) into (6.3). A direct calculation of $h_L(u) = \tilde{f}_1^+(s \mid u)$, with the aid of (6.2), yields

$$h_L(u) = \frac{\lambda}{s + \lambda}\left[e^{-su} + \frac{1 - e^{-s}}{1 - e^{-\lambda}}e^{\lambda(u-1)}\right]. \tag{6.7}$$

Substituting (6.6) and (6.7) into (6.3) results in

$$ce^{\lambda u + (\lambda z/s)e^{-su}} + \frac{1}{z}\sum_{n=1}^{\infty} e^{-nsu}\prod_{k=1}^{n}\frac{\lambda z}{\lambda + ks}$$

$$= \frac{\lambda}{\lambda + s}\left[e^{-su} + \frac{1 - e^{-s}}{1 - e^{-\lambda}}e^{\lambda(u-1)}\right]$$

$$+ \frac{cz\lambda}{1 - e^{-\lambda}}\int_0^1 e^{\lambda v + (\lambda z/s)e^{-sv} - sv - \lambda(v-u)}dv$$

$$+\frac{\lambda}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\left[\prod_{k=1}^{n}\frac{\lambda z}{\lambda+ds}\right]\int_0^1 e^{-nsv-sv-\lambda\langle v-u\rangle}dv.$$

Since c does note depend on u, set in particular, $u=0$ in the equation above, yielding

$$ce^{(\lambda z/s)}+\tfrac{1}{z}\sum_{n=1}^{\infty}\prod_{k=1}^{n}\frac{\lambda z}{\lambda+ks}$$

$$=\frac{\lambda}{\lambda+s}\frac{e^{\lambda}-e^{-s}}{e^{\lambda}-1}$$

$$+\frac{cz\lambda}{1-e^{-\lambda}}\int_0^1 e^{(\lambda z/s)e^{-sv}-sv}dv$$

$$+\frac{\lambda}{1-e^{-\lambda}}\sum_{n=1}^{\infty}\frac{1-e^{-\lambda-(n+1)s}}{\lambda+(n+1)s}\prod_{k=1}^{n}\frac{\lambda z}{\lambda+ks}.$$

Using the evaluation

$$\int_0^1 e^{(\lambda z/s)e^{-sv}-sv}dv=\frac{e^{(\lambda z/s)}-e^{(\lambda z/s)e^{-s}}}{\lambda z}$$

in the preceding equation, we conclude

$$c=\frac{\lambda\frac{e^{\lambda}-e^{-s}}{\lambda+s}+\sum_{n=1}^{\infty}\left(\lambda\frac{e^{\lambda}-e^{-(n+1)s}}{\lambda+(n+1)s}-\frac{e^{\lambda}-1}{z}\right)\prod_{k=1}^{n}\frac{\lambda z}{\lambda+ks}}{e^{(\lambda z/s)e^{-s}+\lambda}-e^{(\lambda z/s)}}. \tag{6.8}$$

The generating function $\widetilde{L}_X^+(z,s\mid u)$ is now given by (6.6) with c given in (6.8).

The solutions for $\widetilde{M}_X^+(s\mid u)$ and $\widetilde{G}_X^+(z,s\mid u)$ are largely similar. However, for $\widetilde{M}_X^+(s\mid u)$, we have a straightforward solution in terms of $\widetilde{L}_X^+(z,s\mid u)$ as given by (3.7).

7. The Peakedness of TES$^+$ Traffic Processes

The peakedness functional provides a partial characterization of the burstiness of an ergodic traffic stream by gauging its effect when offered to an infinite server group. In practice, peakedness is typically used to approximate a solution for blocking and delay statistics in finite-server queues.

Let $\{X_n\}$ be a stationary sequence of interarrival times with a general probability law, arrival rate $\lambda=1/E[X_n]<\infty$, and expectation function $M(t)$ (recall Section 1). Assume that the corresponding traffic stream is offered to an infinite server group consisting of independent servers with common service time distribution F. Let $B(t)$ be the number of busy servers at time t, and assume that its limiting statistics exist. The peakedness functional, z_X, associated with the traffic process $\{X_n\}$, given by

$$z_X[F] = \lim_{t \uparrow \infty} \frac{Var[B(t)]}{E^2[B(t)]}, \tag{7.1}$$

maps the space of all service time distributions to non-negative numbers.

Let $\{F_\mu\}$ be a parametric family of service time distributions, indexed by the service rate $\mu = 1/\int_0^\infty x dF_\mu(x)$. It is convenient to standardize the $F_\mu(x)$ to unit rate by defining $F_1(x) = F_\mu(x/\mu)$, and to replace the peakedness functional $z_X[F_\mu]$ from (7.1) by the corresponding peakedness function

$$z_{X,F_1}(\mu) = z_X[F_\mu]. \tag{7.2}$$

Interestingly, if $\{G_\mu\}$ is any other parametric family of service time distributions, then the corresponding peakedness functions $z_{X,F_1}(\mu)$ and $z_{X,G_1}(\mu)$ contain equivalent information on the traffic process $\{X_n\}$, in the sense that the two are connected by a known transformation (see Jagerman [7]). In particular, for an exponential service time distribution, $F_1(x) = 1 - e^{-x}$, the corresponding peakedness function (7.2) is dented by $z_{X,exp}(\mu)$, and has the representation (see Eckberg [3]),

$$z_{X,exp}(\mu) = 1 - \frac{\lambda}{\mu} + \mu \widetilde{M}(\mu). \tag{7.3}$$

Consider the auxiliary peakedness function

$$z_{X,exp}(\mu, x) = 1 - \frac{\lambda}{\mu} + \mu \widetilde{M}(\mu \mid x), \tag{7.4}$$

from which (7.3) can be obtained by integration with respect to the interarrival time density $f_X(x)$. Substituting the integral equation (3.4) into (7.4) yields the integral equation

$$z_{X,exp}(\mu, x) = 1 - \frac{\lambda}{\mu} + \frac{\lambda}{\mu}\widetilde{f}_1(\mu \mid x) + \int_0^\infty z_{X,exp}(\mu, x) e^{-\mu y} f_1(y \mid x) dy. \tag{7.5}$$

For the remainder of this section, we specialize the discussion to TES$^+$ processes $\{X_n^+\}$. In this case, the peakedness value, $z_{exp}^+(0)$, assumes a particularly simple form. However, before stating the main result, we shall need the following simple facts which will serve to simplify the proof.

Proposition 1: *For any TES$^+$ process $\{X_n^+\}$ of the form* (1.3),

$$z_{exp}^+(\mu) = \int_0^1 z_{exp}^+(\mu, D(v)) dv, \tag{7.6}$$

$$\frac{d}{d\mu} z_{exp}^+(\mu) = \int_0^1 \frac{\partial}{\partial \mu} z_{exp}^+(\mu, D(v)) dv, \tag{7.7}$$

$$\widetilde{f}_1^+(\mu \mid D(u)) = 1 - \mu \int_0^1 D(v) g_U^+(v \mid u) dv + o(\mu), \tag{7.8}$$

$$\tilde{f}_X^+(\mu) = 1 - \frac{\mu}{\lambda} + \frac{1}{2} m_2^+ \mu^2 + o(\mu^2), \tag{7.9}$$

where $m_2^+ = E[(X_n^+)^2]$.

Proof: (7.6) follows immediately from (7.4), because the marginal density of background TES processes is uniform on $[0,1)$.

To prove (7.7) we show that $z_{exp}^+(\mu, x)$ is analytic at $\mu = 0$, permitting the interchange of integration and differentiation. To this end, use the asymptotic expansion of $\tilde{M}_X^+$ from (4.6) in the general relation (7.4) to deduce

$$z_{exp}^+(\mu, D(u)) \sim 1 + b_X^+(u) + O(\mu), \quad \mu \to 0,$$

proving that, in fact, $z_{exp}^+(\mu, x)$ is analytic for $Re[\mu] \geq 0$. Moreover, at $\mu = 0$,

$$b_X^+(u) = z_{exp}^+(0, D(u)) - 1,$$

and from (7.6),

$$b_0^+ = \int_0^1 b_X^+(u)du = z_{exp}^+(0) - 1.$$

This determines the constant b_0^+ (which was left undetermined in Theorem 2) in terms of $z_{exp}^+(0)$. The determination of $z_{exp}^+(0)$ will be given later in Theorem 3.

Equation (7.8) is obtained by expanding $\tilde{f}_1^+(\mu \mid D(u)) = \int_0^1 e^{-\mu D(v)} g_U^+(v \mid u)\, dv$ in powers of μ around 0. Similarly, (7.9) is obtained by expanding $\tilde{f}_X^+(\mu) = E[e^{\mu X_n^+}]$ in powers of μ around 0. □

The main result now follows.

Theorem 3: *For any* TES^+ *process* $\{X_n^+\}$ *of the form (1.3), we have the representations*

$$z_{exp}^+(0) = \frac{1 + (c_X^+)^2}{2} + (c_X^+)^2 \sum_{\tau=1}^{\infty} \rho_X^+(\tau) = \frac{1 + (c_X^+)^2 \pi s_X^+(0)}{2}, \tag{7.10}$$

where $(c_X^+)^2 = Var[X_n^+]/E^2[X_n^+]$ *is the squared coefficient of variation corresponding to* f_X^+.

Proof: Substituting $x = D(u)$, $y = D(v)$ and (4.2) into (7.5) yields

$$z_{exp}^+(\mu, D(u)) = 1 - \frac{\lambda_X^+}{\mu} + \frac{\lambda_X^+}{\mu} \tilde{f}_1^+(\mu \mid D(u))$$
$$+ \int_0^1 z_{exp}^+(\mu, D(v)) e^{-\mu D(v)} g_U^+(v \mid u) dv. \tag{7.11}$$

To obtain $z_{exp}^+(0, D(u))$, substitute (7.8) into (7.11) and set $\mu = 0$. This yields the integral equation

$$z_{exp}^+(0, D(u)) = 1 - \lambda_X^+ \int_0^1 D(v) g_U^+(v \mid u) dv$$
$$+ \int_0^1 z_{exp}^+(0, D(v)) g_U^+(v \mid u) dv, \tag{7.12}$$

in the unknown function $z_{exp}^{+}(0, D(u))$. Since $g_U^{+}(v \mid u) = g_U(v-u)$ is periodic in each argument, we make use of Fourier series to represent the solution of (7.12). Substituting the representation (4.1) for $g_U^{+}(v \mid u)$ into the integral equation (7.12) and interchanging summation and integration, we obtain after some manipulation

$$z_{exp}^{+}(0, D(u)) = 1 - \sum_{\nu=-\infty}^{\infty} \lambda_X^{+} \tilde{f}_V(i2\pi\nu)\tilde{D}(-i2\pi\nu)e^{-i2\pi\nu u}$$

$$+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(i2\pi\nu)e^{-i2\pi\nu u}\int_0^1 e^{i2\pi\nu v} z_{exp}^{+}(0, D(v))dv. \tag{7.13}$$

Next, we let $\hat{z}_\nu = \int_0^1 e^{-i2\pi\nu v} z_{exp}^{+}(0, D(v))dv$, and conclude that $z_{exp}^{+}(0, D(v))$ has the Fourier series representation

$$z_{exp}^{+}(0, D(u)) = \sum_{\nu=-\infty}^{\infty} \hat{z}_\nu e^{i2\pi\nu u}. \tag{7.14}$$

The representation (7.14) is now used on both sides of (7.13) to obtain, after conjugation,

$$\sum_{\nu=-\infty}^{\infty} \hat{z}_\nu e^{i2\pi\nu u} = 1 - \sum_{\nu=-\infty}^{\infty} \lambda_X^{+} \tilde{f}_V(-i2\pi\nu)\tilde{D}(i2\pi\nu)e^{i2\pi\nu u}$$

$$+ \sum_{\nu=-\infty}^{\infty} \tilde{f}_V(-i2\pi\nu)\hat{z}_\nu e^{i2\pi\nu u}. \tag{7.15}$$

The Fourier coefficients, $\hat{z}_\nu$, can now be deduced to be

$$\hat{z}_\nu = \begin{cases} z_{exp}^{+}(0), & \nu = 0 \\ -\dfrac{\lambda_X^{+} f_V(-i2\pi\nu)}{1-\tilde{f}_V(-i2\pi\nu)}\tilde{D}(i2\pi\nu), & \nu \neq 1 \end{cases} \tag{7.16}$$

where the first case above follows by setting (7.14) in (7.6) for $\nu = 0$, and the second case above results from equating Fourier coefficients in (7.15) for $\nu \neq 0$. Furthermore, equating coefficients there for $\nu = 0$ yields the relation $\hat{z}_0 = 1 - \lambda_X^{+} \tilde{D}(0) + \hat{z}_0$, which is consistent with the fact that $\tilde{D}(0) = 1/\lambda_X^{+}$, but $\hat{z}_0 = z_{exp}^{+}(0)$ cannot yet be deduced. Substituting, however, the Fourier coefficients, $\hat{z}_\nu$, from (7.16) into the right-hand side of (7.14) now yields the relation

$$z_{exp}^{+}(0, D(u)) = z_{exp}^{+}(0) - \lambda_X^{+} \sum_{\substack{\nu=-\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(-i2\pi\nu)}{1-\tilde{f}_V(-i2\pi\nu)}\tilde{D}(i2\pi\nu)e^{i2\pi\nu u}, \tag{7.17}$$

with $z_{exp}^{+}(0)$ as yet undetermined.

To determine $z_{exp}^{+}(0)$, integrate (7.11) with respect to u, and use the relation (7.6) to obtain

$$z_{exp}^{+}(\mu) = 1 - \frac{\lambda_X^{+}}{\mu}[1 - \tilde{f}_1^{+}(\mu)] + \int_0^1 z_{exp}^{+}(\mu, D(v))e^{-\mu D(v)}dv. \tag{7.18}$$

Next, substitute (7.9) into the right-hand side of (7.18) and expand the resulting equation in powers of μ around 0, which yields

$$z^+_{exp}(0) + \mu\frac{d}{d\mu}z^+_{exp}(0) + o(\mu)$$

$$= \tfrac{1}{2}\lambda^+_X \mu m_2^+ + \int_0^1 \left[z^+_{exp}(0, D(v)) + \mu\frac{\partial}{\partial\mu}z^+_{exp}(0, D(v))\right]\left[1 - \mu D(v)\right] dv + o(\mu).$$

Simplifying the above equation with the aid of (7.6) and (7.7) and dividing by μ now gives the following condition equation for $z^+_{exp}(0, D(u))$,

$$\int_0^1 z^+_{exp}(0, D(u))D(u)du = \tfrac{1}{2}\lambda^+_X m_2^+. \tag{7.19}$$

The value of $z^+_{exp}(0)$ is finally obtained by substituting (7.17) into (7.19) resulting in

$$z^+_{exp}(0) = \tfrac{1}{2}(\lambda^+_X)^2 m_2^+ + (\lambda^+_X)^2 \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(-i2\pi\nu)}{1 - \tilde{f}_V(-i2\pi\nu)} \mid \tilde{D}(i2\pi\nu) \mid^2$$

$$= \tfrac{1}{2}(\lambda^+_X)^2 m_2^+ + (\lambda^+_X)^2 \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)} \mid \tilde{D}(i2\pi\nu) \mid^2, \tag{7.20}$$

where the second equality is justified by the fact that all quantities in (7.20) are real except for the terms in the infinite sums.

The first equality of (7.10) follows from (7.20), noting that

$$\sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)} \mid \tilde{D}(i2\pi\nu) \mid^2 = \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \sum_{\tau = 1}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu) \mid \tilde{D}(i2\pi\nu) \mid^2$$

$$= \sum_{\tau = 1}^{\infty} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \tilde{f}_V^{\tau}(i2\pi\nu) \mid \tilde{D}(i2\pi\nu) \mid^2 = Var[X_n^+] \sum_{\tau = 1}^{\infty} \rho^+_X(\tau),$$

where the second equality is justified by the absolute convergence of the series, and the third by appeal to (1.6). Finally, the second equality of (7.10) follows from the identity

$$\sum_{\tau = 1}^{\infty} \rho^+_X(\tau) = \frac{\pi s^+_X(0) - 1}{2},$$

in view of (1.2). □

Theorem 3 reveals an interesting connection between the peakedness $z^+_{exp}(0)$ and the index of dispersion notion of traffic burstiness or variability [2, 4, 5]. Let $\{X_n\}$ be a stationary sequence of interarrival times. The index of dispersion for intervals (IDI) is the sequence $\{I_n\}$, defined by

$$I_n = \frac{Var(X_{j+1} + \ldots + X_{j+n})}{nE[X_j]} = c_X^2\left[1 + 2\sum_{\tau = 1}^{n-1}(1 - \tfrac{\tau}{n})\rho_X(\tau)\right]. \tag{7.21}$$

The limit $I_X = \lim_{n\uparrow\infty} I_n$ of (7.21) is

$$I_X = c_X^2\left[1 + 2\sum_{\tau = 1}^{\infty}\rho_X(\tau)\right]. \tag{7.22}$$

For a TES$^+$ process, $\{X_n^+\}$, a comparison of (7.22) with (7.10) yields the relation

$$I_X^+ = 2z_{exp}^+(0) - 1. \tag{7.23}$$

Thus, for TES$^+$ processes, $z_{exp}^+(0)$ and I_X^+ convey the same information. However, for a general traffic process, $\{X_n\}$, the peakedness function, $z_{X,exp}(\mu)$, contains much more information, and especially the traffic impact on a server system. It is, therefore, desirable to obtain at least an approximate solution of the integral equation (7.11), for $\mu > 0$.

To this end, define

$$d_\nu(\mu) = \int_0^1 e^{-i2\pi\nu v - \mu D(v)} dv, \tag{7.24}$$

and for the auxiliary peakedness function, $z_{exp}^+(\mu, D(u))$ in (7.4), assume an approximation of the form

$$z_{exp}^+(\mu, D(u)) \simeq z_{exp}^+(\mu) + \frac{\lambda_X^+}{\mu} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} b_\nu d_\nu(\mu) e^{i2\pi\nu u}, \tag{7.25}$$

for some coefficients b_ν, as yet undetermined. (7.25) clearly satisfies (7.6), independently of the choice of the b_ν. Next, we make the essential assumption that the coefficients b_ν depend only on ν and require that the approximation (7.25) satisfy (7.18). Accordingly, on substituting the former into the latter, we get

$$z_{exp}^+(\mu) \simeq \frac{1}{1 - \tilde{f}_X^+(\mu)} - \frac{\lambda_X^+}{\mu} + \frac{\lambda_X^+}{\mu} \frac{1}{1 - \tilde{f}_X^+(\mu)} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} b_\nu \,|\, d_\nu(\mu) \,|^2. \tag{7.26}$$

In order to determine the b_ν, we further require that sending $\mu \downarrow 0+$ in (7.26), result in a limit that exactly equals the correct value, $z_{exp}^+(0)$, as given in Theorem 3. We show that this requirement yields

$$\lim_{\mu \downarrow 0+} z_{exp}^+(\mu) = \tfrac{1}{2}(\lambda_X^+)^2 m_2^+ + (\lambda_X^+)^2 \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} b_\nu \,|\, \tilde{D}(i2\pi\nu) \,|^2. \tag{7.27}$$

To see that, use (7.9) to obtain the evaluation $\frac{1}{1 - \tilde{f}_X^+(\mu)} - \frac{\lambda_X^+}{\mu} = \frac{1}{2}(\lambda_X^+)^2 m_2^+ + o(\mu)$, and expand (7.24) in powers of μ about 0 to deduce the relation $d_\nu = -\mu\tilde{D}(i2\pi\nu) + o(\mu)$. Equation (7.27) follows from the preceding two relations. A comparison of (7.27) with (7.20) now determines the b_ν as

$$b_\nu = \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)}.$$

Substituting the b_ν above into (7.25) yields the final approximation,

$$z_{exp}^+(\mu, D(u)) \simeq z_{exp}^+(\mu) + \frac{\lambda_X^+}{\mu} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1 - \tilde{f}_V(i2\pi\nu)} d_\nu(\mu) e^{iz\pi\nu u}, \tag{7.28}$$

and similarly, from (7.26),

$$z^+_{exp}(\mu) \simeq \frac{1}{1-\tilde{f}^+_X(\mu)} - \frac{\lambda^+_X}{\mu}$$

$$+ \frac{\lambda^+_X}{\mu} \frac{1}{1-\tilde{f}^+_X(\mu)} \sum_{\substack{\nu = -\infty \\ \nu \neq 0}}^{\infty} \frac{\tilde{f}_V(i2\pi\nu)}{1-\tilde{f}_V(i2\pi\nu)} \, | \, d_\nu(\mu) \, |^2. \tag{7.29}$$

It is interesting to observe that the first two terms,

$$\frac{1}{1-\tilde{f}^+_X(\mu)} - \frac{\lambda^+_X}{\mu},$$

in (7.29) constitute the exact peakedness function for the corresponding renewal traffic (i.e., a sequence of independent identically distributed inter-arrival times with common marginal density f^+_X). Thus, these first two terms represent the contribution of the marginal density, f^+_X, to the burstiness of $\{X^+_n\}$, while the third term in (7.29) arises from the dependence structure $\{X^+_n\}$. Equation (7.29) thus captures the effect of both first-order and higher-order statistics on the burstiness of a TES$^+$ traffic process. This interpretation is consistent with the fact that a renewal process entirely lacks second-order effects (its autocorrelation function is, zero for positive lags), and consequently, renewal traffic owes its peakedness to first-order statistics, exclusively. Interestingly, the case $\mu = 0$ in Equation (7.10) exhibits precisely two contributions, namely, those corresponding to first-order and second-order statistics, but no higher-order ones.

Acknowledgements: We thank Bhaskar Sengupta for the simple derivation of the integral equation for $G(z,t \mid y)$.

References

[1] Cox, D.R., *Renewal Theory*, Methuen and Co., London 1962.

[2] Cox, D.R., and Lewis, P.A.W., *The Statistical Analysis of Series of Events*, Methuen and Co., London 1968.

[3] Eckberg, A.E., Generalized Peakedness of Teletraffic Processes, *Tenth International Teletraffic Conference*, Montreal, Canada 1983.

[4] Fendick, K.W., Saksena, V.R. and Whitt, W., Dependence in packet queues, *IEEE Trans. on Communications* **37:11** (1989), 1173-1183.

[5] Gusella, R., Characterizing the variability of arrival processes with indexes of dispersion, *IEEE J. on Selected Areas in Communications* **9:2** (1991), 203-211.

[6] Hardy, G.H. and Rogosinski, W.W., *Fourier Series*, Cambridge University Press 1956.

[7] Jagerman, D.L., Methods in traffic calculations, *Bell System Technical Journal* **63:7** (1984), 1283-1310.

[8] Jagerman, D.L. and Melamed, B., The transition and autocorrelation structure of TES processes. Part I: General theory, *Stochastic Models* **8:2** (1992), 193-219.

[9] Jagerman, D.L. and Melamed, B., The transition and autocorrelation structure of TES processes. Part II: Special cases, *Stochastic Models* **8:3** (1992), 499-527.

[10] Jagerman, D.L. and Melamed, B., Spectral analysis of basic TES processes, *ORSA J. on Computing* (1993), to appear.

[11] Jerri, A.J., *Introduction to Integral Equations with Applications*, Marcel Dekker 1985.

[12] Lighthill, M.J., *Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press 1960.

[13] Ritger, P.D. and Rose, N.J., *Differential Equations with Applications*, McGraw-Hill 1968.

[14] Tricomi, F.G., *Integral Equations*, Interscience Publishers, New York 1957.

[15] Van Der Pol, B. and Bremmer, H., *Operational Calculus Based on the Two-Sided Laplace Integral*, Cambridge University Press 1950.

[16] Zabreyko, P.P., Koshelev, A.I., Krasnosel'skii, M.A., Mikhlin, S.G., Rakovshchik, L.S. and Stet'senko, V.Ya., *Integral Equations -- A Reference Text*, Noordhoff International Publishing, Leyden, The Netherlands 1975.