J. of Inequal. & Appl., 1997, Vol. 1 pp. 11–24
Reprints available directly from the publisher
Photocopying permitted by license only

Characterizations of Chaotic Order via Generalized Furuta Inequality*

TAKAYUKI FURUTA

Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, 1–3 Kagurazaka, Shinjuku, Tokyo 162, Japan

(Received 2 May 1996)

A characterization of chaotic order is given by using generalized Furuta inequality and its application to related norm inequalities is given as a precise estimation of our previous paper [15]. Also parallel results related to generalized Furuta inequality are given by using nice characterization of chaotic order by Fujii *et al.* [7].

Keywords: Chaotic order; Löwner-Heinz inequality; Furuta inequality.

AMS 1991 Subject Classification: 47A63

1 INTRODUCTION

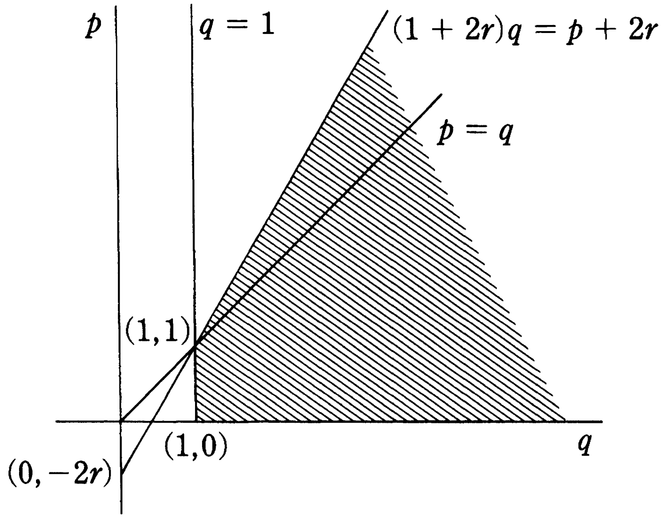
In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible.

As an extension of the Löwner-Heinz theorem [17, 20], we established the following *order preserving operator inequality* in [9].

THEOREM F (Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$

*Dedicated to Professor P.R. Halmos on his 80th birthday with respect and affection.



and

(ii) $(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1 + 2r)q \geq p + 2r$.

Alternative proofs are given in [3, 10] and [18] and also an elementary one-page proof in [11]. Applications and related results are shown in (cf. [4, 5, 12] and [13]).

We remark that the Furuta inequality yields the Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) in Theorem F: if $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0, 1]$.

The domain surrounded by p, q and r in the Figure is the best possible one for the Furuta inequality in [21].

Recently Ando-Hiai [2] established various log-majorization results to ensure excellent and useful inequalities for unitarily invariant norms.

We established the following extension of the Furuta inequality which interpolates an inequality equivalent to the main result of Ando-Hiai log-majorization results and the Furuta inequality itself.

THEOREM A (Generalized Furuta inequality) [6, 14]. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for each* $t \in [0, 1]$ *and* $p \geq 1$,

$$F_{p,t}(A, B, r, s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r *and* s *for any* $s \geq 1$ *and* $r \geq t$ *and the following inequality holds*

$$\begin{aligned} A^{1-t} &= F_{p,t}(A, A, r, s) \\ &\geq F_{p,t}(A, B, r, s) \end{aligned}$$

for any $s \geq 1$, $p \geq 1$ *and* r *such that* $r \geq t \geq 0$.

An immediate consequence of Theorem A, we showed the following result.

THEOREM B [14]. *If* $A \geq B \geq 0$ *with* $A > 0$, *then for each* $t \in [0, 1]$,

$$\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}A^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}$$

holds for any $s \geq 0$, $p \geq 0$, $q \geq 1$ *and* $r \geq t$ *with* $(s-1)(p-1) \geq 0$ *and* $(1-t+r)q \geq (p-t)s+r$.

We write $A \gg B$ if $logA \geq logB$ which is called the chaotic order [5] and it is well known in Ando [1] that $A \gg B$ holds if and only if $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ holds for all $p \geq 0$. As an extension of this characterization, we have the following result.

THEOREM C [5, 13]. *Let* A *and* B *be positive invertible operators. Then the following properties are mutually equivalent:*

(I) $A \gg B$(i.e., $\log A \geq \log B$).
(II) $A^P \geq (A^{p/2}B^pA^{p/2})^{1/2}$ for all $p \geq 0$
(III) $A^u \geq (A^{u/2}B^pA^{u/2})^{\frac{u}{p+u}}$ for all $p \geq 0$ and $u \geq 0$.

We recall that the Schatten q-norms are defined by

$$\|A\|_q = \left(\sum_j s_j^q(A)\right)^{1/q} \qquad \text{for } 1 \leq q \leq \infty,$$

where $s_j(A)$ are the singular values of the compact operator A arranged in decreasing order $s_1(A) \geq s_2(A) \geq \ldots$. When $q = \infty$, the norm $||A||_\infty$ coincides with the operator norm $||A|| = s_1$, the norm $||A||_2$ is called the Hilbert-Schmidt norm and $||A||_1$ is called the trace norm.

2 A CHARACTERIZATION OF CHAOTIC ORDER AND ITS APPLICATION TO RELATED NORM INEQUALITIES

THEOREM 2.1 *The following properties* (I) *and* (II) *are mutually equivalent:*

(I) $A \gg B$ *(i.e.,* $\log A \geq \log B$).

(II) *There exists the unique unitary operator* U_p *for all* $p \geq 0$ *such that* $U_p \longrightarrow I$ *as* $p \longrightarrow +0$, *and*

$$B^p \leq U_p A^p U_p^* \qquad \text{for all } p \geq 0.$$

Theorem 2.1 can be generalized as follows by scrutinizing our previous paper [15].

THEOREM 2.2 *Let* A *and* B *be positive invertible operators. Then the following properties* (I), (II), (III) *and* (IV) *are mutually equivalent:*

(I) $A \gg B$ *(i.e.,* $\log A \geq \log B$*).*

(II) *For* $p \geq u > 0$, $s \geq 1$, $\alpha \in [0, 1]$ *and* $\beta \geq -u\alpha$, *there exists the unique unitary operator* $U = U_{p,\beta,u\alpha,s}$ *such that* $U_{p,\beta,u\alpha,s} \longrightarrow I$ *as* p, β *and* $u\alpha \longrightarrow +0$, *and*

$$A^{\frac{\beta}{2}}(A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}} \leq U A^{(u\alpha+p)s+\beta} U^*.$$

(III) *For* $p \geq 0$, *and* $\beta \geq 0$, *there exists the unique unitary operator* $U = U_{p,\beta}$ *such that* $U_{p,\beta} \longrightarrow I$ *as* p *and* $\beta \longrightarrow +0$, *and*

$$A^{\frac{\beta}{2}} B^p A^{\frac{\beta}{2}} \leq U A^{p+\beta} U^*.$$

(IV) *For* $p \geq 0$, *there exists the unique unitary operator* $U = U_p$ *such that* $U_p \longrightarrow I$ *as* $p \longrightarrow +0$ *,and*

$$B^p \leq U A^p U^*.$$

COROLLARY 2.3 *Let* A *and* B *be positive invertible operators such that* $A \gg B$ (i.e., $\log A \geq \log B$). *Assume that* f *is a continuous increasing function such that* f *on* R_+ *with* $f(0) = 0$. *Let* $||S||_q$ *denote Schatten* q*-norm of an operator* S *for* $q \geq 1$.

(I) *For any* $p \geq u > 0$, $s \geq 1$ *and* $\alpha \in [0, 1]$,

$$||f\{A^{\frac{\beta}{2}}(A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}}\}||_q \leq ||f(A^{(u\alpha+p)s+\beta})||_q$$

holds for all $\beta \geq -u\alpha$.

(II) *For any* $p > 0$,

$$||f(A^{\frac{\beta}{2}} B^p A^{\frac{\beta}{2}})||_q \leq ||f(A^{p+\beta})||_q.$$

holds for all $\beta \geq 0$.

We need the following Lemmas in order to give proofs of the results.

LEMMA 2.1 *Let S be an invertible positive operator and let T be an invertible positive contraction. Then there exists the unique unitary operator $U = U_{S,T}$ such that*

(*) $TST \leq USU^*$.

U can be chosen to be I in (*) if and only if S commutes with T.

Proof of Lemma 2.1 Let $TS^{1/2} = U|TS^{1/2}|$ be the polar decomposition of an operator $TS^{1/2}$. Then U is uniquely determined unitary operator since S and T are invertible and $S^{1/2}T = |TS^{1/2}|U^*$. Therefore we have

$$TST = U|TS^{1/2}|^2U^* = US^{1/2}T^2S^{1/2}U^* \leq USU^* \quad \text{since } 0 < T \leq 1,$$

so that we have (*). Then $U = I \longleftrightarrow TS^{1/2} = |TS^{1/2}| \longleftrightarrow (TS^{1/2})^2 = S^{1/2}T^2S^{1/2} \longleftrightarrow TS^{1/2} = S^{1/2}T \longleftrightarrow TS = ST$.

Whence the proof of Lemma 2.1 is complete.

LEMMA 2.2 [12, 14]. *Let A and B be invertible positive operators. Then for any real number λ,*

$$(BAB)^\lambda = BA^{1/2}(A^{1/2}B^2A^{1/2})^{\lambda-1}A^{1/2}B.$$

Proof of Theorem 2.1

(I) $\Longrightarrow$ (II). By Theorem C, we recall that $A \gg B \Longleftrightarrow A^p \geq (A^{p/2}B^pA^{p/2})^{1/2}$ holds for all $p \geq 0 \Longleftrightarrow (B^{p/2}A^pB^{p/2})^{1/2} \geq B^p$ holds for all $p \geq 0$ since the last implication $\Longleftrightarrow$ easily follows by Lemma 2.2. Let $B^{p/2}A^{p/2} = U_pH_p$ be the polar decomposition of an operator $B^{p/2}A^{p/2}$, where $H_p = |B^{p/2}A^{p/2}| = (A^{p/2}B^pA^{p/2})^{1/2}$ and U_p is the unique unitary operator since A and B are both invertible. Then we have

$$\lim_{p \to +0} U_p = \lim_{p \to +0} \{B^{p/2}A^{p/2}(A^{p/2}B^pA^{p/2})^{-1/2}\} = I.$$

Then we obtain

$$\begin{aligned} B^p &\leq (B^{p/2}A^pB^{p/2})^{1/2} \\ &= (U_pH_p^2U_p^*)^{1/2} \\ &= U_pH_pU_p^* \\ &\leq U_pA^pU_p^* \text{ by Theorem C.} \end{aligned}$$

(II) $\Longrightarrow$ (I). As U_p is unitary operator for any $p \geq 0$ by (II), we have

$$\frac{U_p(A^p - I)U_p^*}{p} \geq \frac{B^p - I}{p},$$

tending $p \longrightarrow +0$, we have $\log A \geq \log B$ since $\lim\limits_{p\to+0} \dfrac{T^p - I}{p} = \log T$ for any positive operator T and $U_p \longrightarrow I$ as $p \longrightarrow +0$ by the hypothesis in (II).

Proof of Theorem 2.2.
(I) $\Longrightarrow$(II). (I). First of all, we recall the following (2.1) by Theorem C

$$A \gg B \text{ holds if and only if } A^u \geq (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}} \text{ for all } p \geq 0 \text{ and } u \geq 0. \tag{2.1}$$

Put $A_1 = A^u$ and $B_1 = (A^{\frac{u}{2}}B^pA^{\frac{u}{2}})^{\frac{u}{p+u}}$ in (2.1). Then $A_1 \geq B_1 \geq 0$ by (2.1). By Theorem B , for each $t \in [0, 1]$ and all $p \geq 0$ and $u \geq 0$,

$$A_1^{\frac{(p_1-t)s+r}{q}} \geq \{A_1^{\frac{r}{2}}(A_1^{\frac{-t}{2}}B_1^{p_1}A_1^{\frac{-t}{2}})^sA_1^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{2.2}$$

holds for any $s \geq 1, p_1 \geq 1, q \geq 1$ and $r \geq t$ with $(1-t+r)q \geq (p_1-t)s+r$.

Put $p_1 = \frac{p+u}{u} \geq 1$, $q = 2$ and also put $\alpha = 1 - t$ in (2.2), then for each $\alpha \in [0, 1]$ and all $p \geq 0$ and $u > 0$,

$$A^{\frac{(u\alpha+p)s+ur}{2}} \geq \{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{1}{2}} \tag{2.3}$$

holds for any $s \geq 1$ under the following conditions (2.4) and (2.5):

$$r \geq 1 - \alpha \tag{2.4}$$

$$2(\alpha + r)u \geq (u\alpha + p)s + ur. \tag{2.5}$$

If (2.5) holds, then we have the following inequality since $p \geq u > 0$ and $s \geq 1$

$$
\begin{aligned}
2(\alpha + r)u &\geq (u\alpha + p)s + ur \\
&\geq u\alpha + p + ur \\
&\geq u(\alpha + 1 + r)
\end{aligned}
$$

so that $\alpha + r \geq 1$, that is, (2.5) ensures (2.4) and therefore (2.3) holds under only the condition (2.5). Let β be defined by:

$$
\beta = \frac{ur - (u\alpha + p)s}{2}. \tag{2.6}
$$

Then (2.5) is equivalent to the following (2.7)

$$
\beta \geq -u\alpha. \tag{2.7}
$$

Let T be defined by

$$
T = A^{\frac{-(u\alpha+p)s-\beta}{2}} \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{1}{2}} A^{\frac{-(u\alpha+p)s-\beta}{2}}. \tag{2.8}
$$

It turns out that T is an invertible positive contraction by (2.3) and (2.6), and by (2.8) we have

$$
A^{\frac{(u\alpha+p)s+\beta}{2}} T A^{\frac{(u\alpha+p)s+\beta}{2}} = \{A^{\frac{ur}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{ur}{2}}\}^{\frac{1}{2}}. \tag{2.9}
$$

Taking square of both sides of (2.9) and refining via (2.6), we obtain

$$
T A^{(u\alpha+p)s+\beta} T = A^{\frac{\beta}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}}. \tag{2.10}
$$

An operator T in (2.8) can be written as $T = T_{p,\beta,u\alpha,s}$ since $ur = 2\beta + (u\alpha + p)s$ by (2.6). Put $S = S_{p,\beta,u\alpha,s} = A^{(u\alpha+p)s+\beta}$. Then $S = S_{p,\beta,u\alpha,s} \longrightarrow I$ as p, β and $u\alpha \longrightarrow +0$ and also $T = T_{p,\beta,u\alpha,s} \longrightarrow I$ as p, β and $u\alpha \longrightarrow +0$ by (2.8) and (2.6). Then by Lemma 2.1 and (2.10), there exists a unique unitary operator $U = U_{p,\beta,u\alpha,s}$ such that $U = U_{p,\beta,u\alpha,s} \longrightarrow I$ as p, β and $u\alpha \longrightarrow +0$, and $TST \leq USU^*$, that is,

$$
\begin{aligned}
A^{\frac{\beta}{2}} (A^{\frac{u\alpha}{2}} B^p A^{\frac{u\alpha}{2}})^s A^{\frac{\beta}{2}} &= T A^{(u\alpha+p)s+\beta} T \\
&\leq U A^{(u\alpha+p)s+\beta} U^*.
\end{aligned} \tag{2.11}
$$

Whence we obtain (II) under the conditions required.

(II) $\Longrightarrow$(III). Put $u\alpha = 0$ and $s = 1$ in (II) and also replace $p > 0$ by $p \geq 0$ by continuity of an operator.

(III)$\Longrightarrow$(IV). Put $\beta = 0$ in (III).

(IV)$\Longrightarrow$(I). (I) follows from (IV) by Theorem 2.1.

Whence the proof of Theorem 2.2 is complete.

Proof of Corollary 2.3.

Essentially we have only to follow the proof of [15, Theorem 1], but for the sake of completeness here we cite its proof.

(I) Applying Kosaki's nice technique [19] to (II) of Theorem 2.2, we obtain by [16, Lemma 1.1] and [16, (2.2) and (2.3)]

$$\mu_n\{A^{\frac{\beta}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{\beta}{2}}\} \leq \mu_n(U^*A^{(p-t)s+\beta}U) \leq \mu_n(A^{(p-t)s+\beta})$$

for $n = 1, 2...$, where $\{\mu_n(\cdot)\}_{n=1,2,...}$ are singular values, so that

$$\begin{aligned}\mu_n\{f[A^{\frac{\beta}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{\beta}{2}}]\} &= f\{\mu_n[A^{\frac{\beta}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{\beta}{2}}]\} \\ &\leq f\{\mu_n(A^{(p-t)s+\beta})\} = \mu_n\{f(A^{(p-t)s+\beta})\}\end{aligned}$$

and by summing up over n on Schatten q-norm for $q \geq 1$, then for any $p \geq u > 0$, $s \geq 1$ and $\alpha \in [0, 1]$,

$$||f\{A^{\frac{\beta}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{\beta}{2}}\}||_q \leq ||f(A^{(u\alpha+p)s+\beta})||_q$$

holds for all $\beta \geq -u\alpha$, that is, we obtain the desired estimate (I) of Corollary 2.3.

(II) We have only to put $u\alpha = 0$ and $s = 1$ in (I).

Whence the proof of Corollary 2.3 is complete.

3 PARALLEL RESULTS RELATED TO GENERALIZED FURUTA INEQUALITY

Very recently, Fujii, Jiang and Kamei [7] obtained very nice characterization of chaotic order and they also applied its results to the Furuta inequality.

In this chapter, as a continuation of [7] we shall obtain parallel results related to Theorem A which interpolates Theorem F and Ando-Hiai log majorization.

At first, we shall state the following two parallel results related to Theorem A.

THEOREM 3.1 *If* $\log A \geq \log B$ *, then for any* $\delta > 0$*, there exists an* $\alpha = \alpha_\delta \in (0, 1]$ *and for each* $t \in [0, \alpha]$ *and* $p \geq \alpha$*,*

$$F_{p,t}(A, B, r, s) = e^{\frac{(\alpha-t+r)ps\delta}{(p-t)s+r}} A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r *and* s *for any* $s \geq 1$ *and* $r \geq t$ *and* $A^{\alpha-t} \geq F_{p,t}(A, B, r, s)$ *holds, that is,*

$$e^{\frac{(\alpha-t+r)ps\delta}{(p-t)s+r}} A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}} \tag{3.1}$$

for any $s \geq 1$, $p \geq \alpha$ *and* $r \geq t$.

THEOREM 3.2 *If* $\log A > \log B$*, then there exists an* $\alpha \in (0, 1]$ *and for each* $t \in [0, \alpha]$ *and* $p \geq \alpha$*,*

$$G_{p,t}(A, B, r, s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r *and* s *for any* $s \geq 1$ *and* $r \geq t$ *and* $A^{\alpha-t} \geq G_{p,t}(A, B, r, s)$ *holds, that is,*

$$A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}} \tag{3.2}$$

for any $s \geq 1$, $p \geq \alpha$ *and* $r \geq t$.

As an immediate consequence of Theorem 3.2, we have the following corollary.

COROLLARY 3.3 *The following properties are mutually equivalent:*

(i) $\log A \geq \log B$.
(ii) *For any* $\delta > 0$*, there exists an* $\alpha = \alpha_\delta \in (0, 1]$ *and the following inequality holds for each* $t \in [0, \alpha]$*,*

$$e^{\frac{(\alpha-t+r)ps\delta}{(p-t)s+r}} A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

for any $s \geq 1$, $p \geq \alpha$ *and* $r \geq t$.

(iii) *For any* $\delta > 0$, *there exists an* $\alpha = \alpha_\delta \in (0, 1]$ *and the following inequality holds:*

$$e^{\frac{(\alpha+r)p\delta}{p+r}} A^{\alpha+r} \geq (A^{r/2}B^pA^{r/2})^{\frac{\alpha+r}{p+r}}$$

for any $p \geq \alpha$ *and* $r \geq 0$.

(iv) *For any* $\delta > 0$, *there exists an* $\alpha = \alpha_\delta \in (0, 1]$ *and the following inequality holds:*

$$e^{\frac{p\delta}{q}} A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{\frac{1}{q}}$$

for any $p \geq \alpha$, $r \geq 0$ *and* $q \geq 1$ *with* $(\alpha + r)q \geq p + r$.

(v) $A^{\frac{p+r}{q}} \geq (A^{r/2}B^pA^{r/2})^{\frac{1}{q}}$ *holds for any* $p \geq 1$, $r \geq 0$ *and* $q \geq 1$ *with* $rq \geq p + r$.

(vi) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$ *holds for any* $p \geq 1$ *and* $r \geq 0$.

(vii) $A^r \geq (A^{r/2}B^pA^{r/2})^{\frac{r}{p+r}}$ *holds for any* $p \geq 0$ *and* $r \geq 0$.

(viii) $A^r \geq (A^{r/2}B^rA^{r/2})^{\frac{1}{2}}$ *holds for any* $r \geq 0$.

We need the following nice results in order to give proofs of the results.

THEOREM D [7]. $\log A \geq \log B$ *holds if and only if for any* $\delta > 0$ *there exists an* $\alpha = \alpha_\delta \in (0, 1]$ *such that* $(e^\delta A)^\alpha > B^\alpha$.

THEOREM E [7]. $\log A > \log B$ *holds if and only if there exists an* $\alpha \in (0, 1]$ *such that* $A^\alpha > B^\alpha$.

Proof of Theorem 3.1 $\log A \geq \log B$ holds if and only if for any $\delta > 0$ there exists an $\alpha = \alpha_\delta \in (0, 1]$ such that $A^\alpha > (e^{-\delta}B)^\alpha$ by Theorem D. Put $A_1 = A^\alpha$ and $B_1 = (e^{-\delta}B)^\alpha$. As $A_1 > B_1$ holds by the hypothesis, Theorem A ensures that for each $t_1 \in [0, 1]$ and $p_1 \geq 1$

$$D_{p_1,t_1}(A_1, B_1, r_1, s) = A_1^{-r_1/2}\{A_1^{r_1/2}(A_1^{-t_1/2}B_1^{p_1}A_1^{-t_1/2})^sA_1^{r_1/2}\}^{\frac{1-t_1+r_1}{(p_1-t_1)s+r_1}}A_1^{-r_1/2} \tag{3.3}$$

is a decreasing function of both r_1 and s for any $s \geq 1$ and $r_1 \geq t_1$, and the following inequality holds:

$$A_1^{1-t_1} = D_{p_1,t_1}(A_1, A_1, r_1, s) \geq D_{p_1,t_1}(A_1, B_1, r_1, s) \tag{3.4}$$

for any $s \geq 1$, $p_1 \geq 1$ and $r_1 \geq t_1$. Put $r_1 = \dfrac{r}{\alpha}$, $t_1 = \dfrac{t}{\alpha}$ and $p_1 = \dfrac{p}{\alpha}$. Then $p_1 \geq 1$, $t_1 \in [0, 1]$ and $r_1 \geq t_1$ since $t \in [0, \alpha]$, $p \geq \alpha$ and $r \geq t$ by the hypothesis and

$$\frac{1-t_1+r_1}{(p_1-t_1)s+r_1}=\frac{\alpha-t+r}{(p-t)s+r}. \tag{3.5}$$

By (3.3), (3.4) and (3.5),

$$F_{p,t}(A,B,r,s)=e^{\frac{-(\alpha-t+r)ps\delta}{(p-t)s+r}}A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for any $s\geq 1$ and $r\geq t$ and $A^{\alpha-t}\geq F_{p,t}(A,B,r,s)$ holds, that is,

$$e^{\frac{(\alpha-t+r)ps\delta}{(p-t)s+r}}A^{\alpha-t+r}\geq\{A^{r/2}(A^{-t/2}B^pA^{-t/2})A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $s\geq 1$, $p\geq\alpha$ and $r\geq t$.

Whence the proof of Theorem 3.1 is complete.

Proof of Theorem 3.2 By the same way as one in the proof of Theorem 3.1, we can give a proof of Theorem 3.2 by Theorem E and Theorem A as follows.

$\log A>\log B$ holds if and only if there exists an $\alpha\in(0,1]$ such that $A^\alpha>B^\alpha$ by Theorem E. Put $A_1=A^\alpha$ and $B_1=B^\alpha$. As $A_1>B_1$ holds by the hypothesis, Theorem A ensures that for each $t_1\in[0,1]$ and $p_1\geq 1$

$$D_{p_1,t_1}(A_1,B_1,r_1,s)=A_1^{-r_1/2}\{A_1^{r_1/2}(A_1^{-t_1/2}B_1^{p_1}A_1^{-t_1/2})^sA_1^{r_1/2}\}^{\frac{1-t_1+r_1}{(p_1-t_1)s+r_1}}A_1^{-r_1/2} \tag{3.6}$$

is a decreasing function of both r_1 and s for any $s\geq 1$ and $r_1\geq t_1$, and the following inequality holds:

$$A_1^{1-t_1}=D_{p_1,t_1}(A_1,A_1,r_1,s)\geq D_{p_1,t_1}(A_1,B_1,r_1,s) \tag{3.7}$$

for any $s\geq 1$, $p_1\geq 1$ and $r_1\geq t_1$. Put $r_1=\frac{r}{\alpha}$, $t_1=\frac{t}{\alpha}$ and $p_1=\frac{p}{\alpha}$. Then $p_1\geq 1$, $t_1\in[0,1]$ and $r_1\geq t_1$ since $t\in[0,\alpha]$, $p\geq\alpha$ and $r\geq t$ by the hypothesis and

$$\frac{1-t_1+r_1}{(p_1-t_1)s+r_1}=\frac{\alpha-t+r}{(p-t)s+r}. \tag{3.8}$$

By (3.6), (3.7) and (3.8),

$$G_{p,t}(A,B,r,s)=A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for any $s \geq 1$ and $r \geq t$ and $A^{\alpha-t} \geq G_{p,t}(A, B, r, s)$ holds, that is,

$$A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$, $p \geq \alpha$ and $r \geq t$.

Whence the proof of Theorem 3.2 is complete.

Proof of Corollary 3.3.

(i) $\Longrightarrow$ (ii). Obtained in (3.1) of Theorem 3.1.
(ii) $\Longrightarrow$ (iii). We have only to put $t = 0$ in (ii) and replace ps by p since $s \geq 1$ and $p \geq \alpha$.
(iii) $\Longleftrightarrow$ (iv). Obvious by Löwner-Heinz inequality.
(iii) $\Longrightarrow$ (vi). Taking $\dfrac{r}{\alpha+r}$ as exponents of both sides of (iii),

$$e^{\frac{r\delta p}{p+r}}A^{r} \geq (A^{r/2}B^{p}A^{r/2})^{\frac{r}{p+r}}$$

holds for $p \geq 1$ and $r \geq 0$, then letting $\delta \longrightarrow 0$, so that we have (vi).
(vi) $\Longleftrightarrow$ (v). Obvious by Löwner-Heinz inequality.
(vi) $\Longrightarrow$ (i). Taking logarithm both sides of (vi) and letting $r \longrightarrow 0$, then we have $logA \geq logB$ since $p \geq 1$.
(i) $\Longleftrightarrow$ (viii) is shown in [1].
(vii) $\Longleftrightarrow$ (viii) is shown in [5, 13].

Whence the proof of Corollary 3.3 is complete.

At the end of this chapter, we cite the following four parallel results (i), (ii), (iii) and (iv) in Remark 3.4 related to Theorem A. In fact (i) is shown by Theorem A, and (ii) is obtained by the same way as one of Theorem 3.2 and also (iii) is shown by Theorem 3.2 and finally (iv) is already obtained by Corollary 3.3.

Remark 3.4 Let A and B be invertible positive operators. Then the following four parallel results hold;

(i) $A \geq B \Longleftrightarrow$ *for each* $t \in [0, 1]$, *and* $p \geq 1$,

$$A^{1-t+r} \geq \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$, *and* $r \geq t$.

(ii) $A^{\alpha} \geq B^{\alpha}$ *holds for some* $\alpha \in (0, 1] \Longleftrightarrow$ *for some* $\alpha \in (0, 1]$, *and for each* $t \in [0, \alpha]$ *and* $p \geq \alpha$,

$$A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$, *and* $r \geq t$.

(iii) $\log A > \log B \Longrightarrow$ *there exists an* $\alpha \in (0, 1]$ *and for each* $t \in [0, \alpha]$ *and* $p \geq \alpha$,

$$A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$, *and* $r \geq t$.

(iv) $\log A \geq \log B \Longleftrightarrow$ *for any* $\delta > 0$, *there exists an* $\alpha = \alpha_{\delta} \in (0, 1]$ *and for each* $t \in [0, \alpha]$ *and* $p \geq \alpha$,

$$e^{\frac{(\alpha-t+r)ps\delta}{(p-t)s+r}}A^{\alpha-t+r} \geq \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $s \geq 1$, *and* $r \geq t$.

It is interesting to point out that there exists a contrast among (i), (ii), (iii) and (iv) in Remark 3.4, that is, as $logt$ is operator monotone function, the corresponding result equivalent to $logA \geq logB$ is somewhat weaker than the corresponding one equivalent to $A \geq B$.

We remark that (ii) in Remark 3.4 in case $t = 0$ is obtained in [8, Theorem 9].

References

[1] T. Ando, On some operator inequality, *Math. Ann.*, **279** (1987), 157–159.

[2] T. Ando and F. Hiai, Log-majorization and complementary Golden-Thompson type inequalities, *Linear Alg. and Its Appl.*, **197**, **198** (1994), 113–131.

[3] M. Fujii, Furuta's inequality and its mean theoretic approach, *J. Operator Theory*, **23** (1990), 67–72.

[4] M. Fujii, T. Furuta and E. Kamei, Operator functions associated with Furuta's inequality, *Linear Alg. and Its Appl.*, **149** (1991), 91–96.

[5] M. Fujii, T. Furuta and E. Kamei, Furuta's inequality and its application to Ando's theorem, *Linear Alg. and Its Appl.*, **179** (1993), 161–169.

[6] M. Fujii and E. Kamei, Mean theoretic approach to the grand Furuta inequality, to appear in *Proc. Amer. Math. Soc.*.

[7] M. Fujii, J.F. Jiang and E. Kamei, *Characterization of chaotic order and its application to Furuta inequality*, preprint.

[8] M. Fujii, J.F. Jiang and E. Kamei, *A Characterization of ordered defined by* $A^{\delta} \geq B^{\delta}$ *via Furuta inequality*, preprint.

[9] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.*, **101** (1987), 85–88.

[10] T. Furuta, A proof via operator means of an order preserving inequality, *Linear Alg. and Its Appl.*, **113** (1989), 129–130.

[11] T. Furuta, Elementary proof of an order preserving inequality, *Proc. Japan Acad.*, **65** (1989), 126.

[12] T. Furuta, Two operator functions with monotone property, *Proc. Amer. Math. Soc.*, **111** (1991), 511–516.

[13] T. Furuta, Applications of order preserving operator inequalities, *Operator Theory: Advances and Applications*, **59** (1992), 180–190.

[14] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization, *Linear Alg. and Its Appl.*, **219** (1995), 139–155.

[15] T. Furuta, Generalizations of Kosaki trace inequalities and related trace inequalities on chaotic order, *Linear Alg. and Its Appl.*, **235** (1996), 153–161.

[16] I. G. Gohberg and M. G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl, Math. Monographs, Vol. **18**, Amer. Math. Soc., 1969.

[17] E. Heinz, Beiträge zur Störungstheorie der Spektralzerlegung, *Math. Ann.*, **123** (1951), 415–438.

[18] E. Kamei, A satellite to Furuta's inequality, *Math. Japon*, **33** (1988), 883–886.

[19] H. Kosaki, On some trace inequality, *Proc. of the Center for Mathematical Analysis, Australian National University*, (1991), 129–134.

[20] K. Löwner, Über monotone Matrixfunktionen, *Math. Z.*, **38** (1934), 177–216.

[21] K. Tanahashi, Best possibility of the Furuta inequality, *Proc. Amer. Math. Soc.*, **124** (1996), 141–146.