Journal of Applied Mathematics and Stochastic Analysis
9, Number 1, 1996, 33-42

SUFFICIENT CONDITIONS FOR OSCILLATIONS OF ALL SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

D.D. BAINOV
Medical University
P.O. Box 45
Sofia 1504 Bulgaria

M.B. DIMITROVA
Technical University
Sliven 8800, Bulgaria

(Received January, 1994; Revised September, 1995)

ABSTRACT

Sufficient conditions are found for oscillation of all solutions of impulsive differential equation with deviating argument.

Key words: Oscillation, Impulsive Differential Equations.

AMS (MOS) subject classifications: 34A37.

1. Introduction

The impulsive differential equations with deviating argument are adequate mathematical models of numerous processes and phenomena in physics, biology and electrical engineering. In spite of wide possibilities for their application, the theory of these equations is developing rather slowly because of considerable difficulties of technical and theoretical character related to their study.

In the recent twenty years, the number of investigations devoted to the oscillatory and non-oscillatory behavior of the solutions of functional differential equations has considerably increased. The large part of the works on this subject published by 1977 is presented in [4]. In monographs [2] and [3], published in 1987 and 1991, respectively, the oscillatory and asymptotic properties of the solutions of various classes of functional differential equations were systematically studied. A pioneering work devoted to the investigation of the oscillatory properties of the solutions of impulsive differential equations with deviating argument was rendered by Gopalsamy and Zhang [1].

In the present paper, sufficient conditions are found for oscillation of all solutions of the equation

$$\begin{aligned} x'(t) - p(t)x(t+h) = 0, \quad t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = b_k x(\tau_k - 0) = b_k x(\tau_k), \end{aligned} \tag{1}$$

where the function $p = p(t)$ is nonnegative and continuous, and $\tau_k (k \in \mathbb{N})$ are fixed moments of impulsive effect.

2. Preliminary Notes

Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$, $p \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$, let h be a positive constant, $\{\tau_k\}_{k=1}^{\infty}$ be a monotone increasing, unbounded sequence of real numbers, and $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers.

Consider the impulsive differential equation with a deviating argument (1) under the condition

$$x(t) = \varphi(t), \quad t \in [0, h), \tag{2}$$

where $\varphi \in C^1([0, h), \mathbb{R}_+)$.

Introduce the following conditions:

H1: $0 < h < \tau_1$.

H2: There exists a positive constant $T > h$ such that $\tau_{k+1} - \tau_k \geq T$, $k \in \mathbb{N}$.

H3: There exists a constant $M > 0$ such that for any $k \in \mathbb{N}$ the inequality $0 \leq M \leq b_k$ is valid.

We construct the sequence

$$\{t_k, k \in \mathbb{N}\} = \{\tau_k, k \in \mathbb{N}\} \cup \{\tau_k - h, k \in \mathbb{N}\}$$

so that $t_k < t_{k+1}, k \in \mathbb{N}$.

Definition 1: By a *solution* of equation (1) under condition (2) we mean any function $x: [0, \infty) \to \mathbb{R}$ for which the following holds true:

1. If $0 \leq t \leq t_1 = \tau_1 - h$, then the function x coincides with the solution of the problem

$$x'(t) - p(t)x(t+h) = 0.$$

2. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_k, k \in \mathbb{N}\} \backslash \{\tau_k - h, k \in \mathbb{N}\}$, then the function x coincides with the solution of the problem

$$x'(t) - p(t)x(t+h) = 0$$

$$x(t_k + 0) = (1 + b_{k_i})x(t_k),$$

where k_i is determined from the equality $\tau_{k_i} = t_k$.

3. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_k - h, k \in \mathbb{N}\} \backslash \{\tau_k, k \in \mathbb{N}\}$, then the function x coincides with the solution of the problem

$$x'(t) - p(t)x(t+h+0) = 0$$

$$x(t_k + 0) = x(t_k).$$

4. If $t_k < t \leq t_{k+1}$, $t_k \in \{\tau_k, k \in \mathbb{N}\} \cap \{\tau_k - h, k \in \mathbb{N}\}$, then the function x coincides with the solution of the problem

$$x'(t) - p(t)x(t+h+0) = 0$$

$$x(t_k + 0) = (1 + b_{k_i})x(t_k),$$

where k_i is determined from the equality $\tau_{k_i} = t_k$.

Definition 2: A nonzero solution x of equation (1) is said to be *nonoscillating* if there exists $t_0 \geq 0$ such that $x(t)$ is of constant sign for $t \geq t_0$. Otherwise, the solution x is said to *oscillate.*

3. Main Results

Theorem 1: *Let the following conditions hold:*

1. *Conditions* H1 *and* H2 *are met.*
2. $\limsup_{i \to \infty} (1 + b_i) \int_{\tau_i - h}^{\tau_i} p(s)ds > 1.$

Then all solutions of equation (1) *oscillate.*

Proof: Let a nonoscillating solution x of equation (1) exist. Without loss of generality we assume that $x(t) > 0$ for $t \geq t_0 > 0$. Then $x(t+h) > 0$ also for $t \geq t_0$.

From (1), it follows that x is a nonincreasing function in $(t_0, \tau_k) \cup \left[\cup_{i=k}^{\infty}(\tau_i, \tau_{i+1})\right]$, where $\tau_k > t_0 > \tau_{k-1}$.

Integrate (1) from $\tau_i - h$ to τ_i $(i \geq k+1)$ and obtain

$$x(\tau_i) - x(\tau_i - h) = \int_{\tau_i - h}^{\tau_i} p(s)x(s+h)ds,$$

$$x(\tau_i) - x(\tau_i - h) \geq x(\tau_i + 0) \int_{\tau_i - h}^{\tau_i} p(s)ds. \tag{3}$$

Since

$$x(\tau_i + 0) = (1 + b_i)x(\tau_i - 0) = (1 + b_i)x(\tau_i) \tag{4}$$

then (3) and (4) yield the inequality

$$x(\tau_i - h) + x(\tau_i)\left[(1 + b_i)\int_{\tau_i - h}^{\tau_i} p(s)ds - 1\right] \leq 0. \tag{5}$$

Inequality (5) is valid only if

$$\limsup_{i \to \infty} (1 + b_i) \int_{\tau_i - h}^{\tau_i} p(s)ds \leq 1,$$

which contradicts condition 2 of Theorem 1. □

Theorem 2: *Let the following conditions hold:*

1. *Conditions* H1-H3 *are met.*

2. $\liminf_{t\to\infty} \int_t^{t+h} p(s)ds > \frac{1}{e(1+M)}$.

Then all solutions of equation (1) *oscillate.*

Proof: Let a nonoscillating solution x of equation (1) exist. Without loss of generality we assume that $x(t) > 0$ for $t \geq t_0 > 0$. Then $x(t+h) > 0$ also for $t \geq t_0$.

From (1) it follows that x is a nondecreasing function in $(t_0, \tau_k) \cup \left[\cup_{i=k}^{\infty}(\tau_i, \tau_{i+1})\right]$, $\tau_{k-1} < t_0 < \tau_k$.

Define the function $w(t) = \frac{x(t+h)}{x(t)}$, $t \geq t_0$, and let $\tau_i \in (t, t+h)$, $t \geq t_0$. Then

$$x(t) \leq x(\tau_i) = \frac{x(\tau_i + 0)}{1+b_i} \leq \frac{x(t+h)}{1+b_i} \leq \frac{x(t+h)}{1+M}.$$

From the last inequality it follows that $w(t) \geq 1 + M$ for $t \geq t_0$.

We shall prove that the function w is bounded from above for $t \geq t_0$.

1. Let $\tau_i \in (t, t+\frac{h}{2})$, $t \geq t_0$. Integrate (1) from t to $t+\frac{h}{2}$ and obtain that

$$x(\tau_i) - x(t) + x(t+\tfrac{h}{2}) - x(\tau_i + 0) = \int_t^{t+h/2} p(s)x(s+h)ds. \tag{6}$$

Since

$$x(\tau_i + 0) = (1+b_i)x(\tau_i) \tag{7}$$

then from (6) and (7) it follows that

$$x(t+\tfrac{h}{2}) = x(t) + \int_t^{t+h/2} p(s)x(s+h)ds + b_i x(\tau_i). \tag{8}$$

From (8) we obtain that

$$x(t+\tfrac{h}{2}) \geq \inf_{s\in[t,t+h/2]} x(s+h) \int_t^{t+h/2} p(s)ds = \inf_{s\in[t+h,t+3h/2]} x(s) \int_t^{t+h/2} p(s)ds. \tag{9}$$

If in the interval $[t+h, t+\frac{3h}{2}]$ there is no point of jump, then

$$\inf_{s\in[t+h,t+3h/2]} x(s) = x(t+h).$$

If in the interval $[t+h, t+\frac{3h}{2}]$ there is a point of jump, τ_{i+1}, then from the inequalities

$$x(t+h) \leq x(\tau_{i+1}) = \frac{x(\tau_{i+1}+0)}{1+b_i} \leq \frac{x(t+\frac{3h}{2})}{1+M}$$

it follows that

$$\inf_{s\in[t+h,t+3h/2]} x(s) = x(t+h).$$

The last inequality and (9) lead to

$$x(t+\tfrac{h}{2}) \geq x(t+h) \int_t^{t+h/2} p(s)ds. \tag{10}$$

Integrating (1) from $t+\frac{h}{2}$ to $t+h$, we get

$$x(t+h)-x(t+\tfrac{h}{2})\geq x(t+\tfrac{3h}{2})\int\limits_{t+h/2}^{t+h}p(s)ds. \tag{11}$$

From (10) and (11) it follows that

$$\frac{x(t+\frac{3h}{2})}{x(t+\frac{h}{2})}\leq\frac{1}{\int\limits_{t}^{t+h/2}p(s)ds\int\limits_{t+h/2}^{t+h}p(s)ds}\leq\text{const.}$$

Thus we proved that the function w is bounded from above.

2. Let $\tau_i\in(t+\frac{h}{2},t+h)$. The boundedness from above of the function w can be proved analogously.

We divide (1) by $x(t)>0$, $t\geq t_0$, integrate from t to $t+h$ and obtain

$$\int\limits_{t}^{\tau_i}\frac{x'(s)}{x(s)}ds+\int\limits_{\tau_i}^{t+h}\frac{x'(s)}{x(s)}ds=\int\limits_{t}^{t+h}p(s)\frac{x(s+h)}{x(s)}ds,$$

$$\ln\left[\frac{1}{1+b_i}w(t)\right]=\int\limits_{t}^{t+h}p(s)w(s)ds. \tag{12}$$

From (12) it follows that

$$\ln\left[\frac{1}{1+M}w(t)\right]\geq\liminf_{t\to\infty}w(t)\int\limits_{t}^{t+h}p(s)ds. \tag{13}$$

Denote $w_0=\liminf_{t\to\infty}w(t)$, $0<w_0<\infty$. Then from (13) we obtain

$$\liminf_{t\to\infty}\int\limits_{t}^{t+h}p(s)ds\leq\frac{\ln[(1+M)^{-1}w_0]}{w_0}\leq\frac{1}{e(1+M)}.$$

The last inequality contradicts condition 2 of Theorem 2. □

Corollary 1: *Let the conditions of Theorem 2 hold. Then:*

1. *The inequality*

$$\begin{aligned}&x'(t)-p(t)x(t+h)\geq 0,\quad t\neq\tau_k,\\&\Delta x(\tau_k)=b_kx(\tau_k)\end{aligned} \tag{14}$$

has no positive solutions.

2. *The inequality*

$$\begin{aligned}&x'(t)-p(t)x(t+h)\leq 0,\quad t\neq\tau_k,\\&\Delta x(\tau_k)=b_kx(\tau_k)\end{aligned} \tag{15}$$

has no negative solutions.

Proof of 2: Let inequality (15) have a negative solution $x(t)$ for $t \geq t_0$ for some $t_0 \geq 0$. From (15) it follows that

$$x'(t) \leq p(t)x(t+h) \leq 0, \tag{16}$$

i.e., x is a nonincreasing function in $(t_0, \tau_k) \cup \left[\cup_{i=k}^{\infty}(\tau_i, \tau_{i+1})\right]$.

From (16) we obtain that

$$\frac{x'(t)}{x(t)} \geq p(t)\frac{x(t+h)}{x(t)}.$$

Analogously to the proof of Theorem 2 we are led to a contradiction with condition 2 of Theorem 2. □

Theorem 3: *Let the following conditions hold:*

1. *Conditions* H1-H3 *are met.*
2. $\liminf_{k\to\infty} \int_{\tau_k - h}^{\tau_k} p(s)ds > \frac{1}{1+M}.$

Then all solutions of equation (1) *oscillate.*

Proof: From (3) analogously to the proof of Theorem 1 we obtain

$$\liminf_{k\to\infty} \int_{\tau_k - h}^{\tau_k} p(s)ds \leq \frac{x(\tau_k)}{x(\tau_k + 0)} = \frac{1}{1+b_k} \leq \frac{1}{1+M}.$$

The last inequality contradicts condition 2 of Theorem 3. □

Corollary 2: *Let the conditions of Theorem* 3 *hold. Then:*

1. *Inequality* (14) *has no positive solutions.*
2. *Inequality* (15) *has no negative solutions.*

The proof of Corollary 2 is carried out analogously to the proof of Corollary 1.

Theorem 4: *Let the following conditions hold:*

1. *Conditions* H1- H3 *are satisfied.*
2. *In each interval of length h there are k points of jump* ($k \in \mathbb{N}$).
3. $\liminf_{t\to\infty} \int_{t}^{t+h} p(s)ds > \frac{1}{e(1+M)^k}.$

Then all solutions of equation 1 *oscillate.*

Proof: Let a nonoscillating solution x of equation (1) exist. Without loss of generality we assume that $x(t) > 0$ for $t \geq t_0 > 0$. Then $x(t+h) > 0$ also for $t \geq 0$.

For any fixed t ($t \geq t_0$) in the interval $(t, t+h)$, let

$$t < \tau_s^{(1)} < \tau_s^{(2)} < \ldots < \tau_s^{(k)} < t+h$$

be k points of jump with respective constants $b_s^{(1)}, b_s^{(2)}, \ldots, b_s^{(k)}$.

Since $x(\tau_s) = \frac{x(\tau_s + 0)}{1+b_s}$, $s \in \mathbb{N}$ and x is a nondecreasing function in $(t, \tau_s^{(1)}) \cup$

$\left[\cup_{i=1}^{k-1}(\tau_s^{(i)}, \tau_s^{(i+1)})\right] \cup (\tau_s^{(k)}, t+h)$, then

$$x(t) \le x(\tau_s^{(1)}) = \frac{x(\tau_s^{(1)}+0)}{1+b_s^{(1)}} \le \dots \le \frac{x(t+h)}{\prod_{i=1}^{k}(1+b_s^{(i)})}. \tag{17}$$

From (17) it follows that

$$\frac{x(t+h)}{x(t)} \ge (1+M)^k.$$

Introduce the function $w(t) = \frac{x(t+h)}{x(t)}$, $t \ge t_0$.

We shall prove that the function w is bounded from above for $t \ge t_0$.

Let the interval $[t, t+\frac{h}{2}]$ contain l points of jumps, and let the interval $[t+\frac{h}{2}, t+h]$ contain r points of jumps ($l+r=k$).

Integrate (1) from t to $t+\frac{h}{2}$ and obtain that

$$x(t+\tfrac{h}{2}) - x(t) = \int_t^{t+h/2} p(s)x(s+h)ds + \sum_{i=1}^{l} b_s^{(i)} x(\tau_s^{(i)}). \tag{18}$$

From (18) it follows that

$$x(t+\tfrac{h}{2}) \ge x(t+h)\int_t^{t+h/2} p(s)ds. \tag{19}$$

Integrate (1) from $t+\frac{h}{2}$ to $t+h$ and obtain that

$$x(t+h) - x(t+\tfrac{h}{2}) = \int_{t+h/2}^{t+h} p(s)x(s+h)ds + \sum_{i=l+1}^{k} b_s^{(i)} x(\tau_s^{(i)}). \tag{20}$$

From (20) it follows that

$$x(t+h) \ge x(t+\tfrac{3h}{2})\int_t^{t+h} p(s)ds. \tag{21}$$

From (19) and (21) we obtain that

$$\frac{x(t+\frac{3h}{2})}{x(t+\frac{h}{2})} \le \frac{1}{\int_t^{t+h/2} p(s)ds \int_{t+h/2}^{t+h} p(s)ds} \le \text{const}.$$

From the last inequality it follows that the function w is bounded from above for $t \ge t_0$.

Denote $w_0 = \liminf_{t\to\infty} w(t)$, $0 < w_0 < \infty$.

Integrate

$$\frac{x'(t)}{x(t)} - p(t)\frac{x(t+h)}{x(t)} = 0$$

from t to $t+h$, $t \ge t_0$, and obtain

$$\ln\frac{x(t+h)}{x(t)} + \sum_{i=1}^{k}\Big[\ln x(\tau_s^{(i)}) - \ln x(\tau_s^{(i)}+0)\Big] = \int_t^{t+h} p(s)\frac{x(s+h)}{x(s)}ds,$$

$$\ln\left[\frac{w(t)}{\prod_{i=1}^{k}(1+b_s^{(i)})}\right]=\int_t^{t+h} p(s)w(s)ds. \tag{22}$$

Assertion (22) leads to the inequality

$$\ln\left[\frac{w(t)}{(1+M)^k}\right]\geq \liminf_{t\to\infty} w(t)\int_t^{t+h} p(s)ds.$$

From the last inequality we obtain that

$$\liminf_{t\to\infty}\int_t^{t+h} p(s)ds\leq\frac{\ln\left[(1+M)^{-k}w_0\right]}{w_0}\leq\frac{1}{e(1+M)^k}$$

which contradicts condition 3 of Theorem 4. □

Corollary 3: *Let the conditions of Theorem* 4 *hold. Then:*

1. *Inequality* (14) *has no positive solutions.*
2. *Inequality* (15) *has no negative solutions.*

The proof of Corollary 3 can be rendered analogously to the proof of Corollary 1 and Theorem 4.

Consider the nonhomogeneous impulsive differential equation with deviating argument:

$$\begin{gathered} x'(t)-p(t)x(t+h)=q(t),\ \ t\neq\tau_k, \\ \Delta x(\tau_k)=b_k x(\tau_k). \end{gathered} \tag{23}$$

Introduce the following condition:

H4: $q\in C(\mathbb{R}_+,\mathbb{R}_+)$.

Theorem 5: *Let the following conditions hold:*

1. *Conditions* H1-H4 *are met.*
2. $\liminf_{k\to\infty}\int_{\tau_k-h}^{\tau_k} p(s)ds>\frac{1}{1+M}$.

Then equation (23) *has no positive solutions.*

Proof: Let $x(t)>0$ be a solution of (23) for $t\geq t_0\geq 0$. Integrate (23) from τ_k-h to τ_k $(\tau_k>t_0+h)$ and obtain

$$x(\tau_k)-x(\tau_k-h)=\int_{\tau_k-h}^{\tau_k} p(s)x(s+h)ds$$

$$+\sum_{\tau_k-h\leq\tau_k^{(s)}\leq\tau_k} b_k^{(s)}x(\tau_k^{(s)})+\int_{\tau_k-h}^{\tau_k} q(s)ds. \tag{24}$$

From (24) it follows that

$$x(\tau_k)\geq x(\tau_k+0)\int_{\tau_k-h}^{\tau_k} p(s)ds.$$

From the last inequality we obtain that

$$\int_{\tau_k - h}^{\tau_k} p(s)ds \leq \frac{x(\tau_k)}{x(\tau_k + 0)} \leq \frac{1}{1+M}$$

which contradicts condition 2 of Theorem 3. □

Introduce the following conditions:

H5: $q \in C([0,\infty),\mathbb{R})$.

H6: There exists a function $v \in (C^1(\mathbb{R}_+,\mathbb{R})$ such that $v'(t) = q(t)$, $t \geq 0$.

H7: There exist constants q_1 and q_2 and two sequences $\{t_i'\}_1^\infty \subset \mathbb{R}_+$ and $\{t_i''\}_1^\infty \subset \mathbb{R}_+$ with $\lim_{i\to\infty} t_i' = \lim_{i\to\infty} t_i'' = \infty$ and $v(t_i') = q_1$, $v(t_i'') = q_2$, $q_1 \leq v(t) \leq q_2$.

Theorem 6: *Let the following conditions hold:*

1. *Conditions* H1, H2, H5-H7 *are satisfied.*
2. $b_k \geq 0$, $k \in \mathbb{N}$.
3. $\limsup_{k\to\infty} \int_{\tau_k}^{\tau_k + h} p(s)ds > 1$.

Then all solutions of equation (23) *oscillate.*

Proof: Let $x(t) > 0$ be a solution of equation (23) for $t \geq t_0 > 0$.

Set

$$z(t) = x(t) - v(t) + q_1.$$

Then from (23) we obtain that

$$\begin{aligned} z'(t) &\geq p(t)z(t+h), \\ \Delta z(\tau_k) &= b_k z(\tau_k) + A_k, \end{aligned} \tag{25}$$

where $A_k = b_k v(\tau_k) - b_k q_1 \geq 0$.

1. Let the inequality (25) have a positive solution $z(t)$ for $t \geq t_1 \geq t_0$. Integrate (25) from τ_k to $\tau_k + h$, $\tau_k \geq t_1$ and obtain that

$$z(\tau_k + h) - z(\tau_k + 0) \geq z(\tau_k + h)\int_{\tau_k}^{\tau_k + h} p(s)ds,$$

$$z(\tau_k + h)\left[\int_{\tau_k}^{\tau_k + h} p(s)ds - 1\right] \leq 0.$$

The last inequality contradicts condition 3 of Theorem 6.

2. Let $z(t) < 0$ for $t \geq t_1$ be a solution of the inequality (25). Then,

$$z(t_i') = x(t_i') - v(t_i') + q_1 = x(t_i') > 0, \quad t_i' \geq t_1.$$ □

Theorem 7: *Let the following conditions hold:*

1. *Conditions* H1-H3, H5-H7 *are met.*
2. $\liminf_{k \to \infty} \int_{\tau_k - h}^{\tau_k} p(s)ds > \frac{1}{1+M}$.

Then all solutions of equation (23) *oscillate.*

Proof: Analogously to the proof of Theorem 6 we obtain (25).

Let $z(t) > 0$ be a solution of (25) for $t \geq t_1 \geq t_0$. Integrate (25) from $\tau_k - h$ to τ_k $(\tau_k > t_1 + h)$ and obtain

$$z(\tau_k) - z(\tau_k - h) \geq z(\tau_k + 0) \int_{\tau_k - h}^{\tau_k} p(s)ds,$$

$$z(\tau_k) \geq \left[(1 + b_k)z(\tau_k) + A_k\right] \int_{\tau_k - h}^{\tau_k} p(s)ds,$$

$$z(\tau_k) \geq (1 + b_k)z(\tau_k) \int_{\tau_k - h}^{\tau_k} p(s)ds.$$

From the last inequality it follows that

$$\int_{\tau_k - h}^{\tau_k} p(s)ds \leq \frac{1}{1 + b_k} \leq \frac{1}{1 + M},$$

which contradicts condition 2 of Theorem 7.

The case when $z(t) < 0$ is considered analogously. □

Acknowledgements

The present investigation was supported by the Bulgarian Ministry of Education, Science and Technologies under Grant MM--422.

References

[1] Gopalsamy, K. and Zhang, B.G., On delay differential equations with impulses, *J. Math. Anal. Appl.* **139**:1 (1989), 110-122.

[2] Györi, I. and Ladas, G., *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford 1991.

[3] Ladde, G.S., Lakshmikantham, V. and Zhang, B.G., *Oscillation Theory of Differential Equations with Deviating Arguments*, Pure and Applied Mathematics **110**, Marcel Dekker, 1987.

[4] Shevelo, V.N., *Oscillations of Solutions of Differential Equations with Deviating Arguments*, Naukova Dumka, Kiev 1978 (in Russian).