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MIXED PROBLEM WITH BOUNDARY INTEGRAL CONDITIONS FOR A CERTAIN PARABOLIC EQUATION

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ABSTRACT

The present article is devoted to a proof of the existence and uniqueness of a solution of a mixed problem with boundary integral conditions for a certain parabolic equation. The proof is based on an energy inequality and on the fact that the range of the operator generated by the problem is dense.

Key words: Parabolic Equation, Boundary Integral Conditions, Energy Inequality.

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1. Introduction

In the rectangle $Q = (0,b) \times (0,T)$, we consider the equation

$$\mathcal{L}u = \frac{\partial u}{\partial t} + (-1)^m a(t)\frac{\partial^{2m} u}{\partial x^{2m}} = f(x,t), \tag{1.1}$$

where $a(t)$ is bounded, $0 < a_0 \leq a(t) \leq a_1$, and $a(t)$ has the bounded derivative such that $0 < c_0 \leq a'(t) \leq c_1$ for $t \in [0,T]$.

We adhere to equation (1.1) the initial condition

$$\ell u = u(x,0) = \varphi(x) \tag{1.2}$$

and the boundary conditions

$$\int_0^b x^k \cdot u(x,t)dx = 0, \quad k = \overline{0, 2m-1}. \tag{1.3}$$

The importance of problems with integral conditions has been pointed out by Samarskii [9]. Problems which combine local and integral condition for second order parabolic equations are investigated by the potential method [2, 7], by Fourier's method [4-6], and by the energy inequalities method [1, 8, 10].

In this paper, the existence and uniqueness of a solution of problem (1.1)-(1.3) is proved. The proof is based on the method of energy inequalities, presented in [1]. Such problems have not been studied previously.

2. Preliminaries

First, we introduce the appropriate function spaces which will be used in the paper. We denote $B_2^m(0,b)$ by:

$$B_2^m(0,b) := \begin{cases} L^2(0,b) & \text{for } m=0, \\ \{u/\mathfrak{T}^m u \in L^2(0,b)\} & \text{for } m \geq 1, \end{cases} \tag{2.1}$$

where $\mathfrak{T}^m u := \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)d\xi$, $m \geq 1$. For $m \geq 1$, the scalar product in $B_2^m(0,b)$ is defined by:

$$(u,v)_{B_2^m(0,b)} = \int_0^b \mathfrak{T}^m u \mathfrak{T}^m v dx.$$

The associated norm is:

$$\| u \|_{B_2^m(0,b)} = \| \mathfrak{T}^m u \|_{L^2(0,b)} \text{ for } m \geq 1.$$

Lemma 1: *For $m \in \mathbb{N}$, we have*

$$\| u \|^2_{B_2^m(0,b)} \leq \frac{b^2}{2} \| u \|^2_{B_2^{m-1}(0,b)}. \tag{2.2}$$

Proof: The Cauchy-Schwarz inequality gives

$$| \mathfrak{T}^m u |^2 \leq \left| \int_0^x \mathfrak{T}^{m-1} u(\xi,t)d\xi \right|^2 \leq \left(\int_0^x d\xi \right) \cdot \left(\int_0^x \left| \mathfrak{T}^{m-1} u(\xi,t) \right|^2 d\xi \right)$$

$$\leq x \cdot \int_0^x \left| \mathfrak{T}^{m-1} u(\xi,t) \right|^2 d\xi \leq x \cdot \int_0^b \left| \mathfrak{T}^{m-1} u(\xi,t) \right|^2 d\xi.$$

Therefore, we have

$$\| u \|^2_{B_2^m(0,b)} \leq \int_0^b \left| \mathfrak{T}^{m-1} u(\xi,t) \right|^2 d\xi \cdot \int_0^b x\, dx$$

$$= \frac{b^2}{2} \| u \|^2_{B_2^{m-1}(0,b)}. \qquad \square$$

Corollary: *For $m \in \mathbb{N}$, we have*

$$\| u \|^2_{B_2^m(0,b)} \leq \left(\frac{b^2}{2} \right)^m \cdot \| u \|^2_{L^2(0,b)}. \tag{2.3}$$

Remark: Inequalities (2.2) and (2.3) remain valid, if we replace the interval $(0,b)$ by a bounded region Ω of $\mathbb{R}^n$. It suffices to replace b by meas(Ω) (measure of Ω) in (2.2) and (2.3). $\square$

The space $B_2^{m,k}(Q)$ is the space with the finite norm

$$\| u \|^2_{B_2^{m,k}(Q)} = \int_0^T \| u(\cdot,t) \|^2_{B_2^m(0,b)} dt + \int_0^b \| u(x,\cdot) \|^2_{B_2^k(0,T)} dx.$$

The space $B_2^{0,0}(Q)$ coincides with $L^2(Q)$.

We associate with problem (1.1)-(1.3), the operator $L = (\mathfrak{L}, \ell)$ with domain denoted by $D(L) =: E$. The operator L is from E to F; E is Banach space of the functions $u \in L^2(0,b)$ satisfying (1.3), with the finite norm

$$\| u \|_E^2 = \| \frac{\partial u}{\partial t} \|^2_{B_2^{m,0}(Q)} + \| \frac{\partial^{2m} u}{\partial x^{2m}} \|^2_{B_2^{m,0}(Q)} + \sup_{0 \leq \tau \leq T} \| u(x,\tau) \|^2_{L^2(0,b)}, \tag{2.4}$$

where F is the Hilbert space obtained by completing the space $B_2^{m,0}(Q) \times L^2(0,b)$ equipped with the norm

$$\| \mathfrak{F} \|_F^2 = \| f \|^2_{B_2^{m,0}(Q)} + \| \varphi \|^2_{L^2(0,b)}, \mathfrak{F} = (f, \varphi). \tag{2.5}$$

Here, we assumed that the function φ satisfies the conditions in the form (1.3), i.e.,

$$\int_0^b x^k \cdot \varphi \, dx = 0, \quad k = \overline{0, 2m-1}. \tag{2.7}$$

3. Two-Sided A Priori Estimates

Theorem 1: *The following a priori estimate*

$$\| Lu \|_F \leq c \| u \|_E \tag{3.1}$$

holds for any function $u \in E$, where constant c is independent of u.

Proof: Equation (1.1) implies that

$$\| \ell u \|^2_{B_2^{m,0}(Q)} \leq 2 \left(\| \frac{\partial u}{\partial t} \|^2_{B_2^{m,0}(Q)} + a_1^2 \| \frac{\partial^{2m} u}{\partial x^{2m}} \|^2_{B_2^{m,0}(Q)} \right) \tag{3.2}$$

and initial condition (1.2) yields

$$\| \mathfrak{L} u \|^2_{L^2(0,b)} \leq \sup_{0 \leq \tau \leq T} \| u(x,\tau) \|^2_{L^2(0,b)}. \tag{3.3}$$

Combining inequality (3.2) with (3.3), we obtain (3.1) for $u \in E$, with $c := \max(2^{1/2}, 2^{1/2} a_1)$. □

Theorem 2: *For any function $u \in E$, we have the inequality*

$$\| u \|_E \leq c \| Lu \|_F, \tag{3.4}$$

where constant $c > 0$ does not depend on u.

Proof: We consider the scalar product in $L^2(Q^\tau)$, where $Q^\tau := (0,b) \times (0,\tau)$ and $0 \leq \tau \leq T$. Observe that

$$\begin{aligned} &2 \int_{Q^\tau} \left| \mathfrak{T}^m \frac{\partial u}{\partial t} \right|^2 dx\, dt + \int_0^b a(\tau) \,|\, u(x,\tau) \,|^2 dx \\ &= 2 Re\left(\mathfrak{L} u, (-1)^m \mathfrak{T}^{2m} \frac{\partial \overline{u}}{\partial t} \right)_{0, Q^\tau} + \int_0^b a(0) \,|\, \varphi \,|^2 dx + \int_{Q^\tau} a'(t) \,|\, u \,|^2 dx\, dt \end{aligned} \tag{3.5}$$

We estimate the first term on the right-hand side of (3.5). By applying an elementary inequality we have

$$2\,Re\,\left(\pounds u,(-1)^m \Im^{2m}\frac{\partial \overline{u}}{\partial t}\right)_{0,Q^\tau} \leq \|\pounds u\|^2_{B_2^{m,0}(Q^\tau)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{B_2^{m,0}(Q^\tau)}. \tag{3.6}$$

From equation (1.1), we obtain

$$\frac{1}{4}a_0^2\left\|\frac{\partial^{2m}u}{\partial x^{2m}}\right\|^2_{B_2^{m,0}(Q^\tau)} \leq \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{B_2^{m,0}(Q^\tau)} + \frac{1}{2}\|\pounds u\|^2_{B_2^{m,0}(Q^\tau)}. \tag{3.7}$$

Therefore, by formulas (3.5)-(3.7),

$$\frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|^2_{B_2^{m,0}(Q^\tau)} + \frac{1}{4}a_0^2\left\|\frac{\partial^{2m}u}{\partial x^{2m}}\right\|^2_{B_2^{m,0}(Q^\tau)} + a_0\|u(x,\tau)\|^2_{L^2(0,b)}$$

$$\leq \frac{3}{2}\|\pounds u\|^2_{B_2^{m,0}(Q^\tau)} + a_1\|\ell u\|^2_{L^2(0,b)} + c_1\|u\|^2_{L^2(Q^\tau)}.$$

Applying Lemma 7.1 from [3] to the above inequality we get

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{B_2^{m,0}(Q^\tau)} + \left\|\frac{\partial^{2m}u}{\partial x^{2m}}\right\|^2_{B_2^{m,0}(Q^\tau)} + \|u(x,\tau)\|^2_{L^2(0,b)}$$

$$\leq c_2\left(\|\pounds u\|^2_{B_2^{m,0}(Q)} + \|\ell u\|^2_{L^2(0,b)}\right),$$

where

$$c_2 := \frac{\max(3/2, a_1)}{\min(1/2, 1/4\,a_0^2, a_0)}\exp(c_1 T).$$

Since the right-hand side of the above inequality does not depend on τ, we can take the least upper bound of the left side with respect to τ from 0 to T. Thus, inequality (3.4) holds, where $c := c_2^{1/2}$. □

4. Solvability of the Problem

From inequality (3.1), it follows that operator $L: E \rightarrow F$ is continuous, while from inequality (3.4) it follows that the range of operator L is closed in F and, therefore, there is the continuous inverse operator L^{-1} yielding the solution. In other words, this means that operator L is a linear homeomorphism from the space E on the closed set $R(L) \subset F$. To prove that problem (1.1)-(1.3) has a unique solution, it remains to show that $R(L) = F$.

Theorem 3: *Let the conditions of Theorem 2 hold, and let the coefficient $a(t)$ have bounded derivatives up to the second order. Then, for any functions $f \in B_2^{m,0}(Q)$ and $\varphi \in L^2(0,b)$, there is a unique solution $u = L^{-1}\mathcal{F}$ of problem (1.1)-(1.3), where $\mathcal{F} = (f, \varphi)$, and*

$$\|u\|_E \leq c\left(\|f\|_{B_2^{m,0}(Q)} + \|\varphi\|_{L^2(0,b)}\right),$$

where constant c is independent of u.

Proof: To prove Theorem 3, we need the following proposition.

Proposition: *Let $D_0(L) = \{u/u \in D(L), \ell u = 0\}$ and let the conditions of Theorem 3 hold. If for $v \in B_2^{m,0}(Q)$ and for all $u \in D_0(L)$,*

$$(\pounds u, v)_{B_2^{m,0}(Q)} = 0, \tag{4.1}$$

then v vanishes almost everywhere on Q.

Proof of the Proposition: Assume that relation (4.1) holds for any function $u \in D_0(L)$. Using this fact we can express (4.1) in a special form. First define h by the formula

$$h: = -\int_t^T \frac{\partial}{\partial \tau}\Big(a(\tau)\frac{\partial u}{\partial \tau}\Big)d\tau.$$

Let $\frac{\partial u}{\partial t}$ be a solution of

$$a(t)\frac{\partial u}{\partial t} = h \tag{4.2}$$

and let

$$D_s(L): = \{u/u \in D(L): u = 0 \text{ for } t \leq s\}. \tag{4.3}$$

We, now, have

$$v = -\frac{\partial}{\partial t}\Big(a(t)\frac{\partial u}{\partial t}\Big). \tag{4.4}$$

Relations (4.2) and (4.3) imply that u is in $D_0(L)$. It possesses, in fact, a higher order of smoothness, and we have the following result:

Lemma 2: *If the conditions of the proposition are met, then the function u defined by* (4.2) *and* (4.3) *has derivatives with respect to t up to the second order belonging to the space $B_2^{m,0}(Q_s)$, where $Q_s = (0,b) \times (s,T)$.*

Proof of Lemma 2: To prove Lemma 2, we will use the following t-averaging operators: Let $\omega \in C^\infty(\mathbb{R})$, $\omega \geq 0$; $\omega = 0$ in a neighborhood of $t = 0$ and $t = T$, and outside the interval $(0,T)$, and let $\int_{\mathbb{R}} \omega(t)dt = 1$. We consider the operators ρ_ϵ defined by the formula

$$(\rho_\epsilon w)(x,t) = \frac{1}{\epsilon}\int_0^T \omega\Big(\Big(\frac{s-t}{\epsilon}\Big)\Big)\, w(x,s)ds \text{ for } w \in B_2^{m,0}(Q).$$

The above operators have the following properties:

***P1*:** The function $\rho_\epsilon w \in C^\infty(Q)$ and it vanishes in a neighborhood of $t = T$ if $w \in B_2^{m,0}(Q)$, and $\rho_\epsilon u \in D_s(L)$ if $u \in D_s(L)$.

***P2*:** If $w \in B_2^{m,0}(Q)$, then $\| \rho_\epsilon w - w \|_{B_2^{m,0}(Q)} \to 0$ when $\epsilon \to 0$, and $\| \rho_\epsilon w \|_{B_2^{m,0}(Q)} \leq \| w \|_{B_2^{m,0}(Q)}$.

***P3*:** $\frac{d^k}{dt^k}\rho_\epsilon u = \rho_\epsilon \frac{d^k u}{dt^k}$ for $k = 1,2$ if $u \in D_s(L)$.

***P4*:** If $w \in B_2^{m,0}(Q)$ then,

$$\| \frac{\partial}{\partial t}\left(a(t)\rho_\epsilon w - \rho_\epsilon a(t) w\right) \|_{B_2^{m,0}(Q)} \to 0, \text{ when } \epsilon \to 0.$$

Proofs of properties $P1$-$P4$ are similar to the proofs of the corresponding properties obtained in [3] (see Lemma 9.1). □

Applying the operators ρ_ϵ and $\frac{\partial}{\partial t}$ to equation (4.2), we obtain

$$a(t)\frac{\partial}{\partial t}\rho_\epsilon\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(a(t)\rho_\epsilon\frac{\partial u}{\partial t} - \rho_\epsilon a(t)\frac{\partial u}{\partial t}\right) - a'(t)\rho_\epsilon\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\rho_\epsilon h.$$

It follows that

$$\| a(t)\frac{\partial}{\partial t}\rho_\epsilon\frac{\partial u}{\partial t}\|^2_{B_2^{m,0}(Q)} \leq c_3 \left(\| \rho_\epsilon\frac{\partial u}{\partial t}\|^2_{B_2^{m,0}(Q)}\right.$$

$$\left. + \|\frac{\partial}{\partial t}\rho_\epsilon h\|^2_{B_2^{m,0}(Q)} + \|\frac{\partial}{\partial t}\left(a(t)\rho_\epsilon\frac{\partial u}{\partial t} - \rho_\epsilon a(t)\frac{\partial u}{\partial t}\right)\|^2_{B_2^{m,0}(Q)}\right),$$

where $c_3 = \max(3c_1, 3)$.

By virtue of properties $P1$-$P4$ of the t-averaging operators and by inequality (2.3), we have

$$\left(\|\frac{\partial^2 u}{\partial t^2}\|^2_{B_2^{m,0}(Q)} \leq c_4 \|\frac{\partial u}{\partial t}\|^2_{L^2(Q)} + \|\frac{\partial}{\partial t}\rho_\epsilon h\|^2_{B_2^{m,0}(Q)}\right),$$

where $c_4\colon = \max\left(c_3 b^{2m}/(a_0^2 2^m), 1/a_0^2\right)$. This yields the proof of Lemma 2. □

Now, we will prove the proposition. Replace v in (4.1) by its representation (4.4). We have

$$-2\,Re\left(\frac{\partial u}{\partial t}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)}$$

$$-2\,Re\left((-1)^m a(t)\frac{\partial^{2m}}{\partial x^{2m}}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)} = 0. \tag{4.5}$$

We write the remaining two terms of (4.5) in the form:

$$-2\,Re\left(\frac{\partial u}{\partial t}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)}$$

$$= \| a^{1/2}(s)\mathfrak{T}^m\frac{\partial u(x,s)}{\partial t}\|^2_{L^2(0,b)} - \| a'^{1/2}(t)\mathfrak{T}^m\frac{\partial u}{\partial t}\|^2_{L^2(Q_s)}, \tag{4.6}$$

$$-2\,Re\left((-1)^m a(t)\frac{\partial^{2m} u}{\partial x^{2m}}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)}$$

$$= 2\,\| a(t)\frac{\partial u}{\partial t}\|^2_{L^2(Q_s)} + Re\left(a'(T)u(x,T), a(T)\overline{u}\,(x,T)\right)_{L^2(0,b)}$$

$$- \| a'(t)u\|^2_{L^2(Q_s)} - Re\left(a''(t)u, a(t)\overline{u}\,\right)_{L^2(Q_s)}. \tag{4.7}$$

Elementary calculations, starting from (4.6) and (4.7), yield the inequalities

$$a_0 \|\frac{\partial u(x,s)}{\partial t}\|^2_{B_2^m(0,b)} \leq c_1 \|\frac{\partial u}{\partial t}\|^2_{B_2^{m,0}(Q_s)} - 2\,Re\left(\frac{\partial u}{\partial t}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)},$$

$$2a_0^2 \|\frac{\partial u}{\partial t}\|^2_{L^2(Q_s)} + a_0 c_0 \| u(x,T)\|^2_{L^2(0,b)}$$

$$\leq -2\,Re\left((-1)^m a(t)\frac{\partial^{2m} u}{\partial x^{2m}}, \frac{\partial}{\partial t}\left(a(t)\frac{\partial \overline{u}}{\partial t}\right)\right)_{B_2^{m,0}(Q_s)} + (1/2a_1^2 + c_1^2 + 1/2c_5^2)\,\| u\|^2_{L^2(Q_s)},$$

where $c_5\colon = \sup_{0 \leq t \leq T} |\, a''(t)\,|$.

Consequently,

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(Q_s)} + \| u(x,T) \|^2_{L^2(0,b)} + \left\| \frac{\partial u(x,s)}{\partial t} \right\|^2_{B_2^m(0,b)}$$

$$\leq c_6 \left(\| u \|^2_{L^2(Q_s)} + \left\| \frac{\partial u}{\partial t} \right\|^2_{B_2^{m,0}(Q_s)} \right), \tag{4.8}$$

where $c_6 := \max(c_1, 1/2a_1^2 + c_1^2 + 1/2c_5^2)/\min(a_0, 2a_0^2, a_0c_0)$.

Inequality (4.8) is the basic of our proof. To use (4.8), we note that constant c_6 is independent of s. However, function u in (4.8) depends on s. To avoid this difficulty we introduce a new function θ by the formula

$$\theta(x,t) := \int_t^T \frac{\partial u}{\partial \tau} d\tau.$$

Then, $u(x,t) = \theta(x,s) - \theta(x,t)$, $u(x,T) = \theta(x,s)$, and we have

$$\| u \|^2_{L^2(Q_s)} \leq 2 \left(\| \theta(x,t) \|^2_{L^2(Q_s)} + (T-s) \| \theta(x,s) \|^2_{L^2(0,b)} \right).$$

Hence, if $s_0 > 0$ satisfies $0 < 2c_6(T - s_0) \leq 1/2$, then (4.8) implies that

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(Q_s)} + \left\| \frac{\partial u(x,s)}{\partial t} \right\|^2_{B_2^m(0,b)} + \| \theta(x,s) \|^2_{L^2(0,b)}$$

$$\leq 4c_6 \left(\left\| \frac{\partial u}{\partial t} \right\|^2_{B_2^{m,0}(Q_s)} + \| \theta(x,t) \|^2_{L^2(Q_s)} \right), \tag{4.9}$$

for all $s \in [T - s_0, T]$.

We denote the sum of the two terms on the right of (4.9) by $\beta(s)$. Hence, we obtain

$$\left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(Q_s)} - \frac{d\beta(s)}{ds} \leq 4c_6 \beta(s),$$

and, consequently,

$$-\frac{d}{ds}\big(\beta(s) \exp(4c_6 s)\big) \leq 0. \tag{4.10}$$

Integrating (4.10) over (s,T) and taking into account that $\beta(T) = 0$, we obtain

$$\beta(s) \exp(4c_6 s) \leq 0. \tag{4.11}$$

It follows from (4.11), that $v = 0$ almost everywhere on Q_{T-s_0}. Proceeding this way step by step along the rectangle with side s_0, we prove that $v = 0$ almost everywhere on Q. This completes the proof of the proposition. □

Now, we will prove Theorem 3. For this purpose it is sufficient to prove that the range $R(L)$ of L is dense in F.

Suppose that, for some $V = (v, v_0) \in {}^{\perp} R(L)$,

$$(\mathfrak{L}u, v)_{B_2^{m,0}(Q)} + (\ell u, v_0)_{L^2(0,b)} = 0. \tag{4.12}$$

We must prove that $V = 0$. Putting $u \in D_0(L)$ into (4.12) we obtain

$$(\mathfrak{L}u, v)_{B_2^{m,0}(Q)} = 0, \quad u \in D(L).$$

Hence, the proposition implies that $v = 0$. Thus, (4.12) takes the form

$$(\ell u, v_0)_{L^2(0,b)} = 0, \quad u \in D(L).$$

Since the range of operator ℓ is everywhere dense in $L^2(0,b)$, the above relation implies that $v_0 = 0$. Hence, $V = 0$. This proves Theorem 3. □

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