

THE STRUCTURE DISTRIBUTION IN A MIXED POISSON PROCESS

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ABSTRACT

We use a variety of real inversion formulas to derive the structure distribution in a mixed Poisson process. These approaches should prove to be useful in applications, e.g., in insurance where such processes are very popular.

This article is dedicated to the memory of Roland L. Dobrushin.

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1. Introduction

One of the most classical examples of a counting process $\{N(t); t \geq 0\}$ is the homogeneous Poisson process. The probability distribution is given on \mathbb{N} in the form:

$$p_n(t) = P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}.$$

The *risk-parameter* λ gives the average number of events per unit time.

In many applications, however, the Poisson process is too simple to be applicable.

Example: If $N(t)$ is the number of claims up to time t in a specific insurance portfolio, then it has been known to actuaries that the variability in the portfolio - expressed by $\text{Var}\{N(t)\}$ - is much larger than λt , the value corresponding to the strict Poisson case. One reason is that, even when the number of claims for each individual policy follows a Poisson distribution, the averages vary over the portfolio. This means that the value λ for an individual policy is one of the possible values of a random variable Λ . This then leads to the notion of a mixed Poisson process, which is defined as follows (see Lundberg, [11]).

Definition: A *mixed Poisson process* (MPP) $\{N(t); t \geq 0\}$ is a pure birth process with state space \mathbb{N} and counting distribution $p_n(t)$ of the form:

$$p_n(t) = P(N(t) = n) = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^n}{n!} dH(\lambda),$$

where H is the structure distribution given by $H(\lambda) = P\{\Lambda \leq \lambda\}$ with $H(0) = 0$.

Here are a few popular choices for the structure function.

- **Homogeneous Poisson process.** The random variable Λ is degenerate at $\lambda (> 0)$. This is the only MPP that is simultaneously a renewal process. The interclaim times are independent and exponentially distributed.
- **Double Poisson process.** The structure distribution has two different jump points, λ_1 and λ_2 , with corresponding heights, $p_1 \in (0, 1)$ and $p_2 = 1 - p_1$, respectively. The counting distribution is then

$$p_n(t) = p_1 \frac{e^{-\lambda_1 t} (\lambda_1 t)^n}{n!} + p_2 \frac{e^{-\lambda_2 t} (\lambda_2 t)^n}{n!}, \quad t > 0.$$

This kind of process might be used if the population were to be subdivided into females and males.

- **Pólya or Pascal process.** For this example, the structure distribution is $Gamma(\alpha, 1/b)$, where $\alpha, b > 0$, so that

$$dH(\lambda) = \frac{b^\alpha}{\Gamma(\alpha)} e^{-b\lambda} \lambda^{\alpha-1} d\lambda.$$

The resulting counting distribution is then

$$p_n(t) = \binom{\alpha + n - 1}{n} \left(\frac{t}{t+b}\right)^n \left(\frac{b}{t+b}\right)^\alpha, \quad t > 0,$$

which is a Pascal or negative binomial distribution. The two parameters, α and b , allow great flexibility while one fits actual data to this theoretical distribution.

- **Sichel process.** In [12], Sichel introduced a distribution as a mixed Poisson by mixing it with a *general inverse Gaussian distribution* of the form:

$$h(\lambda) = \frac{dH(\lambda)}{d\lambda} = \frac{\mu^{-\theta} \lambda^{\theta-1}}{2K_\theta(\frac{\mu}{\beta})} \exp \left\{ -\frac{\lambda^2 + \mu^2}{2\beta\lambda} \right\},$$

where the three parameters, β, θ and μ , are nonnegative. The function K_θ is the modified Bessel function of the third kind. An explicit form of the probabilities can be obtained so that

$$p_n(t) = \frac{(\mu t)^n}{n!} (1 + 2\beta t)^{-\frac{1}{2}(\theta+n)} \frac{K_{\theta+n}(\frac{\mu}{\beta} \sqrt{1+2\beta t})}{K_\theta(\frac{\mu}{\beta})}.$$

The case where $\theta = -1/2$ is particularly interesting since the general inverse Gaussian distribution simplifies to the classical inverse Gaussian distribution.

The introduction of a general MPP is probably due to Thyron [15] for the general case and to Ammeter [4] for the special case of the Pascal process. The first detailed and fundamental study of MPP's is due to Lundberg [11], who derived the deeper connection between MPP's and continuous-time Markov chains. In particular, Lundberg derived the *binomial criterion* as a characterization of MPP's among Markov processes.

Other contributions are due to Albrecht [1-3], who has been the first to discuss statistical problems connected with MPP's. For more information on MPP's, see Johnson and Kotz [9], Bühlmann [5], Gerber [6, 7] and Grandell [8]. Moment estimators and maximum likelihood estimators for the structure distribution have been derived by Tucker [16] and Simar [13]; they result in discrete estimates for H . Again, Albrecht [3] studied estimators for the case of a mixture of a known finite number of Poisson components. In all these estimators, one uses the number of claims in successive repetitions of the process.

An alternative approach is due to Karr [10], who estimated H by inverting the Laplace transform. In this case, only the time epoch of the first claim in each of the realizations of the MPP is used. Our approach is more in the spirit of Karr's.

2. Estimation of H by Real Inversion Formulas

Before we start deriving inversion formulas, let us introduce an abbreviation. For $0 \leq y_1 \leq y_2 < \infty$, we shall write

$$H\{y_1; y_2\} := \frac{1}{2}H(\{y_1\}) + H(y_1, y_2) + \frac{1}{2}H(\{y_2\}). \tag{1}$$

In the above $H(\{y\})$ is the point mass of H at the point y , while $H(y_1, y_2)$ denotes the mass in the open interval (y_1, y_2) .

2.1 Inversion using the Laplace transform of H

This method is based on the equality:

$$\widehat{H}(\theta) = E\left(\left(1 - \frac{\theta}{t}\right)^{N(t)}\right), \tag{2}$$

where \widehat{H} is the Laplace transform of H .

We know that

$$\begin{aligned} E\left(\left(1 - \frac{\theta}{t}\right)^{N(t)}\right) &= \int_0^\infty e^{-\lambda t} \sum_{n=0}^\infty \frac{\left(\left(1 - \frac{\theta}{t}\right)\lambda t\right)^n}{n!} dH(\lambda) \\ &= \int_0^\infty e^{-\lambda t} e^{(1 - \frac{\theta}{t})\lambda t} dH(\lambda) = \int_0^\infty e^{-\lambda\theta} dH(\lambda) = \widehat{H}(\theta). \end{aligned}$$

2.1.1 Inversion via Poisson variates

The following limiting relation can be derived by looking at a sequence of independent, identically distributed Poisson variables. See, for example, Teugels [14].

$$d_n(u, t) := \sum_{m=0}^{[nt]} e^{-nu} \frac{(nu)^m}{m!} \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \text{als } t < u \\ \frac{1}{2} & \text{als } t = u \\ 1 & \text{als } t > u. \end{cases} \tag{3}$$

Define, for $0 < y_1 < y_2 < \infty$, the expression:

$$I_n(y_1, y_2) := \sum_{m=1+[ny_1]}^{[ny_2]} \frac{(-n)^m}{m!} \widehat{H}^{(m)}(n).$$

Using the definition of \widehat{H} , we can write that

$$\begin{aligned} I_n(y_1, y_2) &= \sum_{m=1+[ny_1]}^{[ny_2]} \frac{(-n)^m}{m!} (-1)^m \int_0^\infty e^{-\lambda n} \lambda^m dH(\lambda) \\ &= \int_0^\infty \left(\sum_{m=1+[ny_1]}^{[ny_2]} \frac{(n\lambda)^m}{m!} e^{-\lambda n} \right) dH(\lambda) \\ &= \int_0^\infty \{d_n(\lambda, y_2) - d_n(\lambda, y_1)\} dH(\lambda). \end{aligned}$$

On the other hand, equality (2) can be invoked to write

$$\widehat{H}^{(m)}(\theta) = (-t)^{-m} E\left\{ \frac{N(t)!}{(N(t)-m)!} \left(1 - \frac{\theta}{t}\right)^{N(t)-m} \right\}.$$

Interchanging summation and expectation, we see that

$$I_n(y_1, y_2) = E \left\{ \sum_{m=1+[ny_1]}^{[ny_2]} \binom{N(t)}{m} \left(\frac{n}{t}\right)^m \left(1 - \frac{n}{t}\right)^{N(t)-m} \right\}.$$

We combine this with the first formula and apply Lebesgue's dominated convergence theorem to arrive at a first inversion formula:

$$H\{y_1, y_2\} = \lim_{n \uparrow \infty} E \left\{ \sum_{m=1+[ny_1]}^{[ny_2]} \binom{N(t)}{m} \left(\frac{n}{t}\right)^m \left(1 - \frac{n}{t}\right)^{N(t)-m} \right\}. \quad (4)$$

For the mass at a point, we can use a slightly different approach than that for $n \rightarrow \infty$:

$$\exp\{-n(v-1-\log v)\} \rightarrow \begin{cases} 1 & \text{als } v = 1 \\ 0 & \text{als } v \neq 1. \end{cases}$$

Following the same argument as above, we get

$$H(\{y\}) = \lim_{n \uparrow \infty} E \left\{ \frac{N(t)!}{(N(t)-n)!} \left(1 - \frac{n}{ty}\right)^{N(t)-n} \left(\frac{e}{ty}\right)^n \right\}.$$

2.1.2 Inversion via Gamma variates

An analogous derivation can be made starting from a sequence of i.i.d. exponential variables. For the latter we have the limiting relation:

$$e_n(u) = \frac{1}{\Gamma(n)} \int_0^{nu} e^{-v} v^{n-1} dv \rightarrow_{n \rightarrow \infty} \begin{cases} 0 & \text{als } u < 1 \\ \frac{1}{2} & \text{als } u = 1 \\ 1 & \text{als } u > 1. \end{cases} \quad (5)$$

Define

$$J_n(y_1, y_2) = \int_{y_1}^{y_2} \frac{(-n)^n}{\Gamma(n)} \widehat{H}^{(n)}\left(\frac{n}{s}\right) \frac{ds}{s^{n+1}}.$$

We easily find from the definition of \widehat{H} that

$$J_n(y_1, y_2) = \int_0^\infty \left\{ e_n\left(\frac{\lambda}{y_1}\right) - e_n\left(\frac{\lambda}{y_2}\right) \right\} dH(\lambda).$$

From (2) we see that

$$J_n(y_1, y_2) = E \left\{ \frac{1}{B(n, N(t)-n+1)} \int_{\frac{n}{ty_2}}^{\frac{n}{ty_1}} z^{n-1} (1-z)^{N(t)-n} dz \right\}.$$

Combining these elements with Lebesgue's theorem, we obtain a second inversion formula:

$$H\{y_1; y_2\} = \lim_{n \uparrow \infty} E \left\{ \frac{1}{B(n, N(t)-n+1)} \int_{\frac{n}{ty_2}}^{\frac{n}{ty_1}} z^{n-1} (1-z)^{N(t)-n} dz \right\}. \quad (6)$$

One advantage of this latter formula is that, if the density h of H exists, then

$$h(y) = \lim_{n \uparrow \infty} E \left\{ \frac{1}{B(n, N(t)-n+1)} \frac{1}{y} \left(\frac{n}{yt}\right)^n \left(1 - \frac{n}{yt}\right)^{N(t)-n} \right\}.$$

2.1.3 Normal approximation

For both of the inversion formulas, (4) and (6), one can apply a normal approximation. Let us illustrate the procedure on (6).

In (6), let $z = a_n u + b_n$, where a_n and b_n are functions of t that are to be determined in the sequel. Rewrite the incomplete beta-integral in the form:

$$\int_{\frac{n}{ty_2}}^{\frac{n}{ty_1}} z^{n-1}(1-z)^{N(t)-n} dz = a_n \int_{\alpha_2(n)}^{\alpha_1(n)} (a_n u + b_n)^{n-1} (\bar{b}_n - a_n u)^{N(t)-n} du,$$

with $\bar{b}_n = 1 - b_n$ and

$$\alpha_i(n) = \frac{1}{a_n} \left(\frac{n}{ty_i} - b_n \right), \quad i = 1, 2.$$

The expression inside the expectation sign in (6) then reads as

$$\frac{1}{B(n, N(t) - n + 1)} \int_{\frac{n}{ty_2}}^{\frac{n}{ty_1}} z^{n-1}(1-z)^{N(t)-n} dz = I_1(n) \int_{\alpha_2(n)}^{\alpha_1(n)} \exp(I_2(n, u)) du,$$

where

$$I_1(n) = \frac{a_n b_n^{n-1} \bar{b}_n^{N(t)-n} N(t)!}{(n-1)!(N(t)-n)!}$$

and

$$I_2(n, u) = (n-1) \log \left(1 + \frac{a_n u}{b_n} \right) + (N(t) - n) \log \left(1 - \frac{a_n}{b_n} u \right).$$

Now choose a_n and b_n such that $\exp(I_2(n, u))$ converges to the key factor in the normal density. We then need both $N(t)$ and n to be large, but also $N(t) - n$ needs to be large. A series expansion of the logarithms yields for I_2 that

$$I_2(n, u) = u a_n \left\{ \frac{n-1}{b_n} - \frac{N(t)-n}{\bar{b}_n} \right\} - \frac{u^2}{2} a_n^2 \left\{ \frac{n-1}{b_n^2} + \frac{N(t)-n}{\bar{b}_n^2} \right\} + O\left(\frac{a_n u}{b_n}\right)^3.$$

The obvious choice for b_n should annihilate the first term on the right. The subsequent choice of a_n is made to reduce the coefficient of $-u^2/2$ to 1. This yields the choices:

$$b_n = \frac{n-1}{N(t)-1}, \quad \bar{b}_n = \frac{N(t)-n}{N(t)-1}, \quad \text{and} \quad a_n^2 = \frac{(n-1)(N(t)-n)}{(N(t)-1)^3}.$$

With the help of these expressions and Stirling's formula, one easily shows that $I_1(n) \sim (2\pi)^{-\frac{1}{2}}$. Introducing the above values for a_n and b_n in α_i yields

$$\alpha_i(n) = \frac{(N(t)-1)^{\frac{3}{2}}}{\sqrt{(n-1)(N(t)-n)}} \left\{ \frac{n}{ty_i} - \frac{n-1}{N(t)-1} \right\} \sim \sqrt{\frac{n}{1-\frac{n}{N(t)}}} \left\{ \frac{N(t)}{ty_i} - 1 \right\}.$$

Combining all of the above results, one obtains a normal approximation to (6):

$$H\{y_1; y_2\} \sim \Phi \left\{ \sqrt{\frac{n}{1-\frac{n}{N(t)}}} \left(\frac{N(t)}{ty_1} - 1 \right) \right\} - \Phi \left\{ \sqrt{\frac{n}{1-\frac{n}{N(t)}}} \left(\frac{N(t)}{ty_2} - 1 \right) \right\}. \tag{7}$$

2.2 Inversion based on the time epochs

As an alternative, we can start from the explicit expression for the distribution of the epoch of the n th event, T_n , and combine this with the limiting relation (5). According to a result by

Lundberg [11], (see alternatively, Albrecht [3]),

$$f_{T_n}(x) = \frac{n}{x} p_n(x) = \int_0^\infty e^{-\lambda x} \lambda^n \frac{x^{n-1}}{(n-1)!} dH(\lambda).$$

For the distribution we get

$$\int_0^w f_{T_n}(u) du = \int_0^\infty dH(\lambda) \int_0^{\lambda w} e^{-v} \frac{v^{n-1}}{(n-1)!} dv = \int_0^\infty e_n\left(\frac{\lambda w}{n}\right) dH(\lambda).$$

Henceforth,

$$\begin{aligned} P\left(y_1 \leq \frac{n}{T_n} < y_2\right) &= P\left(T_n \leq \frac{n}{y_1}\right) - P\left(T_n \leq \frac{n}{y_2}\right) \\ &= \int_0^\infty \left\{ e_n\left(\frac{\lambda}{y_1}\right) - e_n\left(\frac{\lambda}{y_2}\right) \right\} dH(\lambda). \end{aligned}$$

Now, we apply (5) directly to obtain a fourth inversion formula:

$$H\{y_1; y_2\} = \lim_{n \uparrow \infty} P\left(y_1 \leq \frac{n}{T_n} < y_2\right).$$

3. Simulations

3.1 Simulation of an MPP

Another important property of an MPP is that the waiting times between epochs of events, with each $W_n = T_{n-1} - T_n$, are dependent, unless we are dealing with a strict Poisson process. Albrecht [3] has shown that the W_n 's are identically distributed, exchangeable, but positively correlated. More specifically, one has

$$\text{Cov}(W_{n-1}, W_n) = \text{Var}\left(\frac{1}{\Lambda}\right).$$

There are at least two distinct procedures to simulate the time epochs of an MPP.

- (i) A first method to simulate the time epochs of an MPP uses the above interdependence between the waiting times. Recall from Albrecht [3] that

$$P(W_n \leq t | T_{n-1} = t_{n-1}) = 1 - \left(\frac{t_{n-1}}{t + t_{n-1}}\right)^{n-1} \frac{p_{n-1}(t + t_{n-1})}{p_{n-1}(t_{n-1})}.$$

- (ii) A second method is based on the uniformity property of the MPP. Given that $N(t) = n$, the first n epochs, T_1, \dots, T_n , have the same joint distribution as the order statistics of a sample from a uniform distribution on $(0, t)$ (Albrecht [3]). One starts with the simulation of n as a value of $N(t)$ with a given value of t and in accordance with the distribution $p_n(t)$. One then simulates n random numbers in $(0, t)$. After rearrangement, these values give a sample (T_1, \dots, T_n) .

3.2 Illustrations

We perform simulations for the empirical versions of the inversion formulas given in (4), (7) and (8) on homogeneous, double Poisson processes, Pascal processes, and mixtures of Pascal processes with Poisson point masses. In all cases, we considered the distribution function by taking y_1 to be zero. From the simulations, we can conclude that the estimators for the inversions (4) and (8) perform quite well, even for a relatively small number of realizations. On the other hand, the normal approximation in (7) seems to be too rough. Generally, continuous structure functions are better estimated by the inversion formulas than discrete distributions are. By means of example, the results of two simulations are shown in Figures (a) and (b).

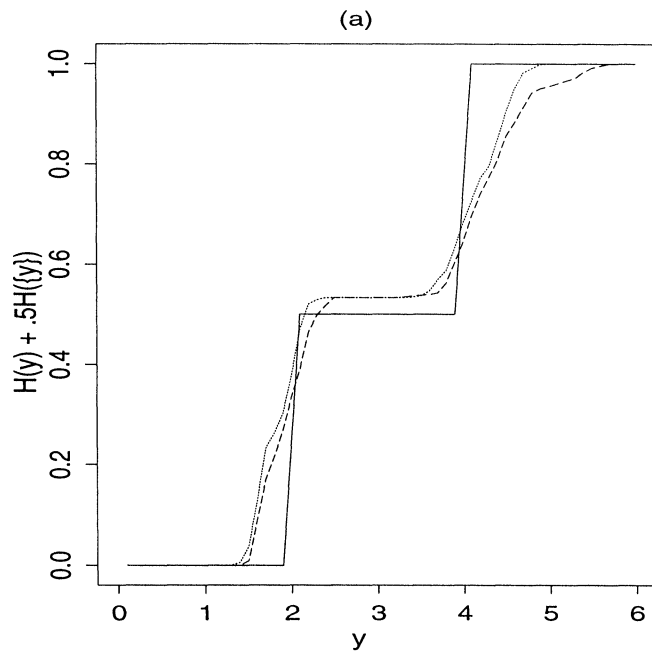


Figure (a)

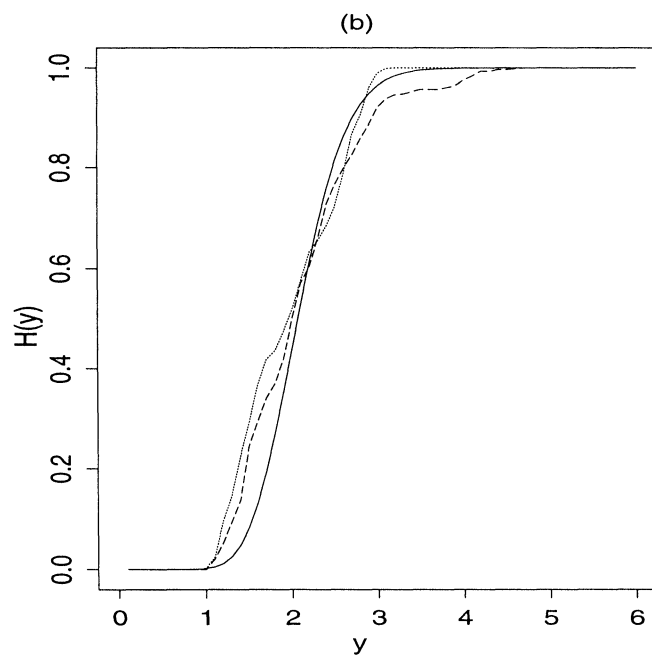


Figure (b)

In both parts, the estimators are based on 15 simulated realizations with fixed $t = 20$. The solid lines represent the theoretical structure function and the dotted lines represent the estimator for (8), while the points are the estimators for inversion formula (4). For Figure (a), we simulated double Poisson processes with parameters $\lambda_1 = 2$, $\lambda_2 = 4$ and $p_1 = .5$. The value of n in (4) was taken to be 20. In the case of the estimator for (8), we considered the averages for n -values from $n = 18$ to $n = N(20)$. Figure (b) concerns simulations of Pascal processes with parameters $a = 21$ and $b = 10$. The value for n in (4) was taken to be 18, whereas for (8) the averages for $n = 15$ to $n = N(20)$ were considered.

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