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FURTHER GENERALIZATION OF GENERALIZED QUASILINEARIZATION METHOD

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ABSTRACT

The question whether it is possible to develop monotone sequences that converge to the solution quadratically when the function involved in the initial value problem admits a decomposition into a sum of two functions, is answered positively. This extends the method of generalized quasilinearization to a large class.

Key words: Quasilinearization Method, Monotone Sequence, Quadratic Convergence.

AMS (MOS) subject classifications: 34A34, 34A40.

1. Introduction

It is well known [1, 2] that the method of quasilinearization offers an approach for obtaining approximate solutions to nonlinear differential equations.

Consider the IVP

$$x' = f(t,x),\ x(0) = x_0 \text{ on } J = [0,T].$$

If $f(t,x)$ is uniformly convex in x for all $t \in [0,T]$, then the method of quasilinearization provides a monotone increasing sequence of approximate solutions that converges quadratically to the unique solution. Moreover, the sequence provides lower bounds for the solution. Recently, the method of quasilinearization has received much attention after the publication of [9]. Since then, there has been a lot of activity in this area and several interesting results have appeared (see, for example, [4, 5, 6, 7, 8, 10, 11]).

In this paper, we show that it is possible to develop monotone sequences that converge to the solution quadratically when f admits a decomposition into a sum of two functions F and G with $F+\psi$ concave and $G+\phi$ convex for some concave function ψ and for some convex function ϕ. Theorem 2.1 extends Theorem 3.1 in [7] in the setup of [6]. However, we follow the direct approach discovered in [5] rather than the complicated multistage algorithmic method reported in [6]. We do not consider the corresponding results given in [7] that are generated by assuming various coupled upper and lower solutions with suitable extra assumption to avoid monotony.

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Clearly, based on our results and those of [7], it is not difficult to construct the proofs of other possible combinations.

2. Main Result

Consider the initial value problem (IVP)

$$x' = f(t,x), \quad x(0) = x_0, \quad t \in J = [0,T] \tag{2.1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$. Let $\alpha_0, \beta_0 \in C'[J, \mathbb{R}]$ such that $\alpha_0 \leq \beta_0$ on J. We define a set

$$\Omega = \{(t,x) : \alpha_0(t) \leq x \leq \beta_0(t), t \in J\}.$$

Theorem 2.1: *Assume that*

A_1) *$\alpha_0, \beta_0 \in C'[J, \mathbb{R}]$ such that $\alpha_0' \leq f(t,\alpha_0)$, $\beta_0' \geq f(t,\beta_0)$ and $\alpha_0 \leq \beta_0$ on J;*

A_2) *$f \in C[\Omega, \mathbb{R}]$, f admits a decomposition $f = F + G$ where $F_x, G_x, F_{x\,x}, G_{xx}$ exist and are continuous satisfying $F_{xx}(t,x) + \psi_{xx}(t,x) \leq 0$ and $G_{xx}(t,x) + \phi_{xx}(t,x) \geq 0$ on Ω, where $\phi, \psi \in C[\Omega, \mathbb{R}]$, $\phi_x(t,x)$, $\psi_x(t,x)$, $\phi_{xx}(t,x)$, $\psi_{xx}(t,x)$ exist, are continuous and $\psi_{xx}(t,x) < 0$, $\phi_{xx}(t,x) > 0$ on Ω.*

Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ which converge uniformly to the unique solution of (2.1) *and the convergence is quadratic.*

Proof: In view of (A_2), we see that

$$F(t,y) \leq H(t,x) - H_x(t,x)(x-y) - \psi(t,y)$$

and

$$G(t,x) \geq M(t,y) + M_x(t,y)(x-y) - \phi(t,x) \tag{2.2}$$

for $x \geq y$, $x, y \in \Omega$, where $H(t,x) = F(t,x) + \psi(t,x)$ and $M(t,x) = G(t,x) + \phi(t,x)$. Also, it is clear that whenever $\alpha_0(t) \leq x_2 \leq x_1 \leq \beta_0(t)$

$$f(t,x_1) - f(t,x_2) \leq L(x_1 - x_2), \text{ for some } L > 0. \tag{2.3}$$

Let α_1, β_1 be the solutions of linear IVPs

$$\alpha_1' = f(t,\alpha_0) + [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)](\alpha_1 - \alpha_0), \; \alpha_1(0) = x_0, \tag{2.4}$$

and

$$\beta_1' = f(t,\beta_0) + [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)](\beta_1 - \beta_0), \; \beta_1(0) = x_0, \tag{2.5}$$

where $\alpha_0(0) \leq x_0 \leq \beta_0(0)$.

We shall prove that $\alpha_0 \leq \alpha_1$ on J. To do this, let $p = \alpha_0 - \alpha_1$ so that $p(0) \leq 0$. Then

$$\begin{aligned} p' &= \alpha_0' - \alpha_1' \\ &\leq f(t,\alpha_0) - [f(t,\alpha_0) + \{M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)\}(\alpha_1 - \alpha_0)] \\ &= [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)]p. \end{aligned}$$

Since $p(0) \leq 0$, we get $\alpha_0(t) \leq \alpha_1(t)$ on J. Next we let $p = \alpha_1 - \beta_0$ and note that $p(0) \leq 0$. Now

$$\begin{aligned} p' &= \alpha_1' - \beta_0' \\ &\leq f(t,\alpha_0) + [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)](\alpha_1 - \alpha_0) - f(t,\beta_0). \end{aligned}$$

But since $\beta_0 \geq \alpha_0$, using (2.2) and (A_2), we have

$$F(t,\alpha_0) \leq H(t,\beta_0) - H_x(t,\beta_0)(\beta_0 - \alpha_0) - \psi(t,\alpha_0)$$

and

$$G(t,\beta_0) \geq M(t,\alpha_0) + M_x(t,\alpha_0)(\beta_0 - \alpha_0) - \phi(t,\beta_0),$$

or

$$F(t,\alpha_0) \leq F(t,\beta_0) - H_x(t,\beta_0)(\beta_0 - \alpha_0) + [\psi(t,\beta_0) - \psi(t,\alpha_0)]$$

and

$$G(t,\beta_0) \geq G(t,\alpha_0) + M_x(t,\alpha_0)(\beta_0 - \alpha_0) - [\phi(t,\beta_0) - \phi(t,\alpha_0)].$$

Now, by using the mean value theorem,

$$\psi(t,\beta_0) - \psi(t,\alpha_0) = \psi_x(t,\xi)(\beta_0 - \alpha_0)$$

and

$$\phi(t,\beta_0) - \phi(t,\alpha_0) = \phi_x(t,\delta)(\beta_0 - \alpha_0),$$

where $\alpha_0 < \xi,\ \delta < \beta_0$. Because $\psi_x(t,x)$ is decreasing in x and $\phi_x(t,x)$ is increasing in x, it follows that

$$\psi(t,\beta_0) - \psi(t,\alpha_0) \leq \psi_x(t,\alpha_0)(\beta_0 - \alpha_0)$$

and

$$\phi(t,\beta_0) - \phi(t,\alpha_0) \leq \phi_x(t,\beta_0)(\beta_0 - \alpha_0).$$

Thus we get

$$F(t,\alpha_0) \leq F(t,\beta_0) - [H_x(t,\beta_0) - \psi_x(t,\alpha_0)](\beta_0 - \alpha_0)$$

and

$$G(t,\beta_0) \geq G(t,\alpha_0) + [M_x(t,\alpha_0) - \phi_x(t,\beta_0)](\beta_0 - \alpha_0)$$

which in turn yields

$$p' \leq [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)]p.$$

Consequently, $p(t) \leq 0$ on J proving $\alpha_1 \leq \beta_0$ on J.

As a result, we have

$$\alpha_0(t) \le \alpha_1(t) \le \beta_0(t) \text{ on } J.$$

Similarly, we can show that $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$ on J. We need to prove that $\alpha_1(t) \le \beta_1(t)$ on J so that it follows

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t) \text{ on } J. \tag{2.6}$$

Since $\alpha_0 \le \alpha_1 \le \beta_0$, using (2.2) and (2.4), we see that

$$\begin{aligned}
\alpha_1' &= f(t,\alpha_0) + [M_x(t,\alpha_0) + H_x(t,\beta_0) - \psi_x(t,\alpha_0) - \phi_x(t,\beta_0)](\alpha_1 - \alpha_0) \\
&= [F(t,\alpha_0) + H_x(t,\beta_0)(\alpha_1 - \alpha_0)] + [G(t,\alpha_0) + M_x(t,\alpha_0)(\alpha_1 - \alpha_0)] \\
&\quad - \psi_x(t,\alpha_0)(\alpha_1 - \alpha_0) - \phi_x(t,\beta_0)(\alpha_1 - \alpha_0) \\
&\le [F(t,\alpha_0) + H_x(t,\alpha_1)(\alpha_1 - \alpha_0)] + [G(t,\alpha_0) + M_x(t,\alpha_0)(\alpha_1 - \alpha_0)] \\
&\quad - \psi_x(t,\alpha_0)(\alpha_1 - \alpha_0) - \phi_x(t,\beta_0)(\alpha_1 - \alpha_0) \\
&\le [F(t,\alpha_1) + \psi(t,\alpha_1) - \psi(t,\alpha_0)] + [G(t,\alpha_1) + \phi(t,\alpha_1) - \phi(t,\alpha_0)] \\
&\quad - \psi_x(t,\alpha_0)(\alpha_1 - \alpha_0) - \phi_x(t,\beta_0)(\alpha_1 - \alpha_0) \\
&\le f(t,\alpha_1) + \psi_x(t,\alpha_0)(\alpha_1 - \alpha_0) + \phi_x(t,\beta_0)(\alpha_1 - \alpha_0) \\
&\quad - \psi_x(t,\alpha_0)(\alpha_1 - \alpha_0) - \phi_x(t,\beta_0)(\alpha_1 - \alpha_0) \\
&= f(t,\alpha_1).
\end{aligned}$$

Here we have used the mean value theorem and the facts that $H_x(t,x)$, $\psi_x(t,x)$ are decreasing in x and $\phi_x(t,x)$ is increasing in x.

Similarly, we can prove that $\beta_1' \ge f(t,\beta_1)$ and therefore by Theorem 1.1.1 [3], it follows that $\alpha_1(t) \le \beta_1(t)$ on J which shows that (2.6) is valid.

Assume now that for some $k > 1$, $\alpha_k' \le f(t,\alpha_k)$, $\beta_k' \ge f(t,\beta_k)$ and $\alpha_k(t) \le \beta_k(t)$ on J, we shall show that

$$\alpha_k(t) \le \alpha_{k+1}(t) \le \beta_{k+1}(t) \le \beta_k(t) \text{ on } J, \tag{2.7}$$

where α_{k+1} and β_{k+1} are the solutions of linear IVP's

$$\begin{gathered}
\alpha_{k+1}' = f(t,\alpha_k) + [M_x(t,\alpha_k) + H_x(t,\beta_k) - \psi_x(t,\alpha_k) - \phi_x(t,\beta_k)](\alpha_{k+1} - \alpha_k), \\
\alpha_{k+1}(0) = x_0,
\end{gathered} \tag{2.8}$$

and

$$\beta_{k+1}' = f(t,\beta_k) + [M_x(t,\alpha_k) + H_x(t,\beta_k) - \psi_x(t,\alpha_k) - \phi_x(t,\beta_k)](\beta_{k+1} - \beta_k),$$

$$\beta_{k+1}(0) = x_0. \tag{2.9}$$

Hence setting $p = \alpha_k - \alpha_{k+1}$, it follows as before that

$$p' \leq [M_x(t,\alpha_k) + H_x(t,\beta_k) - \psi_x(t,\alpha_k) - \phi_x(t,\beta_k)]p \text{ on } J$$

and $p(0) = 0$ which again implies $p(t) \leq 0$ on J. On the other hand, letting $p = \alpha_{k+1} - \beta_k$ yields as before $p' \leq [M_x(t,\alpha_k) + H_x(t,\beta_k) - \psi_x(t,\alpha_k) - \phi_x(t,\beta_k)]p$. This proves that $p(t) \leq 0$, since $p(0) = 0$ and therefore we have $\alpha_k \leq \alpha_{k+1} \leq \beta_k$ on J. In a similar manner, we can prove that

$$\alpha_k \leq \beta_{k+1} \leq \beta_k \text{ on } J.$$

Now using (2.2), (2.8) and the fact $\beta_k \geq \alpha_{k+1} \geq \alpha_k$, we get

$$\begin{aligned}
\alpha'_{k+1} &= f(t,\alpha_k) + [M_x(t,\alpha_k) + H_x(t,\beta_k) - \psi_x(t,\alpha_k) - \phi_x(t,\beta_k)](\alpha_{k+1} - \alpha_k) \\
&= [F(t,\alpha_k) + H_x(t,\beta_k)(\alpha_{k+1} - \alpha_k)] + [G(t,\alpha_k) + M_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k)] \\
&- \psi_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k) - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k)] \\
&\leq [F(t,\alpha_k) + H_x(t,\alpha_{k+1})(\alpha_{k+1} - \alpha_k)] + [G(t,\alpha_k) + M_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k)] \\
&- \psi(t,\alpha_k)(\alpha_{k+1} - \alpha_k) - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) \\
&\leq [F(t,\alpha_{k+1}) + \psi(t,\alpha_{k+1}) - \psi(t,\alpha_k)] + [G(t,\alpha_{k+1}) + \phi(t,\alpha_{k+1}) - \phi(t,\alpha_k)] \\
&- \psi_x(t,\alpha_k)(\alpha_{k+1} - \alpha_k) - \phi_x(t,\beta_k)(\alpha_{k+1} - \alpha_k) \\
&= f(t,\alpha_{k+1}).
\end{aligned}$$

Here we have used the mean value theorem and the facts that $H_x(t,x)$, $\psi_x(t,x)$ are decreasing in x and $\phi_x(t,x)$ is increasing in x.

Similar arguments yield $\beta'_{k+1} \geq f(t,\beta_{k+1})$ and hence Theorem 1.1.1 [3] shows that $\alpha_{k+1}(t) \leq \beta_{k+1}(t)$ on J which proves (2.7).

Hence we obtain by induction

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \beta_n \leq \ldots \leq \beta_2 \leq \beta_1 \leq \beta_0 \text{ on } J.$$

Now using standard arguments, it is easy to show that the sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ converge uniformly and monotonically to the unique solution $x(t)$ of (2.1) on J. Finally, to prove quadratic convergence, we consider

$$p_{n+1}(t) = x(t) - \alpha_{n+1}(t) \geq 0, \quad q_{n+1}(t) = \beta_{n+1}(t) - x(t) \geq 0.$$

Note that $p_{n+1}(0) = 0 = q_{n+1}(0)$. We then have

$$\begin{aligned}
p'_{n+1} &= x' - \alpha'_{n+1} \\
&= f(t,x) - [f(t,\alpha_n) + \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)](\alpha_{n+1} - \alpha_n)]
\end{aligned}$$

$$= \{H(t,x) - H(t,\alpha_n)\} + \{M(t,x) - M(t,\alpha_n)\} - \{\psi(t,x) - \psi(t,\alpha_n)\}$$

$$- \{\phi(t,x) - \phi(t,\alpha_n)\} - \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}(\alpha_{n+1} - \alpha_n)]$$

$$\leq H_x(t,\alpha_n)(x - \alpha_n) + M_x(t,x)(x - \alpha_n) - \psi_x(t,x)(x - a_n) - \phi_x(t,\alpha_n)(x - \alpha_n)$$

$$- \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}(\alpha_{n+1} - x + x - \alpha_n)$$

$$= H_{xx}(t,\xi)(\alpha_n - \beta_n)(x - \alpha_n) + M_{xx}(t,\delta)(x - \alpha_n)^2 - \psi_{xx}(t,\gamma)(x - \alpha_n)^2$$

$$+ \phi_{xx}(t,\theta)(\beta_n - \alpha_n)(x - \alpha_n) + \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}p_{n+1}$$

$$= \{\phi_{xx}(t,\theta) - H_{xx}(t,\xi)\}p_n(q_n + p_n) + \{M_{xx}(t,\delta) - \psi_{xx}(t,\gamma)\}p_n^2$$

$$+ \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}p_{n+1}$$

$$\leq \{\phi_{xx}(t,\theta) - H_{xx}(t,\xi)\}p_n(q_n + p_n) + \{M_{xx}(t,\delta) - \psi_{xx}(t,\gamma)\}p_n^2$$

$$+ \{G_x(t,\alpha_n) + F_x(t,\beta_n)\}p_{n+1},$$

where $\alpha_n < \xi$, $\theta < \beta_n$ and $\alpha_n < \delta$, $\gamma < x$.

Hence, we obtain

$$\begin{aligned} p'_{n+1} &\leq (A + B + C)p_n(q_n + p_n) + (A + C + D)p_n^2 + (E + K)p_{n+1} \\ &\leq (A + B + C)(2p_n^2 + q_n^2) + (A + C + D)p_n^2 + (E + K)p_{n+1} \\ &= Qp_n^2 + Rq_n^2 + Sp_{n+1}, \end{aligned}$$

where on Ω,

$$|\phi_{xx}(t,x)| < A, \ |F_{xx}(t,x)| < B, \ |\psi_{xx}(t,x)| < C, \ |G_{xx}(t,x)| < D,$$

$$|G_x(t,x)| < E, \ |F_x(t,x)| < K, \ Q = 3A + 2B + 3C + D, \ R = A + B + C$$

and

$$S = E + K. \tag{2.10}$$

Now Gronwall's inequality implies

$$0 \leq p_{n+1}(t) \leq \int_0^t e^{S(t-s)}[Qp_n^2(s) + Rq_n^2(s)]ds \quad \text{on } J$$

which yield the desired result

$$\max_J |x(t) - \alpha_{n+1}(t)| \leq \frac{e^{ST}}{S}\left[Q \max_J |x(t) - \alpha_n(t)|^2 + R \max_J |\beta_n(t) - x(t)|^2\right].$$

Similarly,

$$q'_{n+1} = \beta'_{n+1} - x'$$

$$= f(t,\beta_n) + \left[M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\right](\beta_{n+1} - \beta_n) - f(t,x)$$

$$= \{H(t,\beta_n) - H(t,x)\} + \{M(t,\beta_n) - M(t,x)\} + \{\psi(t,x) - \psi(t,\beta_n)\}$$

$$+ \{\phi(t,x) - \phi(t,\beta_n)\} + \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}(\beta_{n+1} - \beta_n)$$

$$\leq H_x(t,x)(\beta_n - x) + M_x(t,\beta_n)(\beta_n - x) - \psi_x(t,\beta_n)(\beta_n - x) - \phi_x(t,x)(\beta_n - x)$$

$$+ \{M_x(t,\alpha_n) + H_x(t,\beta_n) - \psi_x(t,\alpha_n) - \phi_x(t,\beta_n)\}(\beta_{n+1} - x - x - \beta_n)$$

$$\leq \{\phi_{xx}(t,\theta) - H_{xx}(t,\xi)\}(\beta_n - x)^2 + \{M_{xx}(t,\delta) - \psi_{xx}(t,\gamma)\}(\beta_n - \alpha_n)(\beta_n - x)$$

$$+ \{G_x(t,\alpha_n) + F_x(t,\beta_n)\}(\beta_{n+1} - x)$$

$$= \{\phi_{xx}(t,\theta) - H_{xx}(t,\xi)\}q_n^2 + \{M_{xx}(t,\delta) - \psi_{xx}(t,\gamma)\}(q_n + p_n)q_n$$

$$+ \{G_x(t,\alpha_n) + F_x(t,\beta_n)\}q_{n+1}$$

where $\alpha_n < \xi$, $\theta < \beta_n$ and $\alpha_n < \delta$, $\gamma < x$.

Hence, we obtain

$$q'_{n+1} \leq (A+B+C)q_n^2(A+C+D)q_n(q_n + p_n) + (E+K)q_{n+1}$$

$$\leq (A+B+C)q_n^2 + (A+C+D)(2q_n^2 + p_n^2) + (E+K)q_{n+1}$$

$$= Q^* q_n^2 + R^* p_n^2 + Sq_{n+1}$$

where the constants A, B, C, D, E, K and S are as in (2.10) and $Q^* = 3A + B + 3C + 2D$, $R^* = A + C + D$.

An application of Gronwall's inequality yields

$$0 \leq q_{n+1}(t) \leq \int_0^t e^{S(t-s)}\left[Q^* q_n^2(s) + R^* p_n^2(s)\right]ds \quad \text{on } J$$

and hence

$$\max_J \mid \beta_{n+1}(t) - x(t) \mid \; \leq \frac{e^{ST}}{S}\left[Q^* \max_J \mid \beta_n(t) - x(t) \mid^2 + R^* \max_J \mid x(t) - \alpha_n(t) \mid^2\right].$$

This completes the proof of the theorem.

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