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SOLUTIONS OF INITIAL VALUE PROBLEMS FOR A PAIR OF LINEAR FIRST ORDER ORDINARY DIFFERENTIAL SYSTEMS WITH INTERFACE-SPATIAL CONDITIONS

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ABSTRACT

Solutions of initial value problems associated with a pair of ordinary differential systems (L_1, L_2) defined on two adjacent intervals I_1 and I_2 and satisfying certain interface-spatial conditions at the common end (interface) point are studied.

Key words: Interface-Spatially Mixed Conditions, Ordinary Differential Systems, Equations, Initial Value Problems, Linearly Independent Solutoins, Fundamental Systems.

AMS (MOS) subject classifications: 34AXX, 34A10, 34A15.

1. Introduction

In the studies of acoustic waveguides in ocean [1], optical fiber transmission [4], soliton theory [3], etc., we encounter a new class of problems of the type

$$L_1 f_1 = \sum_{k=0}^{n} P_k \frac{df_1^k}{dt^k} = \theta f_1 \text{ defined on an interval } I_1$$

and

$$L_2 f_2 = \sum_{k=0}^{m} Q_k \frac{df_2^k}{dt^k} = \theta_2 f_2 \text{ defined on an adjacent interval } I_2,$$

where θ_1, θ_2 are constants, intervals I_1 and I_2 have common end (interface) point $t = c$, and the functions f_1, f_2 are required to satisfy certain interface conditions at $t = c$. In most of the cases, the complete set of physical conditions on the system gives rise to self adjoint eigenvalue problems associated with the pair (L_1, L_2). In some cases, however, the physical conditions at the interface may be inadequate to describe the problem in a mathematically sound manner. In such a situation, when the problem is formulated mathematically, it becomes ill-posed, and therefore cannot be solved effectively (uniquely) using existing methods. With the introduction of interface-spatial conditions (entirely a new concept), we shall be able to convert these ill-posed problems into well-posed problems and this justifies their mathematical study.

In a series of papers, we wish to develop a unified approach to these interface-spatial problems for both the regular and the singular cases. In the present paper, for the first time, we shall study the initial value problems (IVPs) for a pair of linear first order ordinary differential systems satisfying certain interface-spatial conditions.

Before proving the main theorems, we introduce a few notations and make some assumptions. For any compact interval J of $\mathbb{R}$ and for any non-negative integer k, let $C^k(J)$ denote the space of k-times continuously differentiable complex-valued functions defined on J. If I is a non-compact interval of $\mathbb{R}$, $C^k(I)$ denotes the collection of all complex-valued functions f defined on I whose restriction $f \mid_J$ to any compact subinterval J of I belongs to $C^k(J)$. Let $AC^k(I)$ denote the space of all complex-valued functions f which have $(k-1)$ derivatives on I, and, the $(k-t)^{th}$ derivative is absolutely continuous over each compact subinterval of I. Let $I_1 = (a,c]$, $I_2 = [c,b)$, $-\infty \leq a < c < b \leq +\infty$, and let $f^{(j)}$ denote the j^{th} derivative of f. For a matrix A, let $R(A)$ and $\rho(A)$ denote the range and rank of A. Let $\mathbb{C}^n$ denote the complex n-dimensional space.

Let $A_1(t)$ $(A_2(t))$ be matrix valued functions of order $n \times n$ $(m \times m)$, whose entries belong to $\mathbb{C}^0(I_1)$ $(\mathbb{C}^0(I_2))$. Let $b_1(t)$ $(b_2(t))$ be a vector-valued function of order $n \times 1$ $(m \times 1)$, whose entries are integrable over every compact subinterval of I_1 (I_2).

Let the functions $P_k \in \mathbb{C}^k(I_1)$ $(k = 0,1,\ldots,n)$ $Q_k \in \mathbb{C}^k(I_2)$ $(k = 0,1,\ldots,m)$ $P_n(t) \neq \emptyset$ on I_1 and $Q_m(t) \neq \emptyset$ on I_2. Let $g_1(g_2)$ be a measurable complex-valued function defined on I_1 (I_2) which is integrable over every compact subinterval of $I_1(I_2)$.

Without loss of generality, we assume $n \geq m$. Let A and B be $m \times n$ and $m \times m$ matrices with complex entries respectively, and $R(A) = R(B)$. Consequently, $\rho(A) = \rho(B) =: d(\leq m)$. Let N be a subspace of $R(A)$, and the dimension of N equals d'. Let $t_i \in I_i$ $(i = 1,2)$, $C = \text{column }(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{C}^n$, and $D = \text{column}(d_0, d_1, \ldots, d_{m-1}) \in \mathbb{C}^m$. Let $Y_1 = \text{column }(y_{11}, y_{12}, \ldots, y_{1n})$ and $Y_2 = \text{column}(y_{21}, y_{22}, \ldots, y_{2m})$.

Consider the following interface-spatially mixed pair of linear first order ordinary differential systems:

$$Y_1' = A_1(t)Y_1 + b_1(t),\ t \in I_1, \tag{1}$$

$$Y_2' = A_2(t)Y_2 + b_2(t),\quad t \in I_2, \tag{2}$$

$$AY_1(c) - BY_2(c) \in N. \tag{3}$$

Also, consider the initial conditions

$$Y_1(c) = C \tag{4}$$

and

$$Y_2(c) = D. \tag{5}$$

We call problems (1)-(3) and (4)((5)) the *interface-spatially mixed initial value problems (IFSIVP) (I) ((II))*.

Consider the following interface-spatially mixed pair of linear ordinary differential equations (of orders n and m):

$$L_1 f_1 \equiv \sum_{k=0}^{n} P_k \frac{d^2 f_1}{dt^2} = g_1,\quad t \in I_1, \tag{6}$$

$$L_2 f_2 \equiv \sum_{k=0}^{m} Q_k \frac{d^2 f_2}{dt^2} = g_2,\quad t \in I_2, \tag{7}$$

$$A\widetilde{f}_1(c) - B\widetilde{f}_2(c) \in N, \tag{8}$$

where

$$\widetilde{f}_1 = \text{column}(f_1, f_1^{(1)}, \ldots, f_1^{(n-1)},$$

and

$$\widetilde{f}_2 = \text{column } (f_2, f_2^{(1)}, \ldots, f_2^{(m-1)}).$$

Also consider the initial conditions

$$f_1^{(j)}(t_1) = c_j \quad (j = 0, 1, \ldots, n-1), \tag{9}$$

$$f_2^{(j)}(t_2) = d_j \quad (j = 0, 1, \ldots, m-1). \tag{10}$$

We call problems (6)-(8) and (9) ((10)) the *interface-spatially mixed initial value problems (IFSIVP) (I′) ((II′))*.

Definition 1: We call a pair of vector-valued functions (Y_1, Y_2), defined on $I_1 \times I_2$, an *interface-spatially mixed* (IFS) *solution* of (1)-(2) if

(*i*) $Y_{1j} \in AC^1(I_1)$ $(j = 1, \ldots, n)$,
(*ii*) Y_1 satisfies equation (1) for almost all $t \in I_1$,
(*iii*) $Y_{2j} \in AC^1(I_2)$ $(j = 1, \ldots, m)$,
(*iv*) Y_2 satisfies equation (2) for almost all $t \in I_2$,

and

(*v*) the pair (Y_1, Y_2) satisfies relation (3).

Definition 2: We call a pair of complex-valued functions (f_1, f_2), defined on $I_1 \times I_2$, an *interface-spatially mixed* (IFS) *solution* of (6)-(7) if

(*i*) $f_1 \in AC^n(I_1)$ and satisfies equation (6) for almost all $t \in I_1$,
(*ii*) $f_2 \in AC^m(I_2)$ and satisfies equation (7) for almost all $t \in I_2$

and

(*iii*) the pair (f_1, f_2) satisfies relation (8).

Definition 3: We call a pair of vector-valued functions (Y_1, Y_2), defined on $I_1 \times I_2$, an *interface-spatially mixed solution of* $IFSIVP(I)$ $((II))$ if

(*i*) (Y_1, Y_2) is an IFS solution of (1)-(2)

and

(*ii*) $Y_1(Y_2)$ satisfies condition (4) ((5)).

Definition 4: We call a pair of complex-valued functions (f_1, f_2), defined on $I_1 \times I_2$, an *interface-spatially mixed solution of IFSIVP*(I') $((II'))$ if

(*i*) (f_1, f_2) is a IFS solution of (6)-(7)

and

(*ii*) (f_1, f_2) satisfies condition (9) ((10)).

Definition 5: We say that a collection of non-trivial pairs $(Y_{11}, Y_{12}), \ldots, (Y_{p1}, Y_{p2})$ are *linearly independent* if for any set of scalars $\alpha_1, \ldots, \alpha_p$,

$$\sum_{i=0}^{p} \alpha_i (Y_{i1}, Y_{i2}) = (0, 0)$$

implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_p = 0$.

Similarly, we define the linear independency of a collection of pairs $(f_{11}, f_{12}), \ldots, (f_{p1}, f_{p2})$.

Definition 6: By an *IFS fundamental system for the IFSIVP*(I) $((II))$, we mean a set of linearly independent IFS solutions of IFSIVP(I) $((II))$ which span the IFS solution space of IFSIVP(I) $((II))$.

Similarly, we define a *fundamental system for the IFSIVP*(I') $((II'))$.

2. Main Theorems

Theorem 1: *(a) If either $b_1(t) \neq 0$, $b_2(t) \neq 0$, or C is a nonzero vector, then the IFSVP(I) has an IFS fundamental system consisting of "$m-d+d'+1$" linearly independent IFS solutions of IFSIVP(I). If $b_1(t) \equiv 0$, $b_2(t) \equiv 0$, and C is a zero vector, then the IFSIVP(I) has an IFS fundamental system consisting of "$m-d+d'$" linearly independent IFS solutions of IFSIVP(I).*

(b) If either $b_1(t) \neq 0$, $b_2(t) \neq 0$, or D is a nonzero vector, then the IFSIVP(II) has a fundamental system consisting of "$n-d+d'+1$" linearly independent IFS solutions of IFSIVP(II). If $b_1(t) \equiv 0$, $b_2(t) = 0$, and D is a zero vector, then the IFSIVP(II) has an IFS fundamental system consisting of "$n-d+d'$" linearly independent IFS solutions of IFSIVP(II).

Proof: Since the components of $b_1(t)$ are measurable complex-valued functions integrable on I_1 by Theorem 2.1 [2], there exists a unique vector-valued function $\phi(t) = \text{column}(\phi_1(t), \phi_2(t), \ldots, \phi_n(t))$ defined on I_1 with $\phi_j \in AC^1(I_1)$ such that

$$\phi'(t) = A_1(t)\phi(t) + b_1(t),\ t \in I_1,$$
$$\phi(t_1) = C.$$

Let $\phi(c) = \eta$. Since $R(A) = R(B)$, there exists a vector $\beta^0 \in \mathbb{C}^m$ such that $A\eta = B\beta^0$. If $A\eta \neq 0$, β^0 is a nonzero vector, and if $A\eta = 0$, then we take β^0 to be zero vector. Since $\rho(B) = d$, there exist $(m-d)$ linearly independent vectors $\beta^1, \beta^2, \ldots, \beta^{m-d} \in \mathbb{C}^m$ which are solutions of $B\beta = 0$. Clearly, $\beta^0, \beta^0 + \beta^1, \ldots, \beta^0 + \beta^{m-d}$ are $(m-d+1)$ or $(m-d)$ linearly independent solutions of $A\eta = B\beta$, affected by $A\eta \neq 0$ or $A\eta = 0$.

Also, since the components of $b_2(t)$ are measurable complex-valued functions integrable on I_2, there exists a unique vector-valued function $\psi_0(t) = \text{column } (\psi_{01}(t), \ldots, \psi_{0n}(t))$ defined on I_2 with $\psi_{0j}(t) \in AC^1(I_2)$ such that

$$\psi'(t) = A_2(t)\psi_0(t) + b_2(t),\quad t \in I_2,$$
$$\psi_0(c) = \beta^0.$$

Let $\psi_i(t) = \text{column}(\psi_{i1}(t), \ldots, \psi_{in}(t))$, defined on I_2 with $\psi_{ij} \in AC^1(I_2)$, be the unique vector-valued function such that

$$\psi_i(t) = A_2(t)\psi_i(t),\quad t \in I_2,$$
$$\psi_i(c) = \beta^i,\quad i = 1, \ldots, m-d.$$

Clearly, $\psi_1, \ldots, \psi_{m-d}$ are linearly independent and if $\beta^0 \neq 0$, then $\psi_0, \ldots, \psi_{m-d}$ are also linearly independent.

Choose a basis $\alpha^1, \ldots, \alpha^{d'}$ for N, and let $\beta = \beta^{m-d+i}$ be a solution of

$$-B\beta^{m-d+i} = \alpha^i \quad (i = 1, \ldots, d').$$

Since α^i are linearly independent $\beta^{m-d+d'}s$ are also linearly independent. In fact, $\beta^1, \ldots, \beta^{m-d+d'}$ are linearly independent.

Again, let $\psi_i(t)$, defined on I_2, be a unique vector-valued function such that

$$\psi_i(t) = A_2(t)\psi_i(t),\quad t \in I_2,$$
$$\psi_i(c) = \beta^i \quad (i = m-d+1, \ldots, m-d+d').$$

Clearly, $\psi_1, \ldots, \psi_{m-d+d'}$ are linearly independent.

Now, define

$$(Y_{01}, Y_{02}) = (\phi, \psi_0),$$

$$(Y_{i1}, Y_{i2}) = (\phi, \psi_0 + \psi_i) \quad (i = 1, \ldots, m - d + d').$$

Clearly, each pair (Y_{i1}, Y_{i2}) $(i = 0, 1, \ldots, m - d + d')$ is an IFS solution of (1)-(2). Moreover, if $b_1(t) \neq 0$, $b_2(t) \neq 0$, or $C \neq 0$, then the pair (ϕ, ψ_0) is nontrivial.

Claim: For $b_1(t) \neq 0$, $b_2 \neq 0$ or $C \neq 0$, $\{(Y_{i1}, Y_{i2}), i = 0, \ldots, m - d + d'\}$ is an IFS fundamental system for the IFSIVP(I).

Let $\sum_{i=0}^{m-d+d'} a_i(Y_{i1}, Y_{i2}) = (0,0)$, where $a_i s$ are scalars. Then

$$\sum_{i=0}^{m-d+d'} a_i Y_{i1} = 0 \quad \text{and} \quad \sum_{i=0}^{m-d+d'} a_i Y_{i2} = 0. \tag{11}$$

Consequently, we get

$$\sum_{i=1}^{m-d+d'} a_i[A\phi(c) - B(\psi_0(c) + \psi_i(c))] + a_0(A\phi(c) - B\psi_0(c)) = 0,$$

i.e.,

$$\sum_{i=m-d+i}^{m-d'} a_i(-B\psi_i(c)) = 0,$$

$$\sum_{i=m-d+1}^{m-d+d'} a_i\alpha^i = 0, \text{ which implies that } a_i = 0 \quad (i = m-d+1, \ldots, m-d+d').$$

Hence, relation (11) becomes

$$\sum_{i=0}^{m-d} a_i\phi = 0 \text{ and } \sum_{i=1}^{m-d} a_i(\psi_0 + \psi_i) + A_0\psi_0 = 0. \tag{12}$$

Again, from relation (12), we get

$$\left(\sum_{i=0}^{m-d} a_i\right)\psi_0(c) + \sum_{i=1}^{m-d} a_i\psi_i(c) = 0,$$

i.e.,

$$\left(\sum_{i=0}^{m-d} a_i\right)\beta^0 + \sum_{i=1}^{m-d} a_i\beta^i = 0. \tag{13}$$

If $\beta^0 \neq \emptyset$, then $\beta^0, \beta^1, \ldots, \beta^{m-d}$, are linearly independent and hence $a_i = 0$ $(i = 0, 1, \ldots, m-d)$. If $\beta^0 = 0$, then relation (13) gives $a_i = 0$ $(i = 1, \ldots, m-d)$ and from relation (12) we get $a_0(\phi, \psi_0) = (0,0)$, which implies that $a_0 = 0$. Thus, (Y_{i1}, Y_{i2}) $(i = 0, 1, \ldots, m-d+d')$ are linearly independent.

Now, let (Y_1, Y_2) be any solution of the IFSIVP(I). We note that $Y_1 = \phi$.

Case (i): Suppose that $AY_1(c) - BY_2(c) = 0$. Furthermore, since $A\phi(c) - B\psi_0(c) = 0$, we get $B(Y_2(c) - \psi_0(c)) = 0$, which implies that $Y_2(c) - \psi_0(c)$ belongs to the null space of B. Therefore, there exist constants a_i $(i = 1, \ldots, m-d)$ such that

$$Y_2(c) - \psi_0(c) = \sum_{i=1}^{m-d} a_i\beta^i,$$

i.e.,

$$Y_2(c) = \beta^0 + \sum_{i=1}^{m-d} a_i\beta^i = \left(1 - \sum_{i=1}^{m-d} a_i\right)\beta^0 + \sum_{i=1}^{m-d} a_i(\beta^0 + \beta^i)$$

$$= \left(1 - \sum_{i=1}^{m-d} a_i\right)\psi_0(c) + \sum_{i=1}^{m-d} a_i(\psi_0(c) + \psi_i(c))$$

$$= \left(1 - \sum_{i=1}^{m-d} a_i\right)Y_{02}(c) + \sum_{i=1}^{m-d} a_iY_{i2}(c).$$

Thus, by the uniqueness of the solution of IVPs for a system of ordinary differential equations, we have

$$(Y_1, Y_2) = (1 - \sum_{i=1}^{m-d} a_i)(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d} a_i(Y_{i1}, Y_{i2}).$$

Case (*ii*): Suppose that $AY_1(c) - BY_2(c) = \xi = \sum_{i=1}^{d'} a_{i+m-d}\alpha_i$, where $a_i s$ are scalars.

Define a pair (K_1, K_2) by

$$(K_1, K_2) = (1 - \sum_{i=m-d+1}^{m-d+d'} a_i)(Y_{01}, Y_{02}) + \sum_{i=m-d+1}^{m-d+d'} a_i(Y_{i1}, Y_{i2}). \tag{14}$$

Then (K_1, K_2) is an IFS solution of IFSIVP(I). Consequently, we get

$$B(Y_2(c) - K_2(c)) = 0.$$

Therefore, there exist scalars a_i $(i = 1, \ldots, m-d)$ such that

$$Y_2(c) - K_2(c) = \sum_{i=1}^{m-d} a_i\beta^i,$$

i.e.,

$$\begin{aligned} Y_2(c) &= K_2(c) + \sum_{i=1}^{m-d} a_i\beta^i \\ &= K_2(c) - (\sum_{i=1}^{m-d} a_i)\beta^0 + \sum_{i=1}^{m-d} a_i(\beta^0 + \beta^i) \\ &= K_2(c) - (\sum_{i=1}^{m-d} a_i)\psi_0(c) + \sum_{i=1}^{m-d} a_i(\psi_0(c) + \psi_i(c)) \\ &\ldots\ldots\ldots\ldots \end{aligned} \tag{15}$$

Thus,

$$\begin{aligned} (Y_1, Y_2) &= (K_1, K_2) - \sum_{i=1}^{m-d} a_i(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d} a_i(Y_{i1}, Y_{i2}) \\ &= (1 - \sum_{i=1}^{m-d+d'} a_i)(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d+d'} a_i(Y_{i1}, Y_{i2}). \end{aligned}$$

Hence, the claim is proved. If $b_1(t) \equiv 0$, $b_2(t) \equiv 0$, and $C = 0$, then (ϕ, ψ_0) is a trivial pair and the pairs (Y_{i1}, Y_{i2}) $(i = 1, \ldots, m-d+d')$ form an IFS fundamental system for the IFSIVP(I).

This completes the proof of part (a). Part (b) can proved similarly.

Theorem 2: *There exist exactly "$n + m - d + d'$" linearly independent* (IFS) *solutions of*

$$Y_1' = A_1(t)Y_1, \quad t \in I_1, \tag{16}$$

$$Y_2' = A_2(t)Y_2, \quad t \in I_2, \tag{17}$$

satisfying the interface-spatial conditions

$$AY_1(c) - BY_2(c) \in N. \tag{18}$$

Proof: Since $\rho(A) = \rho(B) =: d$, there exists a basis $\{\eta^1, \ldots, \eta^n\}$ for $\mathbb{C}^n$ such that $\{\eta^1, \ldots, \eta^{n-d}\}$ forms a basis for the null-space of A, and a basis $\{\beta^1, \ldots, \beta^m\}$ for $\mathbb{C}^m$ such that $\{\beta^{d+1}, \ldots, \beta^m\}$ forms a basis for the null space of B.

Let $\widehat{Y}_{i1}$ (whose components belong to $AC^1(I_1)$) be the unique solution of

$$Y_1' = A_1(t)Y_1, \quad t \in I_1,$$
$$Y_1(c) = \eta^i \quad (i = 1,\dots,n).$$

Since $R(A) = R(B)$, for each $i = n-d+1,\dots,n$, there exist scalars $\theta^i_j \quad (j = 1,\dots,d)$ such that

$$A\eta^i = \sum_{j=1}^{d} \theta^i_j B\beta^j.$$

Let $\widehat{Y}_{i2}$ (with components belonging to $AC(I_2)$) be the unique solution of

$$Y_2' = A_2(t)Y_2,\ t \in I_2,$$
$$Y_2(c) = B\beta^{i-n+d} \quad (i = n+1,\dots,n+m-d).$$

Let $(\alpha^1,\dots,\alpha^{d'}\}$ be a basis for N and choose $\widehat{\beta}^i \in \mathbb{C}^m$ such that

$$-B\widehat{\beta}^i = \alpha^i \quad (i = 1,\dots,d').$$

Let $\widehat{Y}_{i2}$ (with components belonging to $AC^1(I_2)$) be the unique solution of

$$Y_2' = A_2(t)Y_2, \quad t \in I_2,$$
$$Y_2(c) = \widehat{\beta}^{i-n-m+d} \quad (i = n+m-d+1,\dots,n+m-d+d').$$

Define the pairs

$$(Y_{i1}, Y_{i2}) = \begin{cases} (\widehat{Y}_{i1}, 0) \quad (i = 1,\dots,n-d), \\ (\widehat{Y}_{i1}, \widehat{Y}_{i2}) \quad (i = n-d+1,\dots,n), \\ (0, \widehat{Y}_{i2}) \quad (i = n+1,\dots,n+m-d+d'). \end{cases}$$

Clearly each pair (Y_{i1}, Y_{i2}) is a nontrival IFS solution of (16)-(18).

Claim: $(Y_{i1}, Y_{i2}) \quad (i = 1,\dots,n+m-d+d')$ form an IFS fundamental system for the IFS solutions of (16)-(18).

First, we shall show that the pairs (Y_{i1}, Y_{i2}) are linearly independent. To this end, let

$$\sum_{i=1}^{n+m-d+d'} a_i(Y_{i1}, Y_{i2}) = (0,0), \text{ where } a_i s \text{ are scalars.}$$

Then,

$$\sum_{i=1}^{n} a_i\widehat{Y}_{i1} = 0 \text{ and } \sum_{i=n-d+1}^{n+m-d+d'} a_i\widehat{Y}_{i2} = 0. \tag{19}$$

Since $\widehat{Y}_{i1}(c) \quad (i = 1,\dots,n+m-d+d')$ are linearly independent, from the first equation of relation (19) we get $a_i = 0 \ (i = 1,\dots,n)$. Consequently, the second equation reduces to

$$\sum_{i=n+1}^{n+m-d+d'} a_i Y_{i2} = 0. \tag{20}$$

Evaluating the above expression at $t = c$ and then applying the matrix B to the resulting expression, we get

$$\sum_{i=n+m-d+1}^{n+m-d+d'} a_i\alpha^{i-n-m+d} = 0,$$

which implies that $a_i = 0$, for $i = n+m-d+1,\dots,n+m-d+d'$. Thus, relation (20) reduces to

$\sum_{i=n+1}^{n+m-d} a_i \widehat{Y}_{i2} = 0$, and since $\widehat{Y}_{i2}(c)$ $(i = n+1, \ldots, n+m-d)$ are linearly independent (this fact can be easily verified), it follows that

$$a_i = 0 \quad (i = n+1, \ldots, n+m-d).$$

This proves the linear independency of (Y_{i1}, Y_{i2})s.

Next, let (Y_1, Y_2) be any IFS solution of (16)-(18). Choose scalars a_i $(i = 1, \ldots, n)$ such that

$$Y_1(c) = \sum_{i=1}^{n} a_i \eta^i. \tag{21}$$

Case (1): Suppose that $AY_1(c) - BY_2(c) = 0$.

Define the pair $(K_1, K_2) = \sum_{i=1}^{n} a_i (Y_{i1} Y_{i2})$.

Then $K_1(c) = \sum_{i=1}^{n} a_i Y_{i1}(c) = Y_1(c)$. Hence, $Y_1 = K_1$ and $B(Y_2(c) - K_2(c)) = 0$, which implies that

$$Y_2(c) = K_2(c) + \sum_{i=n+1}^{n+m-d} a_i \beta^{i-n+d} \text{ for some scalars } a_i s,$$

i.e.,

$$Y_2(c) = K_2(c) + \sum_{i=n+1}^{n+m-d} a_i Y_{i2}(c).$$

Thus,

$$\begin{aligned}(Y_1, Y_2) &= (K_1, K_2) + \sum_{i=n+1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \\ &= \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}).\end{aligned}$$

Case (2): Suppose that $A(Y_1(c) - BY_2(c) = \xi = \sum_{i=n+m-d+1}^{n+m-d+d'} a_i \alpha^{i-n-m+d}$, where $a_i s$ are scalars.

Define $(H_1, H_2) = \sum_{i=n+m-d+1}^{n+m-d+d'} a_i (Y_{i1}, Y_{i2})$.

Then $A(H_1(c) - Y_1(c)) - B(H_2(c) - Y_2(c)) = 0$, and therefore, by case (1),

$$(Y_1 - H_1, Y_2 - H_2) = \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \text{ for some scalars } a_i s.$$

Thus,

$$\begin{aligned}(Y_1, Y_2) &= (H_1, H_2) + \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \\ &= \sum_{i=1}^{n+m-d+d'} a_i (Y_{i1}, Y_{i2}).\end{aligned}$$

This completes the proof.

Remark 1: The assumption $d' = d$ yields that there are no explicit boundary conditions at the interface point.

If $d' = 0$, then the interface-spatial condition becomes

$$AY_1(c) - BY_2(c) = 0,$$

which is generally called the *interface condition.*

Since higher order ordinary differential equations can be converted into a system of first order equations, Theorems 1 and 2 yield the following results for the pair (L_1, L_2):

Theorem 3: *(a) If either $g_1 \neq 0$, $g_2 \neq 0$, or $c_0, c_1, \ldots, c_{n-1}$ are not all zeros, then the IFSIVP(I′) has a fundamental system consisting of "$m - d + d' + 1$" linearly independent IFS solutions of IFSIVP(I′). If $g_1 \equiv 0$, $g_2 \equiv 0$, and $c_0, c_1, \ldots, c_{n-1}$ are all zeros, then the IFSIVP(I′) has a IFS fundamental system consisting of "$m - d + d'$" linearly independent solutions of IFSIVP(I′).*

(b) If either $g_1 \neq 0$, $g_2 \neq 0$, or $d_0, d_1, \ldots, d_{n-1}$ are not all zeros, then the IFSIVP(II′) has a IFS fundamental system consisting of "$n - d + d' + 1$" linearly independent IFS solutions of IFSIVP(II′). If $g_1 \equiv 0$, $g_2 \equiv 0$, and $d_0, d_1, \ldots, d_{n-1}$ are all zeros, then the IFSIVP(II′) has an IFS fundamental system consisting of "$n - d + d'$" linearly independent IFS solutions of IFSIVP(II′).

Theorem 4: *There exist exactly "$n + m - d + d'$" linearly independent (IFS) solutions of*

$$L_1 f_1 = 0, \quad t \in I_1,$$
$$L_2 f_2 = 0, \quad t \in I_2,$$

satisfying the interface-spatial conditions

$$A\tilde{f}_1(c) - B\tilde{f}_2(c) \in N.$$

Remark 4: For $d' = d$, Theorems 3 and 4 reduce to Theorems 1 and 4 of [6].

For $d' = 0$, Theorems 3 and 4 reduce to Theorems 3 and 6 of [6].

For $d' = 0$ as well as for the $(m \times n)$ matrix A given by

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & m^{th}\text{ column} & & & \\ & & & \uparrow & & & \end{array} \\ A = & \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} \end{array}$$

and B equal to the $(m \times m)$ identity matrix, Theorems 3 and 4 reduce to Theorems 2 and 5 of [6].

3. Physical Examples

Example 1 - Acoustic waveguides in ocean [1]: The following problem is encountered in the study of acoustic waves in the ocean consisting of two layers: an outer layer of finite depth and an inner layer of infinite depth:

$$L_1 f_1 = \frac{d^2 f_1}{dt^2} + k_1^2 f_1 = \lambda f_1, \quad 0 \leq t \leq d_1, \tag{22}$$

$$L_2 f_2 = \frac{d^2 f_2}{dt^2} + k_2^2 f_2 = \lambda f_2, \quad d_1 \leq \leq t \leq +\infty, \tag{23}$$

together with the end point conditions given by

$$f_1(0) = 0, \operatorname*{lt}_{t \to \infty} f_2^{(1)}(t) = 0, \tag{24}$$

and the interface conditions given by

$$f_1(d_1) = f_2(d_1), \quad 1/\rho_1 f_1^1(d_1) = 1/\rho_2 f_2^{(1)}(d_1). \tag{25}$$

Here ρ_1, ρ_2 are constant densities of the two layers, k_1, k_2 are the constants which depend upon the frequency constant and the constant sound velocities c_1, c_2 of the two layers, respectively, λ is an unknown constant, d_1 denotes the depth of the outer layer, and f_1, f_2 stand for the depth eigenfunctions.

In this example, the interface conditions at $t = d_1$ of the two layers can be written in the matrix form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_1 \end{pmatrix} \begin{pmatrix} f_1(d_1) \\ f_1^{(1)}(d_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_1 \end{pmatrix} \begin{pmatrix} f_2(d_1) \\ f_2^{(1)}(d_1) \end{pmatrix}.$$

Here $A = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_2 \end{pmatrix}$, $\operatorname{rank} A = \operatorname{rank} B = 2$, $m = n = d = 2$ and $d' = 0$.

Hence, by Theorem 3 and Remark 2, there exist a unique IFS solution for any IFSIVP associated with (22)-(23) and (25). Also, By Theorem 4 and Remark 2, there exist exactly two linearly independent IFS solutions of problems (22)-(23) and (25).

Example 2 - Optical fiber transmission [4]: In the study of wave optics of step index fiber, we encounter the following problem:

$$L_1 f_1 = \frac{d^2 f_1}{dt^2} + 1/t \frac{df_1}{dt} + (\eta_1^2 k_0^2 - \nu^2/t^2) f_1 = \beta^2 f_1, \quad 0 < t \le a, \tag{26}$$

$$L_2 f_2 = \frac{d^2 f_2}{dt^2} + 1/t \frac{df_2}{dt} + (\eta_2 k_0^2 - \nu^2/t^2) f_2 = \beta^2 f_2, \quad a \le t < \infty, \tag{27}$$

together with the interface conditions at $t = a$, given by

$$\operatorname*{lt}_{t \to 0} \mid f_1(t) \mid \; < +\infty, \operatorname*{lt}_{t \to \infty} \mid f_2(t) \mid \; = 0. \tag{29}$$

Here η_1 and η_2 are the refractive indices of the core and cladding, respectively, β is the wave propagation constant, ν is an integer $k_0 = w/c$, c is the prorogation velocity and w is the wave frequency and f_1 and f_2 are the field (electromagnetic) distributions of core and cladding, respectively.

In this example, relation (28) gives continuity conditions at $t = a$. Here A and B are the 2×2 identity matrices, $n = m = d = 2$ and $d' = 0$. Hence, by Theorem 3 and Remark 2, there exists a unique IFS solution for IFSIVP associated with (26)-(28). Also, by Theorem 4 and Remark 2, there exist exactly two linearly independent IFS (continuous) solutions of (26)-(28).

Example 3 - One-dimensional scattering in quantum theorem [3]: In quantum theory, the one-dimensional time-independent scattering problem with the delta function scattering potential is represented by the problem:

$$L_1 f_1 = \frac{d^2 f_1}{dt^2} + k^2 f_1 = 0, \quad -\infty < t \le 0, \tag{30}$$

$$L_2 f_2 = \frac{d^2 f_2}{dt^2} + (k^2 - \nu_0) f_2 = 0, \quad 0 \le t < +\infty, \tag{31}$$

together with the interface conditions given by

$$f_1(0) - f_2(0) = 0, \tag{32}$$

$$f_1^{(1)}(0) - f_2^{(1)}(0) = -\nu_0 f_1(0), \tag{33}$$

where $k^2 = 2mE/h^2$, ν_0 is a constant, and the functions f_1 and f_2 are associated with the flux density of the particle of the two regions, respectively. Here, m denotes the mass of the particle, E denotes its total energy, and h denotes the Planck constant divided by 2π. In this example, the interface conditions at $t = 0$ of the two regions can be written in the matrix form

$$\begin{pmatrix} 1 & 0 \\ \nu_0 & 1 \end{pmatrix}\begin{pmatrix} f_1(0) \\ f_1^{(1)}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_2(0) \\ f_2^{(1)}(0) \end{pmatrix}.$$

Here

$$A = \begin{pmatrix} 1 & 0 \\ \nu_0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\operatorname{rank} A = \operatorname{rank} B = 2$, $m = n = d = 2$, and $d' = 0$.

Hence, by Theorem 3 and Remark 2, there exists a unique IFS solution of any IFSIVP associated with (30)-(33). Also, by Theorem 4 and Remark 2, there exist exactly two linearly independent IFS solutions of (30)-(33).

Example 4: In this illustrative example, consider the following problem:

$$L_1 f_1 = \frac{d^2 f_1}{dt^2} - k_1^2 f_1 = 0, \quad a \le t \le c, \tag{34}$$

$$L_2 f_2 = \frac{d^2 f_2}{dt^2} + k_2^2 f_2 = 0, \quad c \le t \le b, \tag{35}$$

together with interface condition

$$f_1(c) = f_2(c) \tag{36}$$

and the end point conditions

$$f_1(a) = 0 = f_2(b), \tag{37}$$

where k_1 and k_2 are constants. Problems (34)-(37) can be thought of as the transverse vibrations of a string stretched between a and b, fixed at a and b, with different uniform linear densities (in the portion) between a and c and between c and b, and plucked at the point $t = c$.

In this example, there is only one condition at the interface (i.e., the continuity condition), and no definite relation between the derivatives is available. Therefore, we may take

$$f_1^{(1)}(c) - f_2^{(1)}(c) = \alpha, \quad \alpha \in \mathbb{R}. \tag{38}$$

We note that relation (38) is not at all a restriction on derivatives. Consequently, relation (36) and (38) can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_1(c) \\ f_1^{(1)}(c) \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} f_2(c) \\ f_2^{(1)}(c) \end{pmatrix} \in N, \tag{39}$$

where $N =$ the linear span of $\binom{0}{1}$.

Here, $A = B =$ the (2×2) identity matrix, $n = m = d = 2$, and $d' = 1$. Therefore, by Theorem 3, there exist one or two linearly independent IFS solutions of the IFSIVP associated with problems (34)-(36) depending on whether the initial data is zero or nonzero. Also, by Theorem 4, there exist three linearly independent IFS solutions of problems (34)-(36).

Remark 3: The results of this paper are used in studying the deficiency indices and self-adjoint boundary value problems associated with (L_1, L_2) satisfying interface-spatial conditions which we shall establish elsewhere.

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