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QUANTITATIVE RESULTS FOR PERTURBED STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

The paper is devoted to Itô type stochastic differential equations (SDE's) with "small" perturbations. Our goal is to present strong results showing how "close" are the $2m$-order moments of the solutions of the perturbed SDE's and the unperturbed SDE.

Key words: Stochastic Differential Equation, "Small" Perturbations, Moments of Order $2m$, Approximations of the Solutions.

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1. Introduction. Statement of the Problem

The object of this study are stochastic differential equations (SDE's) of the following type

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dw_s, \quad t \geq t_0 \geq 0. \tag{1}$$

Here $w = (w_t, t \geq 0)$ is a standard Wiener process defined on a given probability space $(\Omega, \mathfrak{F}, P)$, $a(t,x)$ and $b(t,x)$, $t \geq t_0$, $x \in \mathbb{R}^1$, are measurable real-valued functions, and X_{t_0} is a random variable (r.v.) independent of w with $E\{X_{t_0}^2\} < \infty$. Finally, $\int_{t_0}^{t} b(\cdot)dw_s$ is the well-known stochastic integral in Itô sense.

Under general conditions, the SDE (1) has a unique (strong) solution $X = (X_t, t \geq t_0)$, which is a diffusion Markov process with a drift coefficient a and a diffusion coefficient b^2. Let us adopt the following classical conditions: For some constants $K_1 > 0$ and $K_2 > 0$ and all $t \geq t_0$, $x, y \in \mathbb{R}^1$ we have

$$\begin{cases} \mid a(t,x) - a(t,y) \mid + \mid b(t,x) - b(t,y) \mid \leq K_1 \mid x - y \mid \\ a^2(t,x) + b^2(t,x) \leq K_2^2(1 + x^2). \end{cases} \tag{2}$$

Notice that standard references in the area of SDE's are the books by Gihman and Skorohod [4], Arnold [1], Liptser and Shiryaev [5] and Gard [3]. We shall systematically use basic facts from these sources without mentioning.

Now, along with (1), we consider another SDE:

$$X_t^\epsilon = X_{t_0}^{\epsilon_0} + \int_{t_0}^{t} \widetilde{a}\,(s, X_s^\epsilon, \epsilon_1)ds + \int_{0}^{t} \widetilde{b}\,(s, X_s^\epsilon, \epsilon_2)dw_s, \quad t \geq t_0. \tag{3}$$

Here ϵ_0, ϵ_1, ϵ_2 are "small" positive parameters, e.g., each is in the interval $(0,1]$ and ϵ stands for $(\epsilon_0, \epsilon_1, \epsilon_2)$; $X_{t_0}^{\epsilon_0}$, $\widetilde{a}$, and $\widetilde{b}$ are as X_{t_0}, a, and b above; w is the same Wiener process. Thus the SDE (3) has also a unique (strong) solution $X^\epsilon = (X_t^\epsilon,\ t \geq t_0)$.

Our goal is to compare the solutions X^ϵ and X of (3) and (1) in the case when their coefficients are related as follows:

$$\begin{cases} \widetilde{a}\,(t, x, \epsilon_1) = a(t,x) + \alpha(t, x, \epsilon_1) \\ \widetilde{b}\,(t, x, \epsilon_2) = b(t,x) + \beta(t, x, \epsilon_2). \end{cases} \tag{4}$$

The terms $\alpha(\cdot)$ and $\beta(\cdot)$ are called perturbations of the coefficients $a(\cdot)$ and $b(\cdot)$ which explains why (3) is called a "perturbed SDE" while the name "unperturbed SDE" is kept for (1).

Let us suppose that for some fixed natural number m we have $E\{(X_{t_0}^{\epsilon_0})^{2m}\} < \infty$, $E\{(X_{t_0})^{2m}\} < \infty$ and let for all $t \geq t_0$,

$$\begin{cases} E\{\,|\,X_{t_0}^{\epsilon_0} - X_{t_0}\,|^{2m}\} \leq \delta_0(\epsilon_0) \\ \sup_x |\,\alpha(t, x, \epsilon_1)\,| \leq \delta_1(t, \epsilon_1) \\ \sup_x |\,\beta(t, x, \epsilon_2)\,| \leq \delta_2(t, \epsilon_2). \end{cases} \tag{5}$$

Thus we can expect that if the quantities $\delta_0(\epsilon_0)$, $\delta_1(t, \epsilon_1)$, $\delta_2(t, \epsilon_2)$ are small for small ϵ_0, ϵ_1, ϵ_2, then the process X^ϵ is close to X. Recall that X^ϵ and X are $2m$-integrable in the sense that for each $t \geq t_0$ the r.v.'s $|\,X_t^\epsilon\,|^{2m}$ and $|\,X_t\,|^{2m}$ are P-integrable. Thus, the following quantity

$$\Delta_t^\epsilon = E\{\,|\,X_t^\epsilon - X_t\,|^{2m}\}$$

is well-defined and we are interested in conditions guaranteeing that $\Delta_t^\epsilon \to 0$ as $\epsilon \to 0$. This means that for a fixed t, the $2m$-order moments of X_t^ϵ and X_t are close. Furthermore, we describe a few cases when $\Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ on intervals whose length tend to infinity.

2. Preliminary Result

Let us prove first a result which is of independent interest. This result plays a key role for the statements in the next section.

Theorem A: *Suppose conditions* (2) *and* (5) *are satisfied for the SDE's* (1) *and* (3). *Then for any* $t \geq t_0$,

$$\Delta_t^\epsilon \leq \Bigg\{ \delta_0^{1/m}(\epsilon_0) \exp\left[M(t - t_0) + 2\int_{t_0}^{t} \delta_1(s, \epsilon_1)ds \right] + \int_{t_0}^{t} [2\delta_1(s, \epsilon_1) + (2m-1)\delta_2^2(s, \epsilon_2)] \exp\left[M(t-s) + 2\int_{s}^{t} \delta_1(\tau, \epsilon_1)d\tau \right] ds \Bigg\}^m, \tag{6}$$

where $M = 2K + 2(2m-1)K^2$ $(K = \max[K_1, K_2]$, *and* K_1 *and* K_2 *are the constants from (2)).*

Proof: If we write explicitly the difference $Z_t^\epsilon = X_t^\epsilon - X_t$, $t \geq t_0$, apply the Itô formula to $(Z_t^\epsilon)^{2m}$ and take expectations, we find that

$$\Delta_t^\epsilon = E\{(Z_t^\epsilon)^{2m}\} = E\{(Z_{t_0}^{\epsilon_0})^{2m}\} + 2mE\{I_1(t)\} + m(2m-1)E\{I_2(t)\} + 2mE\{I_3(t)\}, \tag{7}$$

where

$$I_1(t) = \int_{t_0}^{t} [\widetilde{a}\,(s, X_s^\epsilon, \epsilon_1) - a(s, X_s)](Z_s^\epsilon)^{2m-1} ds,$$

$$I_2(t) = \int_{t_0}^{t} \left[\widetilde{b}\,(s, X_s^\epsilon, \epsilon_2) - b(s, X_s)\right]^2 (Z_s^\epsilon)^{2m-2} ds,$$

$$I_3(t) = \int_{t_0}^{t} \left[\widetilde{b}\,(s, X_s^\epsilon, \epsilon_2) - b(s, X_s)\right](Z_s^\epsilon)^{2m-1} dw_s.$$

The existence of the $2m$-order moments of X_t^ϵ and X_t, $t \geq t_0$ and the conditions on $\widetilde{b}\,(\cdot)$ and $b(\cdot)$ allow us to use one of the properties of the stochastic integrals, thus concluding that

$$E\{I_3(t)\} = 0.$$

Let us estimate $I_1(t)$ and $I_2(t)$. In view of (2), we see that

$$I_1(t) \leq K \int_{t_0}^{t} |X_s^\epsilon - X_s|\,|Z_s^\epsilon|^{2m-1} ds + \int_{t_0}^{t} \delta_1(s, \epsilon_1)\,|Z_s^\epsilon|^{2m-1} ds.$$

Now we take the expectations of both sides of the last inequality and applying Hölder's inequality to $E\{|Z_s^\epsilon|^{2m-1}\}$ (see Shiryaev [7]) we find that

$$E\{I_1(t)\} \leq \int_{t_0}^{t} \Delta_s^\epsilon ds + \int_{t_0}^{t} \delta_1(s, \epsilon_1)(\Delta_s^\epsilon)^{(2m-1)/(2m)} ds. \tag{8}$$

Similar arguments imply that

$$E\{I_2(t)\} \leq 2K^2 \int_{t_0}^{t} \Delta_s^\epsilon ds + \int_{t_0}^{t} \delta_2^2(s, \epsilon_2)(\Delta_s^\epsilon)^{(m-1)/m} ds. \tag{9}$$

Therefore, from (7), (8) and (9), we get

$$\begin{aligned}\Delta_t^\epsilon \leq \delta_0(\epsilon_0) + mM \int_{t_0}^{t} \Delta_s^\epsilon ds + 2m \int_{t_0}^{t} \delta_1(s, \epsilon_1)(\Delta_s^\epsilon)^{(2m-1)/(2m)} ds \\ + m(2m-1) \int_{t_0}^{t} \delta_2^2(s, \epsilon_2)(\Delta_s^\epsilon)^{(m-1)/m} ds.\end{aligned} \tag{10}$$

Now we use the following elementary inequality

$$v^{r_2} \le v^{r_1} + v$$

which is valid for any nonnegative number v and $0 < r_1 \le r_2 < 1$. Setting

$$r_1 = (m-1)/m, \quad r_2 = (2m-1)/(2m), \text{ and } v = \Delta_s^\epsilon$$

we find that

$$(\Delta_s^\epsilon)^{(2m-1)/(2m)} \le (\Delta_s^\epsilon)^{(m-1)/m} + \Delta_s^\epsilon$$

and hence (10) takes the form

$$\Delta_t^\epsilon \le \delta_0(\epsilon_0) + m\int_{t_0}^{t} [M + 2\delta_1(s,\epsilon_1)]\Delta_s^\epsilon ds$$
$$+ \int_{t_0}^{t} [2m\delta_1(s,\epsilon_1) + m(2m-1)\delta_2^2(s,\epsilon_2)](\Delta_s^\epsilon)^{(m-1)/m} ds. \tag{11}$$

The last tool we need is the following generalized Gronwall-Bellman inequality (see Filatov and Sharova [2]):

If a nonnegative function $u(t)$, $t \ge t_0$, satisfies the integral inequality

$$u(t) \le C + \int_{t_0}^{t} A(s)u(s)ds + \int_{t_0}^{t} B(s)[u(s)]^\gamma ds,$$

where $C \ge 0$, $0 \le \gamma < 1$, and functions $A(t)$ and $B(t)$, $t \ge t_0$, are nonnegative and continuous, then

$$u(t) \le \left\{ C^{1-\gamma}\exp\left[(1-\gamma)\int_{t_0}^{t} A(s)ds\right] + (1-\gamma)\int_{t_0}^{t} B(s)\exp\left[(1-\gamma)\int_{s}^{t} A(\tau)d\tau\right]ds \right\}^{1/(1-\gamma)}.$$

Obviously, it remains to apply this inequality to (11) by letting

$$u(t) = \Delta_t^\epsilon, \quad C = \delta_0(\epsilon_0), \quad \gamma = (m-1)/m,$$

$$A(s) = m[M + 2\delta_1(s,\epsilon_1)],$$

$$B(s) = 2m\delta_1(s,\epsilon_1) + m(2m-1)\delta_2^2(s,\epsilon_2).$$

Thus we arrive at the desired relation (6). Theorem A is proved. □

3. Basic Results. Proofs

Since the magnitude of the perturbations of SDE (1) is determined by the quantities $\delta_0(\epsilon_0)$, $\delta_1(t,\epsilon_1)$ and $\delta_2(t,\epsilon_2)$ (see (4) and (5)) it is natural to impose some conditions on these quantities and see how $\Delta_t^\epsilon = E\{\mid X_t^\epsilon - X_t \mid^{2m}\} \to 0$ as $\epsilon \to 0$ and on which intervals this convergence holds.

Three specific cases will be considered.

In the statements below (Theorems 1, 2 and 3) we assume (with mentioning it again) that the general conditions of Theorem A are satisfied. We also use the constant $M = 2K + 2(2m-1)K^2$.

Theorem 1: *Suppose that for all* $t \geq t_0$,

$$\delta_0(\epsilon_0) = \epsilon_0,\ \delta_1(t, \epsilon_1) = \epsilon_1 \text{ and } \delta_2(t, \epsilon_2) = \epsilon_2. \tag{12}$$

Define the numbers ϵ *and* T_1 *as follows:*

$$\epsilon = \max[\epsilon_0^{1/m}, \epsilon_1, \epsilon_2^2] \text{ and } T_1 = (1-\rho)/(M+\rho),$$

where $\rho \in (0,1)$ *is arbitrarily chosen.*

Then the following relation holds:

$$\sup_t \Delta_t^\epsilon \to 0 \text{ as } \epsilon \to 0 \text{ for } t \in [t_0, t_0 + T_1 \ln(1/\epsilon)).$$

Proof: In Theorem A we have found the upper bound (6) for Δ_t^ϵ and now by using (12) we can go further. After a substitution we arrive at

$$(\Delta_t^\epsilon)^{1/m} \leq \epsilon_0^{1/m} \exp[M(t-t_0) + 2\epsilon_1(t-t_0)] + \int_{t_0}^{t} [2\epsilon_1 + (2m-1)\epsilon_2^2 \exp[M(t-s) + 2\epsilon_1(t-s)]]ds.$$

Since $\epsilon \to 0$ implies that $\epsilon_1 \to 0$, we can assume that $2\epsilon_1 < \rho$. Taking into account that $\epsilon_0^{1/m} \leq \epsilon$, $\epsilon_1 \leq \epsilon$ and $\epsilon_2^2 \leq \epsilon$ we find that

$$(\Delta_t^\epsilon)^{1/m} \leq C_1 \epsilon e^{(M+\rho)t} + (4\epsilon)/(M+\rho), \tag{13}$$

where the constant C_1 depends on t_0 and m but not on t.

Obviously, (13) implies that $\Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ for each t on any finite interval $[t_0, t_1]$ with fixed $t_1 > t_0$. However, we can use (13) and make one step further by extending the time-interval on which $\Delta_t^\epsilon \to 0$. Indeed, if we take $T_1 = (1-\rho)/(M+\rho)$ we find from (13) that

$$\Delta_t^\epsilon \leq [C_1 \epsilon^\rho + (4\epsilon)/(M+\rho)]^m \text{ for any } t \in [t_0, t_0 + T_1 \ln(1/\epsilon))$$

and hence $\sup_t \Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ on the interval $[t_0, t_0 + T_1 \ln(1/\epsilon))$. Note that the length of this interval tends to infinity as $\epsilon \to 0$. Theorem 1 is proved. □

Theorem 2: *Suppose that* $t_0 > 1$ *and let for all* $t \geq t_0$,

$$\delta_0(\epsilon_0) = t_0^{-1/\epsilon_0},\ \delta_1(t, \epsilon_1) = t^{-1/\epsilon_1} \text{ and } \delta_2(t, \epsilon_2) = t^{-1/\epsilon_2}. \tag{14}$$

Define ϵ *and* T_2 *as follows:*

$$\epsilon = \max[m\epsilon_0, \epsilon_1, \epsilon_2/2] \text{ and } T_2 = (1/M)\ln(t_0 - \rho),$$

where ρ *is an arbitrary number in the interval* $(0, t_0)$.

Then,

$$\sup_t \Delta_t^\epsilon \to 0 \text{ as } \epsilon \to 0 \text{ for } t \in [t_0, t_0 + T_2/\epsilon).$$

Proof: From (6) and (14), we find that

$$(\Delta_t^\epsilon)^{1/m} \leq t_0^{-1/(m\epsilon_0)} \exp[M(t-t_0) + 2\epsilon_1(t_0^{1-1/\epsilon_1} - t^{1-1/\epsilon_1})/(1-\epsilon_1 0]$$

$$+ \int_{t_0}^{t} [2s^{-1/\epsilon_1} + (2m-1)s^{-2/\epsilon_2}] \exp[M(t-s) + 2\epsilon_1(s^{1-1/\epsilon_1} - t^{1-1/\epsilon})/(1-\epsilon_1 0] ds.$$

Obviously, we can take $\epsilon_1 < 1/2$, and, since $s^{-1/(m\epsilon_0)} \leq s^{-1/\epsilon}$, $s^{-1/\epsilon_1} \leq s^{-1/\epsilon}$ and $s^{-2/\epsilon_2} \leq s^{-1/\epsilon}$ for any $s > 1$, we derive that

$$(\Delta_t^\epsilon)^{1/m} \leq t_0^{-1/\epsilon} e^{M(t-t_0)+2} + 4m \int_{t_0}^{t} s^{-1/\epsilon} e^{M(t-s)+2} ds.$$

Hence,

$$(\Delta_t^\epsilon)^{1/m} \leq C_2 t_0^{-1/\epsilon} e^{Mt}, \tag{15}$$

where C_2 is a constant depending on t_0 and m but not on t. It follows from (15) that $\Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ for each t on any finite interval $[t_0, t_2]$ with fixed $t_2 > t_0$. Moreover, with $T_2 = (1/M)\ln(t_0 - \rho)$ we easily find that

$$\Delta_t^\epsilon \leq C_2^m (1 - \rho/t_0)^{m/\epsilon} \text{ for all } t \in [t_0, t_0 + T_2/\epsilon).$$

Since $0 < 1 - \rho/t_0 < 1$, the conclusion is that $\sup_t \Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ on the interval $[t_0, t_0 + T_2/\epsilon)$ whose length tends to infinity. Theorem 2 is proved. □

Theorem 3: *Suppose that* $t_0 > 0$ *and let for all* $t \geq t_0$

$$\delta_0(\epsilon_0) = e^{-t_0/\epsilon_0},\ \delta_1(t, \epsilon_1) = e^{-t/\epsilon_1} \text{ and } \delta_2(t, \epsilon_2) = e^{-t/\epsilon_2}. \tag{16}$$

For an arbitrary $\rho \in (0, t_0)$, *define*

$$\epsilon = \max[m\epsilon_0, \epsilon_2/2] \text{ and } T_3 = (t_0 - \rho)/M.$$

Then,

$$\sup_t \Delta_t^\epsilon \to 0 \text{ as } \epsilon \to 0 \text{ for } t \in [t_0, t_0 + T_3/\epsilon).$$

Proof: Taking $\epsilon_1 < 1$ (we can do this since $\epsilon_1 \to 0$) we easily see that $\int_{t_0}^{t} e^{-s/\epsilon_1} ds < 1$ and $\int_{s}^{t} e^{-\tau/\epsilon_1} d\tau < 1$. Then, substituting (16) into (6) and suitably transforming the right-hand side of (6) we finally arrive at the relation:

$$(\Delta_t^\epsilon)^{1/m} \leq C_3 e^{Mt - t_0/\epsilon},$$

where C_3 depends on t_0, m and M but not on t.

Therefore, $\Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ for each t on any finite interval $[t_0, t_3]$ with fixed $t_3 > t_0$. Even more, if $T_3 = (t_0 = \rho)/M$, then

$$\sup_t \Delta_t^\epsilon \leq C_3^m e^{-\rho/\epsilon} \to 0 \text{ as } \epsilon \to 0 \text{ for } t \in [t_0, t_0 + T_3/\epsilon).$$

Again the convergence to zero holds on intervals whose lengths tends to infinity. Theorem 3 is proved. □

4. Additional Remarks

(*a*) In Theorems 1, 2 and 3 not only did we prove that $\sup_t \Delta_t^\epsilon \to 0$ as $\epsilon \to 0$ but, in addition, we can specify the rate of convergence. It is a power rate in Theorem 1 and exponential rate in Theorems 2 and 3. Moreover, we can specify the rate of getting to infinity of the lengths of the corresponding intervals. Obviously, both rates depend on the magnitude of perturbations.

(*b*) Instead of $\Delta_t^\epsilon = E\{ | X_t^\epsilon - X_t |^{2m}\}$, we can consider the quantity

$$\widetilde{\Delta}_t^\epsilon = E\{ \sup_{t \in [t_0, T]} | X_t^\epsilon - X_t |^{2m}\}$$

as a measure of closeness between the processes X^ϵ and X. If we establish a sup-version of Theorem A and use some additional arguments, we can provide conditions under which $\widetilde{\Delta}_T^\epsilon \to 0$ as $\epsilon \to 0$ on finite fixed intervals or on intervals whose lengths tend to infinity.

(*c*) The results of the present paper can be used when studying stability properties of SDE's under perturbations. Another possibility is to look for the so-called expansions of the solution X^ϵ of the perturbed SDE (3) assuming some smoothness of the coefficients $\widetilde{a}(\cdot)$ and $\widetilde{b}(\cdot)$.

(*d*) Similar questions can be raised for more general SDE's driven by arbitrary semimartingales not just by the standard Wiener process (see Protter [6]).

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