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Radial Solutions of Equations and Inequalities Involving the p-Laplacian

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Several problems for the differential equation

$$L_p^\alpha u = g(r, u) \quad \text{with} \quad L_p^\alpha u = r^{-\alpha}(r^\alpha |u'|^{p-2}u')'$$

are considered. For $\alpha = N-1$, the operator L_p^α is the radially symmetric p-Laplacian in $\mathbb{R}^N$. For the initial value problem with given data $u(r_0) = u_0$, $u'(r_0) = u_0'$ various uniqueness conditions and counterexamples to uniqueness are given. For the case where g is increasing in u, a sharp comparison theorem is established; it leads to maximal solutions, nonuniqueness and uniqueness results, among others. Using these results, a strong comparison principle for the boundary value problem and a number of properties of blow-up solutions are proved under weak assumptions on the nonlinearity $g(r, u)$.

Keywords: p-Laplacian; radial solutions; uniqueness; comparison principle; blow-up solutions.

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1 INTRODUCTION

This work is devoted to the study of the nonlinear second order operator

$$L_p^\alpha u = r^{-\alpha}(r^\alpha |u'|^{p-2}u')' = |u'|^{p-2}\left((p-1)u'' + \frac{\alpha}{r}u'\right) \tag{1}$$

and to initial and boundary value problems for equations of the form

$$L_p^\alpha u = f(u) \quad \text{and} \quad L_p^\alpha u = g(r, u).$$

It is always assumed that $p > 1$ and $\alpha \geq 0$. For a function u depending only on $r = |x|$, $x \in \mathbb{R}^N$, the operator L_p^{N-1} is the p-Laplacian $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ in $\mathbb{R}^N$; in particular, $L_2^{N-1}u = u'' + (N-1)u'/r$ is the radial Laplacian (we use the same letter u as a function of $x \in \mathbb{R}^N$ and as a function of $r = |x| \in \mathbb{R}$). In the linear case $p = 2$ we simply write L^α in place of L_2^α. With this notation,

$$L_p^\alpha u = (p-1)|u'|^{p-2} L^{\alpha'} u, \quad \text{where} \quad \alpha' = \alpha/(p-1).$$

A description of the contents of the paper follows. In the theorems the nonlinearity is always of the form $g(r, u)$, but in this overview we formulate some of the results only for the special case $f(u)$.

The first significant new result is given in Theorem 2. It states that the initial value problem

$$L_p^\alpha u = f(u), \; u(r_0) = u_0, \; u'(r_0) = u_0' \tag{2}$$

is uniquely solvable if f is merely continuous, at least in the case $u_0' \neq 0$, $r_0 > 0$ and also in some cases where $r_0 = 0$, $u_0' = 0$. The consequences can be summed up in the statement that the usual assumption that f belongs to C^1 can often be replaced by continuity of f. Uniqueness for the general initial value problem (3) is a subtle problem. This becomes already manifest in the simple "p-linear" equation $L_p^\alpha u + k(r)u^{(p-1)} = h(r)$. In the homogeneous case $h \equiv 0$ the initial value problem is always uniquely solvable (cf. [13]), whereas in the inhomogeneous case this is not true, see Section 2. An extensive list of uniqueness conditions is given in Section 2, together with examples of non-uniqueness. Theorem 3 is a refined version of a comparison theorem for problem (3), where $g(r, u)$ is increasing in u. It gives rise to maximal and minimal solutions, equipped with classical properties. Section 3 contains a strong comparison theorem for the boundary value problem *without* the usual hypothesis of non-vanishing gradients; e.g., if $1 < p \leq 2$ and $g(r, u)$ is locally Lipschitzian and (weakly) increasing in u, then strong comparison holds. In Section 4 blow-up problems of the form

$$L_p^\alpha u = g(r, u), \quad u(r) \to \infty \text{ as } r \to R$$

are discussed. Using Corollary (e) of Theorem 3, it can be shown that the asymptote of a blow-up solution of (3) depends continuously and *strictly monotone* on u_0 and u_0'. This has immediate consequences on the uniqueness of radial blow-up solutions of $\Delta_p u = f(u)$ in a ball in $\mathbb{R}^N$. These results are obtained under weak assumptions on f and g; in particular, differentiability is not required. Both for the strong comparison theorem and the blow-up problem extensive use is made of earlier results on the initial value problem.

Our results apply also to radial and convex C^2-solutions of Monge-Ampère equations $det\, D^2u = \tilde{g}(|x|, u)$, since they satisfy $u''(u'/r)^{N-1} = \tilde{g}(r, u)$, i.e.,

$$L_N^0 = g(r, u), \quad u'(0) = 0 \text{ with } g(r, s) = r^{N-1}\tilde{g}(r, s).$$

NOTATION For simplicity, we write the odd power function in the form $s^{(q)} = |s|^{q-1}s = |s|^q sign\colon s$ (q real); it has the properties

$$s^{(q)}t^{(q)} = (st)^{(q)}, \quad 1/s^{(q)} = (1/s)^{(q)}, \quad (-s)^{(q)} = -s^{(q)},$$

$$|s|^{q_1}s^{(q_2)} = s^{(q_1+q_2)}, \quad \frac{d}{ds}s^{(q)} = q|s|^{q-1}, \quad \frac{d}{ds}|s|^q = qs^{(q-1)}.$$

The inverse function of $s^{(q)}$ is $s^{(1/q)}$.

Monotonicity is used in the weak sense, i.e., f is increasing if $u < v$ implies $f(u) \le f(v)$, and strictly increasing if $u < v$ implies $f(u) < f(v)$.

For a solution u in an interval $J \subset [0, \infty)$ we require that u and $r^\alpha u'^{(p-1)}$ belong to $C^1(J)$; this implies that u'' is continuous as long as $u' \neq 0$.

2 EXISTENCE, UNIQUENESS, CONTINUOUS DEPENDENCE

For the reader's convenience we state and prove an existence theorem of Peano type for the initial value problem

$$L_p^\alpha u = g(r, u), \; u(r_0) = u_0, \; u'(r_0) = u_0'. \tag{3}$$

THEOREM 1 (Existence). *Assume that $g(r, s)$ is continuous and bounded in the strip $S = J \times \mathbb{R}$, where $J = [0, b]$ in the case $r_0 = 0$ and $J = [a, b]$ in the case $0 < a \le r_0 \le b$. Then the initial value problem* (3) *has – under the provision that $u_0' = 0$ in the case $r_0 = 0$ – a solution existing in J.*

COROLLARY *Assume that g is continuous in G, where G is a relatively open subset of* $[0, \infty) \times \mathbb{R}$, *and that* $(r_0, u_0) \in G$. *Then problem* (3) *has a local solution $u(r)$ in some interval. It can be extended (as a solution) to a maximal interval of existence* $[0, \beta_+)$ *or* (β_-, β_+) *with* $0 \le \beta_- < \beta_+ \le \infty$, *where the second case applies only if $r_0 > 0$; the extended solution tends to the boundary of G as $r \to \beta_-$ and $r \to \beta_+$.*

Proof It follows from (3) that

$$r^{\alpha} u'(r)^{(p-1)} - r_0^{\alpha} u_0'^{(p-1)} = \int_{r_0}^{r} \rho^{\alpha} g(\rho, u(\rho))\, d\rho. \tag{3'}$$

Hence problem (3) is equivalent to the fixed point equation $u = Su$, where

$$(Su)(r) = u_0 + \int_{r_0}^{r} \left\{ \left(\frac{r_0}{t}\right)^{\alpha} u_0'^{(p-1)} + \int_{r_0}^{t} \left(\frac{\rho}{t}\right)^{\alpha} g(\rho, u(\rho))\, d\rho \right\}^{\left(\frac{1}{p-1}\right)} dt. \tag{3''}$$

We apply Schauder's fixed point theorem in the Banach space $X = C^0(J)$. Obviously, S maps X into itself and is continuous in the maximum norm, i.e., $u_k \to u$ uniformly in J implies $Su_k \to Su$ uniformly in J. Furthermore, since g and the functions $(r_0/t)^{\alpha}$ and $(\rho/t)^{\alpha}$ are bounded, $|(Su)'| \le K$ for $u \in X$ and $r \in J$. Hence $S(X)$ is a relatively compact subset of X, and Schauder's theorem shows that a fixed point exists. The corollary is derived in a standard way from Theorem 1. ∎

THEOREM 2 (Uniqueness). *Assume that $G \subset S = [0, \infty) \times \mathbb{R}$ is relatively open in S and $g(r, s)$ is continuous in G and locally Lipschitzian with respect to s or r. If $(r_0, u_0) \in G$ and $r_0 > 0$, $u_0' \neq 0$, then problem* (3) *has a unique local solution. The extension $u(r)$ remains unique as long as $u'(r) \neq 0$.*

Proof If $g(r, s)$ is locally Lipschitzian in s, notice that as long as $u' \neq 0$ the differential equation can be written in the form $u'' = \tilde{g}(r, u, u')$ where $\tilde{g}(r, s, s')$ is locally Lipschitzian in s, s' in $G \times (\mathbb{R} \setminus \{0\})$. Uniqueness then follows form a well known classical theorem. Now let g be locally Lipschitzian in r. A solution u satisfies $u'(r_0) \neq 0$; therefore it has an inverse function $r(u)$ of class C^2 in a neighborhood of u_0. It follows from

$$u'r' = 1, \quad u''r'^2 + u'r'' = 0 \quad \text{and} \quad r > 0,$$

where $r' = dr(u)/du$, $u' = u'(r(u)), \ldots$, that $r(u)$ is a solution of the initial value problem

$$(p-1)r'' = \frac{\alpha}{r} r'^2 - r'^{(p+1)} g(r(u), u), \quad r(u_0) = r_0, \quad r'(u_0) = 1/u_0'.$$

Since the right hand side of the differential equation is locally Lipschitzian in r as long as $|r'| > 0$, the theorem follows. ∎

It is well known that existence and uniqueness imply continuous dependence on the initial data. We formulate this result for problem (3), using the notation $u(r; r_0, u_0, u_0')$ for a solution of (3).

COROLLARY *Let g be as in Theorem 2 and let $u(r) = u(r; r_0, u_0, u_0')$ be a solution in a compact interval $I = [a, b]$, where $0 < a \le r_0 \le b$ and $u' \ne 0$ in I. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that for $|r_0 - \tilde{r}_0| < \delta$, $\tilde{r}_0 \in I$, $|u_0 - \tilde{u}_0| < \delta$, $|u_0' - \tilde{u}_0'| < \delta$ the solution $\tilde{u}(r) = u(r; \tilde{r}_0, \tilde{u}_0, \tilde{u}_0')$ exists in I and is uniquely determined, and $|u(r) - \tilde{u}(r)| < \epsilon$, $|u'(r) - \tilde{u}'(r)| < \epsilon$ and $\tilde{u}' \ne 0$ in I.*

For proof, one changes $g(r, s)$ outside a neighbourhood N of the solution u in such a way that g becomes bounded and continuous in $I \times \mathbb{R}$; one may take $N = \{(r, s) : r \in I, |s - u(r)| \le \gamma\} \subset G$. Then the set of all solutions $u(r; \tilde{r}_0, \tilde{u}_0, \tilde{u}_0')$, where the parameters satisfy the above inequalities with $\delta = 1$, is a relatively compact subset of $X = C(I)$ (every solution exists in I). For every sequence $(r_0^k, u_0^k, u_0'^k) \to (r_0, u_0, u_0')$ the corresponding sequence (u_k) of solutions has a uniformly convergent subsequence with limit u, and it follows from (3′) that the sequence of derivatives converges uniformly to u'. Let $\tilde{\lambda} = (\tilde{r}_0, \tilde{u}_0, \tilde{u}_0')$ and $\lambda = (r_0, u_0, u_0')$. Then $(u(r; \tilde{\lambda})), u'(r; \tilde{\lambda})) \to (u(r), u'(r))$ uniformly in I as $\tilde{\lambda} \to \lambda$. The rest is easy. ■

In the next theorem we use the notation $v(a+) < w(a+)$ (or $v < w$ at $a+$) if there exists $\epsilon > 0$ such that $v < w$ in $(a, a + \epsilon)$. For $v, w \in C^1$, this relation holds if $v(a) < w(a)$ or if $v(a) = w(a)$ and $v'(a) < w'(a)$.

THEOREM 3 (Comparison). *Let $I = [a, b]$ and $I_0 = (a, b]$ $(0 \le a < b)$. Assume $v, w \in C^1(I)$ with $v'^{(p-1)}, w'^{(p-1)} \in C^1(I_0)$ satisfy*

$$v(a+) < w(a+), \quad v'(a) \le w'(a), \quad L_p^\alpha v - g(r, v) \le L_p^\alpha w - g(r, w) \quad \text{in} \quad I_0,$$

where $g(r, s)$ is increasing in s. Then

$$v' \le w' \quad \text{in} \quad I, \quad \text{which implies} \quad v < w \quad \text{in} \quad I_0.$$

If (i) *g is strictly increasing in s or* (ii) *$v' < w'$ at $a+$ or* (iii) *the differential inequality is strict at $a+$, then $v' < w'$ in I_0.*

The theorem remains true in the case $I = [b, a]$ $(0 < b < a)$ is an interval to the left of a if the inequalities involving v', w' are reversed and I_0 is the interval $[b, a)$: The differential inequality and $v(a-) < w(a-)$, $v'(a) \ge w'(a)$ imply $v' \ge w'$ and $v < w$ in I_0, and the cases (ii), (iii) *have to be changed accordingly.*

Proof Let $v \le w$ in $I' = [a, c]$, where c is maximal. Then

$$[r^\alpha(w'^{(p-1)} - v'^{(p-1)})]' \ge r^\alpha[g(r, w) - g(r, v)] \ge 0 \quad \text{in} \quad I'. \qquad (*)$$

It follows that $w' - v' \geq 0$ in I' which implies that $w - v$ is positive and increasing in $(a, c]$. This shows that $c = b$. In each of the cases (i)–(iii) the first term in $(*)$ is positive at $a+$, which gives $w' > v'$ in I_0. ■

Remark It is clear that in the case of nonuniqueness of problem (3) the Comparison Theorem cannot hold if in all inequalities of the assumption equality is permitted. But it is remarkable that a strict inequality in one place ($v < w$ at $a+$) suffices without any conditions on g except monotonicity.

COROLLARIES In the following propositions (a)–(h) it is always assumed that $g(r, s)$ is continuous and increasing in s on a set $G \subset S = I \times \mathbb{R}$ which is relatively open in S, and that u, v, w (with graphs belonging to G) satisfy the smoothness assumptions of the theorem; as before, $I = [a, b]$ and $I_0 = (a, b]$. The initial value problem (3_a) is the problem (3) with $r_0 = a$.

Similar propositions hold also to the left of $a > 0$, where $I = [b, a]$ and $I_0 = [b, a)$; an explicit formulation is only given in those cases where the necessary changes are not obvious.

(a) **Upper and lower solutions.** If w satisfies the inequality $L_p^\alpha w \geq g(r, w)$ in I_0, then w is called an *upper solution* (or *supersolution*) to the differential equation $L_p^\alpha u = g(r, u)$; it is an upper solution to the initial value problem (3_a) if, in addition, $w(a+) > u_0$, $w'(a) \geq u_0'$. These inequalities imply that $w > u$ and $w' \geq u'$ in I_0, where $u = u(r; a, u_0, u_0')$. A *lower solution* (*subsolution*) v is defined similarly, with inequalities reversed.

(b) **Maximal and minimal solutions.** Problem (3_a) has a maximal solution $\bar{u} = \bar{u}(r; a, u_0, u_1)$ in a maximal interval of existence $[a, \bar{c})$ $(\bar{c} \leq b)$ or $[a, b]$ and a minimal solution $\underline{u} = \underline{u}(r; a, u_0, u_0')$ in a maximal interval $[a, \underline{c})$ $(\underline{c} \leq b)$ or $[a, b]$. For every other solution u of (3_a) the inequalities $\underline{u} \leq u \leq \bar{u}$, $\underline{u}' \leq u' \leq \bar{u}'$ hold in the interval of existence of both $\bar{u}$ and $\underline{u}$. The maximal solution $\bar{u}$ can be obtained as the limit of the sequence of solutions $u_k(r) = u(r; a, u_0 + 1/k, u_0')$, which is strictly decreasing (this follows from Theorem 3). A similar proposition holds for the minimal solution.

(c) **Comparison with maximal and minimal solutions.** *If w satisfies*

$$L_p^\alpha w \geq g(r, w), \quad w(a) \geq u_0, \quad w'(a) \geq u_0',$$

then $w \geq \underline{u}$ and $w' \geq \underline{u}'$, where $\underline{u} = \underline{u}(r; a, u_0, u_0')$. In particular, if problem (3_a) has a unique solution $u = u(r; a, u_0, u_0')$, then $w \geq u$, $w' \geq u'$. In this case w (with the above properties) is also called an upper solution for the initial value problem. There are again corresponding statements for lower solutions v of (3_a) and "to the left".

This follows from (b) and Theorem 3, applied to w and $u(r; a, u_0 \mp 1/k, u_0')$.

We now describe two techniques which generate upper and lower solutions from a solution. The first one applies in the case where $g(r, s)$ is increasing in r, while the second one requires some kind of Lipschitz continuity in r.

(d) **Shift of solutions.** *Assume that u satisfies the smoothness assumptions in an interval* $I = [a, b]$ $(a \geq 0)$ *and* $u' > 0$ *in* $I_0 = (a, b]$. *Then the function* $u_\delta(r) := u(r + \delta)$ *is defined in* $I_\delta = I - \delta$, *and*

$$(L_p^\alpha u_\delta)(r) > (L_p^\alpha u)(r + \delta) \quad \text{for} \quad \delta > 0 \quad \text{and}$$

$$(L_p^\alpha u_\delta)(r) < (L_p^\alpha)u(r + \delta) \quad \text{for} \quad \delta < 0 \; (r \in I_0 \cap I_\delta). \tag{4}$$

If u is a solution of $L_p^\alpha u = g(r, u)$ and $u' > 0$ in I, where g is increasing in (s and) r, then u_δ is a super- or subsolution to the differential equation in the case $\delta > 0$ or $\delta < 0$, resp.

(e) **Supersolutions by substitution.** *Let* $\phi(r)$ *be of class* C^2 *and* $w(r) = u(\phi(r))$. *Then*

$$w' = u'\phi' \quad \text{and} \quad w'' = u''\phi'^2 + u'\phi''$$

with $\phi' = \phi'(r)$, $u' = u'(\phi(r))$, ... *This implies*

$$L_p^\alpha w = (p-1)|u'\phi'|^{p-2}\left(u''\phi'^2 + u'\phi'' + \frac{\alpha'}{r}u'\phi'\right), \tag{5}$$

with $\alpha' = \alpha/(p-1)$ *and*

$$Dw := L_p^\alpha w - (L_p^\alpha u)(\phi(r)) = (p-1)|u'|^{p-2} \times$$

$$\left(u''(|\phi'|^p - 1) + u'|\phi'|^{p-2}\phi'' + \alpha' u'\left(\frac{\phi'^{(p-1)}}{r} - \frac{1}{\phi(r)}\right)\right). \tag{6}$$

This formula will be used in the generation of a supersolution from a solution u of $L_p^\alpha u = g(r, u)$:

$$L_p^\alpha w \geq g(r, w) \Leftrightarrow Dw \geq g(r, w) - g(\phi(r), w). \tag{7}$$

The same equivalence holds for $\leq$ (in both places) and for strict inequalities.

(f) **Uniqueness under the condition $\mathbf{u_0' = u'(a) = 0}$ $\mathbf{(a \geq 0)}$.** *The solution $u(r; a, u_0, 0)$ of (3_a) is unique in a neighbourhood of a if $g(a, u_0) > 0$, and, in addition,*

(i) *$g \in Lip^-(r)$ for uniqueness to the right,*
(ii) *$g \in Lip^+(r)$ for uniqueness to the left ($a > 0$).*

A function $\psi(r)$ belongs to Lip^+ or Lip^- if the difference quotient $[\psi(r_2) - \psi(r_1)]/(r_2 - r_1)$ is bounded above or below, resp. Obviously, increasing or decreasing functions belong to Lip^- or Lip^+, resp., and $Lip = Lip^+ \cap Lip^-$.

(g) **Uniqueness of solutions with one sign.** *We consider a solution $u(r; a, u_0, u_0')$ of (3_a) with $u_0, u_0' \geq 0 (\leq 0)$ and, in case both initial values vanish, $g(a, 0) > 0 (< 0)$. The solution is unique in a neighbourhood of a if furthermore $g(r, s)/s^{p-1}$ is decreasing (increasing) in s for $s > 0$ ($s < 0$) if $u > 0$ ($u < 0$) in $(a, a + \epsilon]$. For uniqueness in an interval to the left of a the proposition remains valid if the inequality for u_0' is reversed.*

(h) **Maximal solutions in the case $\mathbf{u_0' = 0}$.** *Assume that $g(r, s)$ is increasing in r (e.g., $g(r, s) = f(s)$) and that $g(r, u_0) = 0$ for $a \leq r \leq b$. Then $u \equiv u_0$ is a solution. Assume that there exists a subsolution which is $> u_0$ in $(a, b]$. Then, for $a \leq r_0 < b$, the maximal solution $\bar{u}(r; r_0, u_0, 0)$ and no other solution is $> u_0$ for $r > r_0$. Under these assumptions, all solutions of (3_a) are given by*

$$u(r) = u_0 \quad \text{in} \quad [a, r_0], \quad u(r) = \overline{u}(r; r_0, u_0, 0) \quad \text{in} \quad [r_0, b] \quad (a \leq r_0 \leq b).$$

They fill the area between the curves $s = u_0$ and $s = \bar{u}(r; a, u_0, 0)$ in the r, s-space. When I is an interval $[b, a]$ to the left of a, then g must be decreasing in r, and r_0 satisfies $b \leq r_0 \leq a$.

EXAMPLE If $g(r, s) \geq K s^q$ for $s \geq 0$ and $0 < q < p-1$, then the statement in (h) applies for $u_0 = u_0' = 0$ and $a \geq 0$. For $\mu > (q + 1)/(p - q - 1)$ the function $(r - a)^{\mu+1}$ is a subsolution to the right of a and the function $(a - r)^{\mu+1}$ is a subsolution to the left of a (in case $a > 0$) for the initial value problem $L_p^\alpha u = K u^q$, $u_0 = u_0' = 0$ in a small one-sided neighbourhood of a. The functions are positive to the right or left of a, resp.

Remarks 1. The statement (f)(i) fails to be true under the weaker assumption that $g(r, u_0)$ is positive only in $I_0 = (a, b]$. The initial value problem

$$u'' = 12\sqrt{2}\left(u + \frac{1}{2}r^4\right)^{(1/2)}, u(0) = u'(0) = 0$$

with the solutions $u = -r^4$ and $u = (1 + \sqrt{2})r^4$ is a counterexample.
2. It is not clear whether the statement in (h) about the characterization of the maximal solution as the unique positive solution remains true if the monotonicity of $g(r, s)$ in r is replaced by a local Lipschitz condition in r. A simple counterexample to the assertion in (h) is

$$u'' = g(r, u), \quad \text{where} \quad g(r, s) = \begin{cases} 6r & \text{for } s > r^3 \\ 6s/r^2 & \text{for } 0 \le s \le r^3 \\ 0 & \text{for } s < 0. \end{cases}$$

Solutions are $u(r) = \lambda r^3$, $0 \le \lambda \le 1$; the function g is continuous in $[0, \infty) \times \mathbb{R}$, but not Lipschitzian in r.
3. The example $u'' = 12u^{(1/2)}$ with the three solutions $u(r) = r^4$, 0 and $-r^4$ shows that under the assumptions in (h) the solutions $u(r) \equiv u_0$ is in general not the minimal solution.

Proof (a)–(e) are simple. (f)(i) We consider the case $a = 0$ and use the notation $v_1(r) \approx v_2(r)$ if $v_1/v_2 \to 1$ as $r \to 0$. We use (e) with $\phi(r) = r + \epsilon(\delta + r)$, where $\epsilon, \delta > 0$. Let $g(0, u_0) = \gamma > 0$. Then, for every solution u, $(r^\alpha u'^{(p-1)})' \approx r^\alpha \gamma$, hence $\frac{1}{r}u'^{(p-1)} \approx \frac{\gamma}{\alpha+1}$. Since $L_p^\alpha u \approx \gamma$, it follows from (1) that $(p-1)|u'|^{p-2}u'' \approx \frac{\gamma}{\alpha+1}$; hence there is $c > 0$ such that

$$(p-1)|u'|^{p-2}u'' \ge \frac{\gamma}{2(\alpha+1)} := \gamma_1 \quad \text{in} \quad [0, c].$$

The expression Dw in (6) consists of three terms, D_1, D_2, D_3 where $D_2 = 0$ because $\phi'' = 0$, and $D_3 > 0$ because $\phi' = 1 + \epsilon > 1$ and $\phi(r) > r$. The first term allows now the estimate

$$Dw > D_1 w > \gamma_1[(1+\epsilon)^p - 1] > \gamma_1 p\epsilon \quad \text{as long as} \quad \phi(r) \le c.$$

Let $-L \le 0$ be a lower bound for the difference quotients of $g(r, s)$ in a neighborhood of $(0, u_0)$. Then the right side of (7) is bounded above by $L(\phi(r) - r) = L\epsilon(\delta + r) \le 2\epsilon L\delta$ if r is restricted to $0 \le r \le \delta$. This shows that for $0 < \epsilon < 1$ the function w is an upper solution in $[0, \delta]$ if the constant $\delta > 0$ satisfies

$$\gamma_1 p \ge 2L\delta \quad \text{and} \quad 3\delta \le c;$$

furthermore, w satisfies $w(0) = u(\epsilon\delta) > u(0)$ and $w'(0) = u'(\epsilon\delta)(1+\epsilon) > 0 = u'(0)$.

If v is another solution, then $w > v$ and, by letting $\epsilon \to 0$, $u \geq v$ in $[0, \delta]$. As before, it follows by a symmetric argument that $u = v$ in an interval $[0, \delta_1]$.

In the case where $a > 0$ we have $(p-1)|u'|^{p-1}u'' \approx \gamma$, and we may use $\phi(r) = r + \epsilon(\delta + (r-a))$ for the proof which is similar.

(f)(ii) For uniqueness to the left one uses the same technique with $\phi(r) = r - \epsilon(1 + \delta(a-r) + \beta(a-r)^2)$. One chooses first $\delta > 0$ so large that $D_1 > 2L\epsilon$ ($L \geq 0$ is now an upper bound for the difference quotients of g), then β so large that $D_2 + D_3 > 0$ in a small left neighbourhood of a where $r - \phi(r) < 2\epsilon$.

(g) If $u_0 \neq 0$ or $u_0' \neq 0$ then u has one sign in $(a, a+\epsilon]$. If both u_0, u_0' vanish, it follows from $g(a, 0) \neq 0$ and (3′) that u has one sign in $(a, a+\epsilon]$. Let u, v be two solutions on $I = [a, a+\epsilon]$ for sufficiently small ϵ. If u_0, u_0' do not both vanish, then u/v is bounded below on I by a positive constant. If $u_0 = u_0' = 0$ then it follows from (3″) and the mean value theorem for integrals that

$$\frac{u(r)}{v(r)} = \frac{g(\rho_1, u(\rho_1))^{1/(p-1)} \int_a^r J(s)^{1/(p-1)}\,ds}{g(\rho_2, v(\rho_2))^{1/(p-1)} \int_a^r J(s)^{1/(p-1)}\,ds} \to 1 \quad \text{as } r \to a,$$

where $J(s) = \int_a^s (\rho/s)^\alpha\,d\rho$ and $\rho_1, \rho_2 \in [a, r]$. In case u, v are positive in $(a, a+\epsilon]$, there exists a large $\lambda_0 > 1$ with $\lambda u \geq v$ in I for all $\lambda \geq \lambda_0$. Let $\lambda^* = \inf\{\lambda \geq 1 : \lambda u \geq v \quad \text{in} \quad I\}$ and suppose for contradiction that $\lambda^* > 1$. Then, by assumption, $g(r, \lambda^* u) \leq \lambda^{*\,p-1} g(r, u)$ on I and hence

$$L_p^\alpha(\lambda^* u) - g(r, \lambda^* u) \geq \lambda^{*\,p-1}(L_p^\alpha u - g(r, u)) = 0 = L_p^\alpha v - g(r, v) \quad \text{in} \quad I.$$

Since $g(r, s)$ is increasing in s and $\lambda^* u > v$ at $a+$, the comparison theorem shows that $\lambda^* u > v$ in $(a, a+\epsilon]$, contradicting the minimality of λ^*. Hence $\lambda^* = 1$ and $u \geq v$ for any two solutions u, v of (3_a). In case u, v are negative in $(a, a+\epsilon]$, the proof is similar with $\lambda^* = \inf\{\lambda \geq 1 : \lambda u \leq v \quad \text{in} \quad I\}$. Notice that now by assumption $g(r, \lambda^* u) \geq \lambda^{*\,p-1} g(r, u)$ on I.

(h) If a solution $u(r)$ is $> u_0$ in I_0, then u_δ ($\delta > 0$) satisfies $u_\delta(a) > u_0$, $u_\delta'(a) \geq u_0' = 0$. As before, this implies $u_\delta > v$ for $\delta > 0$ and every solution v. Hence $u \geq v$, i.e., u is the maximal solution $\bar{u}(r; a, u_0, 0) =: \bar{u}$. This solution produces, according to (d), a subsolution $v(r) = \bar{u}(r-(r_0-a))$. Hence $\bar{u}(r; r_0, u_0, 0) \geq v(r) > u_0$ for $r > r_0$. According to the reasoning at the beginning of the proof, applied to $r_0 = a$, every other solution $> u_0$ equals the maximal solution. ■

A Summary on Uniqueness

THEOREM 4 *Under each of the following conditions, uniqueness for the initial value problem* ($a \geq 0$)

$$L_p^\alpha u = g(r, u), \quad u(a) = u_0, \quad u'(a) = u_0'$$

is guaranteed in a neighbourhood of a.

It is assumed that the functions $g(r, s)$, $h(r, s)$, defined in a neighbourhood $U(a, u_0) \subset [0, \infty) \times \mathbb{R}$, and $k(r)$, defined in a neigbourhood $U(a) \subset [0, \infty)$, are continuous. We write $g(r, s) \in Lip(s)$ if $g(r, s)$ is locally Lipschitzian in s on $U(a, u_0)$; $g(r, s) \in Lip(r)$ is defined analogously. The spaces of locally q-Hölder continuous functions are denoted by $Lip^q(s)$ $(0 < q \leq 1)$. For one-sided Lipschitz conditions we use the terms Lip^+ or Lip^- (see Corollary (f)) if the difference quotients are (locally) bounded above or below, resp.

Initial condition			*valid for*	*Properties of* $g(r, s)$
(α)	$u_0' \neq 0$	(i)	$p > 1$	$g \in \mathrm{Lip}\,(r)$
	(hence $a > 0$)	(ii)	$p > 1$	$g \in \mathrm{Lip}\,(s)$
		(iii)	$1 < p \leq 2$	$g(r, s) \in \mathrm{Lip}^{\,q}(s), 0 < q \leq 1$
(β)	$u_0' = 0$	(i)	$p > 1$	$g(a, u_0) > 0$, g incr. s, $g \in \mathrm{Lip}^{\,-}(r)$
		(ii)	$p > 1$	$g(a, u_0) < 0$, g incr. s, $g \in \mathrm{Lip}^{\,+}(r)$
		(iii)	$1 < p \leq 2$	$g \in \mathrm{Lip}\,(s)$
		(iv)	$1 < p \leq 2$	$g \geq 0$, $g(r, s) \in \mathrm{Lip}^{\,p-1}(s)$
		(v)	$p \geq 2$	$g(a, u_0) \neq 0$, $g \in \mathrm{Lip}\,(s)$
		(vi)	$p \geq 2$	$g(r, s) = h(r, s)^{p-1} + k(r)$, $h, k \geq 0$, $h \in \mathrm{Lip}\,(s)$
(γ)	$u_0, u_0' \in \mathbb{R}$ $u_0' = 0$ if $a = 0$		$p > 1$	$g(r, s) = k(r)s^{(p-1)}$ (p-linear case)
(δ)	$u_0 = u_0' = 0$	(i)	$p > 1$	$\lvert g(r, s)\rvert \leq K\lvert s\rvert^{p-1}$
		(ii)	$p > 1$	$g(r, s)s < 0$ for $s \neq 0$, $g(r, 0) = 0$, $\lvert g_r(r, s)\rvert \leq K\lvert g(r, s)\rvert$

In (β)(iv), (vi) the sign condition on $g(r, s)$ and $h(r, s)$, $k(r)$ may be reversed.

Reading guide. The properties of g as stated apply to the uniqueness to the right; swap Lip^- and Lip^+ in (β)(i), (ii) for uniqueness to the left (the other cases remain unchanged).

Remark The cases (α)(ii) and (β)(iii) and (v) have recently appeared in a paper of Franchi, Lanconelli and Serrin [5]. For $\alpha = 0$, DelPino, Manásevich and Murúa [3] have given uniqueness conditions contained in the above list under the overall growth condition $|g(r,s)| \leq K|s|^{p-1}$.

In order to treat initial value problems where the right hand side vanishes at $r = a$ we need the following:

COROLLARY *If $g(r,s) = l(r)\tilde{g}(r,s)$, where $\tilde{g}(r,s)$ satisfies* (β)(i), (ii) *or* (v), *then the corresponding initial value problem is uniquely solvable to the right of a (to the left of a for $a > 0$), if l is continuous in a neighbourhood of a, $l(r) > 0$ for $r > a$ $(r < a)$ and if in the cases* (β)(i), (ii) *l is increasing (decreasing).*

Remark In all other cases a factor $l(r)$ is already allowed in the above list.

Proof of Theorem 4. Since we only prove local uniqueness, we may assume boundedness of g. The proofs are only given for uniqueness to the right. The changes for uniqueness to the left in case $a > 0$ are obvious.

(α). Conditions (i), (ii) give uniqueness by Theorem 2. Case (iii) is easily reduced to the case where $u_0 = 0$. The operator S in (3″) has then the form

$$(Su)(r) = \int_a^r A(t;u)^{(\frac{1}{p-1})}\,dt$$

where $A(t;u) = (a/t)^\alpha u_0'^{(p-1)} + \int_a^t (\rho/t)^\alpha g(\rho, u(\rho))d\rho \to u_0'^{(p-1)} \neq 0$ as $t \to a$. We proceed like in McKenna, Reichel, Walter [9] and investigate the operator S on the complete metric space $C([a, r_0])$ with the metric $d(u,v) = \max|u^{(q)} - v^{(q)}|$. W.l.o.g. we may assume $u_0' > 0$ and hence $A(t;u) > u_0'^{p-1}/2 > 0$ for t close to a; otherwise we consider $-S$. By the positivity of $A(t;u)$, we obtain the following estimate for t close to a. We write $(Su)^{(q)} = \|U\|$ and $(Sv)^{(q)} = \|V\|$, where $\|\cdot\|$ is the $L_{1/q}$-norm on $[a,r]$, $U = A(t;u)^{\frac{q}{p-1}}$ and $V = A(t;v)^{\frac{q}{p-1}}$:

$$
\begin{aligned}
|(Su)^{(q)} - (Sv)^{(q)}|(r) &= |\|U\| - \|V\|| \leq \|U - V\| \\
&= \left| \int_a^r |A(t; u)^{\frac{q}{p-1}} - A(t; v)^{\frac{q}{p-1}}|^{1/q}\, dt \right|^q \\
&\leq KL(r_0 - a) \left| \int_a^r |u(t)^{(q)} - v(t)^{(q)}|^{1/q}\, dt \right|^q \\
&\leq KL(r_0 - a)^{q+1}\, d(u, v),
\end{aligned}
$$

where L is the q-Hölder constant of $g(r, s)$ and K is a Lipschitz constant for $S^{\frac{q}{p-1}}$ near $u_0'^{(p-1)}$. Hence for r_0 sufficiently close to a, the operator S is a contraction on $(C([a, r_0]), d)$ and has a unique fixed point.

(β). Condition (β)(i) guarantees uniqueness by Corollary (f)(i) and (ii). For (β)(ii) one needs to observe that the function $\tilde{u}(r) = -u(r)$ satisfies $L_p^\alpha \tilde{u} = \tilde{g}(r, \tilde{u})$ with $\tilde{g}(r, s) = -g(r, -s)$; for $\tilde{g}$ (β)(i) is applicable.

Uniqueness under (iii) follows from the observation that $s^{(1/(p-1))}$ is differentiable on $\mathbb{R}$ if $1 < p \leq 2$. The proof is then similar to (α)(iii) by estimating

$$
|A(t; u)^{\left(\frac{1}{p-1}\right)} - A(t; v)^{\left(\frac{1}{p-1}\right)}| \leq \frac{K}{p-1} |A(t; u) - A(t; v)|,
$$

where $K = ((r_0 - a) \max |g(r, s)|)^{(2-p)/(p-1)}$, and using the Lipschitz continuity of $g(r, s)$ in s. Conditions (iv) and (vi) are taken from McKenna, Reichel and Walter [9] and are based on suitable contraction mapping arguments.

For the proof of (v) we assume $g(a, u_0) > 0$ and observe that the expression

$$
\begin{aligned}
A(t; u) &= \int_a^t \left(\frac{\rho}{t}\right)^\alpha g(\rho, u(\rho))\, d\rho \\
&= \frac{g(\sigma_1, u(\sigma_1))}{\alpha + 1} \frac{t^{\alpha+1} - a^{\alpha+1}}{t^\alpha}, \qquad a \leq \sigma_1 \leq t
\end{aligned}
$$

is positive for $a < t < r_0$ if r_0 is close to a by the assumption. Hence by $(t^{\alpha+1} - a^{\alpha+1})/t^\alpha \geq t - a$ and by the mean-value theorem we get

$$
\begin{aligned}
&|A(t; u)^{\left(\frac{1}{p-1}\right)} - A(t; v)^{\left(\frac{1}{p-1}\right)}| \leq \\
&\qquad \frac{1}{p-1} \left(\frac{(t-a)}{\alpha + 1} \frac{g(a, u_0)}{2} \right)^{\frac{2-p}{p-1}} |A(t; u) - A(t; v)|
\end{aligned}
$$

which results in

$$|(Su)(r) - (Sv)(r)| \leq \left(\frac{g(a, u_0)}{2(\alpha+1)}\right)^{\frac{2-p}{p-1}} \frac{L}{p}(r_0 - a)^{p/(p-1)} \max_{a \leq r \leq r_0} |u - v|$$

for $a \leq r \leq r_0$ and r_0 sufficiently close to a; here L is the Lipschitz constant of g. Again S is a contraction operator on $C([a, r_0])$, equipped with the maximum-norm.

The invertability of the sign condition imposed on g in (iv), (vi) is evident, since $-S$ is a contraction if and only if S is a contraction.

(γ). This condition was found by Walter [13] and is proved in the context of Sturm-Liouville problems by Prüfer's transformation.

(δ). Under condition (i) it follows from ($3''$) that a solution u satisfies

$$|u(r)| \leq K^{\frac{1}{p-1}} \int_a^r (t-a)\,dt \max_{a \leq r \leq r_0} |u|$$

and hence $u \equiv 0$ on a sufficiently small interval $[a-\epsilon, a+\epsilon]$ $(a > 0)$ or $[0, \epsilon]$ $(a = 0)$. For the proof of (ii) we define $G(r,s) = \int_0^s g(r,\sigma)\,d\sigma$, which is non-positive by assumption, and find

$$\begin{aligned} u' L_p^\alpha u &= u'^{(p-1)}((p-1)u'' + \frac{\alpha}{r}u') \\ &= \gamma(|u'|^p)' + \frac{\alpha}{r}|u'|^p \\ &= (G(r,u))' - G_r(r,u), \end{aligned}$$

with $\gamma = (p-1)/p$. Substituting $v = |u'|^p$ we obtain the linear first order equation

$$\gamma(v' + \frac{\bar{\alpha}}{r}v) = (G(r,u))' - G_r(r,u), \quad v(a) = 0,$$

with $\bar{\alpha} = \alpha/\gamma$. Solving this equation for $v \geq 0$ and integrating the first term by parts, we get

$$\begin{aligned} \gamma v(r) &= \int_a^r \Big(G(t,u(t))' - G_r(t,u(t))\Big)\frac{t^{\bar{\alpha}}}{r^{\bar{\alpha}}}\,dt \\ &= G(r,u(r)) - \bar{\alpha}\int_a^r G(t,u(t))\frac{t^{\bar{\alpha}-1}}{r^{\bar{\alpha}}}\,dt - \int_a^r G_r(t,u(t))\frac{t^{\bar{\alpha}}}{r^{\bar{\alpha}}}\,dt. \end{aligned}$$

If $u \not\equiv 0$ in a neighbourhood of a, we may choose the sequence $r_n \to a$ such that $G(r_n, u(r_n)) = \min_{[a,r_n]} G(t, u(t)) < 0$. Then we get

$$\gamma v(r_n) \leq |G(r_n, u(r_n))|(-1+\bar{\alpha} \int_a^{r_n} \frac{t^{\bar{\alpha}-1}}{r^{\bar{\alpha}}} dt+(r_n-a)K) < 0 \text{ for } n \text{ large,}$$

in contradiction to $v \geq 0$. Hence $v = |u'|^p \equiv 0$ in a neighbourhood of a. ■

Proof of the corollary. We only indicate where the differences to the original proofs are. Suppose $\tilde{g}(r, s)$ satisfies (β)(i) and $l(r)$ is increasing. We prove uniqueness to the right for $L_p^\alpha u = g(r, u)$, $u_0' = 0$ where $g(r, s) = l(r)\tilde{g}(r, s)$. Let us go back to the proof of Corollary 3(f)(i) with $a = 0$ and $\tilde{g}(a, u_0) > 0$. The estimate for Dw now becomes

$$Dw > \gamma p\epsilon\, l(\phi(r)) \text{ as long as } \phi(r) \leq c.$$

If $-L$ is a lower bound for the difference quotients of $\tilde{g}(r, s)$ in a neighbourhood of $(0, u_0)$, then we have the estimate (notice $\phi(r) > r$)

$$g(r, w) - g(\phi(r), w) \leq l(\phi(r))\big(\tilde{g}(r, w) - \tilde{g}(\phi(r), w)\big) \leq l(\phi(r))2\epsilon L\delta.$$

In order to get $Dw \geq g(r, w) - g(\phi(r), w)$, the function $l(\phi(r)) > 0$ drops out and the proof goes as before. For uniqueness to the left the estimate $Dw \geq g(r, w) - g(\phi(r), w)$ is obtained by using the decreasing character of $l(r)$ together with $\phi(r) < r$. If $\tilde{g}(r, s)$ satisfies (β)(ii), the proof is obtained by considering solutions $v = -u$ of $L_p^\alpha v = -l(r)\tilde{g}(r, -v)$ where now $-\tilde{g}(r, -s)$ satisfies (β)(i).

Suppose now that $\tilde{g}(r, s)$ satisfies (β)(v) with $g(a, u_0) > 0$. As in the proof of (v) the positivity of $A(t; u) = \int_a^t (\rho/t)^\alpha g(\rho, u(\rho)d\rho$ for $a < t < r_0$ follows from the positivity $\tilde{g}(a, u_0)$ and of $l(\rho)$ for $\rho > a$. Hence the estimate

$$A(t; u) \geq \frac{\tilde{g}(a, u_0)}{2} \int_a^t \left(\frac{\rho}{t}\right)^\alpha l(\rho)\, d\rho$$

holds for $a < t < r_0$ with r_0 close to a. Denoting $I(t) = \int_a^t (\rho/t)^\alpha l(\rho)d\rho$ we get by the mean-value theorem

$$|A(t; u)^{(\frac{1}{p-1})} - A(t; v)^{(\frac{1}{p-1})}| \leq \frac{1}{p-1}\left(I(t)\frac{\tilde{g}(a, u_0)}{2}\right)^{\frac{2-p}{p-1}} |A(t; u) - A(t; v)|.$$

With the Lipschitz property of $\tilde{g}(r, s)$ this results in

$$|(Su)(r) - (Sv)(r)| \leq \left(\frac{\tilde{g}(a, u_0)}{2}\right)^{\frac{2-p}{p-1}} L(r_0 - a) \max_{[a,r_0]} I(t)^{\frac{1}{p-1}} \max_{[a,r_0]} |u - v|.$$

This gives the contraction property for S on $[a, r_0]$ for r_0 close to a. ∎

Remark If, for given initial conditions, $g(r, s)$ satisfies one of the above uniqueness conditions and is furthermore increasing in s, then comparison between an upper and a lower solution holds even if equality is permitted in the initial values; cf. Corollary (c) to Theorem 3.

We furnish our results with two

Counterexamples. For $q > 0$ the problem

$$L_p^\alpha u = u^{(q)} - 1, \quad u(0) = 1, \quad u'(0) = 0$$

has the trivial solution $u \equiv 1$. For $1 < p \leq 2$ this is the only solution by (β)(iii). For $p > 2$ the function $v(r) = 1 + \epsilon r^{1+\gamma}$ is a subsolution if $\gamma > 2/(p-2)$ and $\epsilon > 0$ is sufficiently small. Hence the initial value problem has at least two solutions, since the maximal solution $\bar{u}(r; 0, 1, 0)$ is > 1 for $r > 0$.

This counterexample shows, that uniqueness may fail if in (β)(i), (ii) and (v) only the condition $g(a, u_0) > 0$, < 0 and $\neq 0$, resp., is violated and if in (β)(vi) only the condition $k(r) \geq 0$ is dropped. Furthermore it shows that the equivalent of (β)(iii) does not hold for $p > 2$. Finally it shows, that in (γ) the homogeneity is essential.

If the Lipschitz continuity of $g(r, s)$ in (β)(iii), of $h(r, s)$ in (β)(vi) or the $p-1$-Hölder continuity of $g(r, s)$ in (β)(iv) is dropped, then uniqueness may fail, as shown by the example following Corollary (h), where

$$L_p^\alpha u = u^{(q)}, \quad u(a) = 0, \quad u'(a) = 0$$

has nontrivial solutions for $0 < q < p - 1$ to the right and left (if $a > 0$). This example also shows that in (δ)(i) the growth exponent $p - 1$ cannot be decreased and in (δ)(ii) uniqueness fails if the sign condition $g(r, s)s < 0$ for $s \neq 0$ is reversed. Notice that (δ)(ii) gives uniqueness for

$$L_p^\alpha u = -u^{(q)}, \quad u(a) = 0, \quad u'(a) = 0.$$

3 A STRONG COMPARISON PRINCIPLE

For an interval $I = [a, b]$ $(0 \le a < b)$ we define $I_1 = (a, b)$ if $a > 0$ and $I_1 = [0, b)$ if $a = 0$. We consider pairs of functions $v, w \in C^1(I)$, $r^\alpha v'^{(p-1)}, r^\alpha u'^{(p-1)} \in C^1(I_1)$, which satisfy

$$L_p^\alpha v - g(r, v) \ge L_p^\alpha w - g(r, w) \quad \text{in} \quad (a, b), \tag{8}$$

$$v(b) \le w(b) \ \text{ and } \ v(a) \le w(a) \ \text{ if } \ a > 0, \ \ v'(0) = w'(0) = 0 \ \text{ if } \ a = 0. \tag{9}$$

If $g(r, s)$ is (weakly) increasing in s, then the well known comparison principle states that $v \le w$ in $[a, b]$; cf. Tolksdorf [11], Walter [12]. Here we address the question, under what conditions the weak comparison $v \le w$ (WCP) can be strengthened to the strong comparison $v < w$ or $v \equiv w$ (SCP).

Remark For $a > 0$ we want the strong comparison $v < w$ to hold on (a, b) whereas for $a = 0$ it is required to hold on $[0, b)$. Note that for $\alpha = N - 1$ the interval (a, b) represents an open annulus and $[0, b)$ an open ball in N-space.

We formulate the corresponding Hopf version (H) of (SCP) at the boundary points b and a (for $a > 0$):

$$v < w \quad \text{in} \quad I_1 \quad \text{and} \quad v(b) = w(b) \quad \text{implies} \quad v'(b) > w'(b). \tag{H$_b$}$$

$$v < w \quad \text{in} \quad I_1 \quad \text{and} \quad v(a) = w(a) \quad \text{implies} \quad v'(a) < w'(a). \tag{H$_a$}$$

In Walter [12], essentially the following counterexample for $p > 2$ is given, which shows that (SCP) and (H) can fail, even if the increasing function $g(r, s)$ is Lipschitzian as a function of $s^{(p-1)}$. Consider the equation

$$L_p^\alpha u = u^{(q)} - 1$$

for $q > 0$. We have seen in the counterexamples in Section 2 that the initial value problem for the above equation with $u_0 = 1$, $u_0' = 0$ has two solutions $u \equiv 1$ and $\bar{u}(r; 0, 1, 0) > 1$ for $r > 0$. By Corollary 3(h), the maximal solution $\bar{u}(r; a, 1, 0)$ is > 1 for $r > a$ $(a \ge 0)$. If we take $v \equiv 1$ and $w = \bar{u}(r; a, 1, 0)$, this example shows that both (SCP) and (H$_a$) fail. The following condition from Tolksdorf [11] or Walter [12] is known to guarantee (SCP) [and (H)]

Case $a > 0$: $v' \ne 0$ or $w' \ne 0$ in (a, b) [in $[a, b]$], $g(r, s)$ is increasing and locally Lipschitzian in s.

The main weakness of this result is the assumption on the non-vanishing of the derivatives, which is in general not controllable. Our approach is based on the following simple idea: Since we have the weak comparison $w \geq v$ in I, the strong comparison $w > v$ in I_1 fails only if there exists a touching point $r_0 \in I_1$ with $w(r_0) = v(r_0)$ and $w'(r_0) = v'(r_0)$. If we furthermore suppose for contradiction that $w \not\equiv v$, then we may take r_0 to be a point of a strict one-sided local zero-minimum of $w - v$. We determine a continuous function $q(r)$ which satisfies

$$L_p^\alpha v - g(r, v) \geq q(r) \geq L_p^\alpha w - g(r, w) \quad \text{in} \quad I_1 \tag{8'}$$

and consider the initial value problem

$$L_p^\alpha u = g(r, u) + q(r), \;\; u(r_0) = v(r_0) = w(r_0), \;\; u'(r_0) = v'(r_0) = w'(r_0). \tag{10}$$

The function v is a supersolution and the function w is a subsolution to this problem. Assuming that (10) has a unique solution in a neighbourhood U of r_0, we obtain from Theorem 3, Corollary (c) that $w \leq u \leq v$, which leads to $v \equiv w$ in U, a contradiction. Summing up, we have

THEOREM 5 *Suppose v, w satisfy (8)–(9) and $g(r, s)$ is continuous in $(r, s) \in [a, b] \times \mathbb{R}$ and increasing in $s \in \mathbb{R}$. Then* (SCP) *holds if all problems* (10) *with $r_0 \in I_1$ are uniquely solvable. In particular* (SCP) *holds if the function $\tilde{g}(r, s) = g(r, s) + q(r)$ satisfies for initial values $u_0 = v(r_0)$ and $u_0' = v'(r_0)$ a uniqueness condition of Theorem* 4.

The assertions (H_a), (H_b) are proved by the same argument where a strict one-sided zero-minimum of $w - v$ at $r_0 = a$ or $r_0 = b$ with $v'(r_0) = w'(r_0)$ is led to a contradiction. We need q to be defined and continuous in $[a, a + \epsilon]$ or $[b - \epsilon, b]$, resp.

For illustration, we give some explicit assumptions which imply (SCP) and (H). We use the notation $Pu = L_p^\alpha u - g(r, u)$; naturally, $g(r, s)$ is increasing in s.

(a) $1 < p \leq 2$, $g \in Lip(s)$ (no condition on q).
(b) The p-linear case, $g(r, s) = k(r)s^{(p-1)}, k \geq 0$. Take $q(r) = 0$, i.e., $Pv \geq 0 \geq Pw$ (for $q = -1$, we have a counterexample)
(c) $g(r, w(r)) + q(r) \neq 0$ in I_1, g and $q \in Lip(r)$.
(d) $1 < p \leq 2$, $g(r, w(r)) + q(r) \geq 0$ or ≤ 0 in I_1, $g(r, s) \in Lip^{p-1}(s)$.
(e) $p \geq 2$, $g(r, w(r)) + q(r) \neq 0$ in I_1, $g \in Lip(s)$.

(f) $p \geq 2, g(r,s) = h(r,s)^{p-1} + k(r), h \in Lip(s), h$ and $k+q$ non-negative in I_1.

(g) If $v' \neq 0$ or $w' \neq 0$ in I_1 $(a > 0)$, then it suffices that g satisfies (α) (note that (α)(ii) is the condition of Tolksdorf and Walter stated above).

4 BLOW-UP SOLUTIONS

Our next theorem deals with blow-up solutions for the equation

$$L_p^\alpha u = g(r,u), \quad u(r) \to \infty \text{ as } r \to R, \tag{11}$$

in particular $L_p^\alpha u = f(u)$. We introduce an assumption (A) consisting of five parts:

(A1) $f(s)$ is continuous, nonnegative and increasing in $[s_0, \infty)$.

(A2) **The generalized Keller condition.** The function $F(s) = \int_{s_0}^s f(t)\,dt$ satisfies

$$\int^\infty \frac{ds}{F(s)^{1/p}} < \infty. \tag{12}$$

(A3) $g(r,s)$ is continuous, nonnegative and increasing in s in the set $I \times [s_0, \infty)$, $I = [a,b]$ with $a \geq 0$.

(A4) There exist $f(s)$ satisfying (A1)(A2) and positive constants c_1, c_2 such that

$$c_1 f(s) \leq g(r,s) \leq c_2 f(s) \quad \text{in} \quad I \times (s_1, \infty), \quad \text{where} \quad s_1 > s_0.$$

(A5) g satisfies a condition of Lipschitz type

$$|g(r_1,s) - g(r_2,s)| \leq L|r_1 - r_2| g(r_1,s) \quad \text{in} \quad I \times [s_0, \infty).$$

Remarks 1. Condition (A2) has been given by Keller [6] in the classical case $p = 2$; it is a necessary condition for blow-up. Under more restrictive assumptions, but for general N-dimensional domains, the blow-up problem has recently been studied by Bandle and Marcus [1, 2], Lazer and McKenna [7, 8] for $p = 2$ and by Diaz and Letelier [4] for general $p > 1$. McKenna, Reichel and Walter [9] have treated the radial case for $f(u) = |u|^q$ and general $p > 1$.

2. It follows from (A) that $\lim f(s)/s^{p-1} = \infty$ and $\lim g(r,s)/s^{p-1} = \infty$ for $s \to \infty$ uniformly in I. For proof one may adapt Lemma A and B in Bandle, Marcus [2].

3. The function $g(r, s) = h(r) f(s)$ satisfies (A) if f satisfies (A1) (A2) and h is continuous and positive and $|h(r_1) - h(r_2)| \leq L|r_1 - r_2|$.
4. Condition (A) is satisfied for $f(s) = s^q$ and, more general, for $f(s) = s^q k(s)$, if k is continuous, positive and increasing and $q > p-1$. It also holds for $g(r, s) = \lambda s^q + k(r, s)$ if k is such that (A) holds and $k(r, s)s^{-q} \to 0$ as $s \to \infty$ uniformly in I.

LEMMA *If f satisfies* (A1) (A2), *then*

$$\frac{f(s)}{F(s)^{\frac{p-1}{p}}} \to \infty \ \ as \ \ s \to \infty.$$

Proof In an interval $J = [s_1, \infty)$ the function $\psi(s) = F(s)^{1/p}$ has the following properties:

$$\psi \in C^1(J), \quad \psi > 0, \quad \psi' > 0, \quad \psi^p \ \text{ convex and } \int^\infty \frac{ds}{\psi(s)} < \infty.$$

We have to prove that $\psi'(s) \to \infty$ as $s \to \infty$. Assume first that $\psi \in C^2(J)$. Then convexity of ψ^p implies $(\psi^p)'' \geq 0$ or $\psi\psi'' + (p-1)\psi'^2 \geq 0$, which implies

$$\left(\frac{1}{\psi'}\right)' = -\frac{\psi''}{\psi'^2} \leq \frac{p-1}{\psi}.$$

By integration, one obtains

$$\frac{1}{\psi'(b)} - \frac{1}{\psi'(a)} \leq (p-1)\int_a^b \frac{ds}{\psi(s)} \quad (s_1 < a < b). \tag{*}$$

If ψ' were bounded, then ψ would grow at most linearly with the effect that $\int^\infty (1/\psi) ds = \infty$. Hence $\sup \psi' = \infty$. Let M be positive and let $a \in J$ be such that

$$\int_a^\infty \frac{ds}{\psi(s)} < \frac{1}{M} \quad \text{and} \quad \psi'(a) > M.$$

Then it follows from the inequality $(*)$ that

$$\frac{1}{\psi'(s)} \leq \frac{1}{\psi'(a)} + \frac{p-1}{M} < \frac{1}{M} + \frac{p-1}{M} = \frac{p}{M} \quad \text{in} \quad [a, \infty).$$

This inequality shows that $\lim_{s\to\infty} \psi'(s) = \infty$. Now assume that ψ is only of class C^1. We approximate $F = \psi^p$ by smooth functions F_α using the mollifier technique:

$$F_\alpha(s) = \int_{\mathbb{R}} F(s + \alpha t)\phi(t)\, dt,$$

where $\phi \geq 0$, $\operatorname{supp} \phi = [-1, 1]$, $\phi \in C^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \phi(t)\, dt = 1$. Since F belongs to C^1, differentiation gives

$$F'_\alpha(s) = \int_{\mathbb{R}} F'(s + \alpha t)\phi(t)\, dt.$$

Due to convexity, $F'(s)$ is increasing, and this property is inherited by F'_α. Hence the inequality $(*)$ holds for the functions $\psi_\alpha(s) := F_\alpha(s)^{1/p}$. In the limit as $\alpha \to 0+$, we get $F_\alpha \to F$, $F'_\alpha \to f$, hence $\psi_\alpha \to \psi$ and $(\psi_\alpha^p)' \to (\psi^p)'$, which leads easily to $\psi'_\alpha \to \psi'$. This shows that $(*)$ holds under the assumption of the Lemma, which now follows as before. ■

The first part of the proof was contributed by Prof. M. Plum, which is gratefully acknowledged.

THEOREM 6 *Suppose that, for $i = 1, 2$, u_i is a solution of* (11) *in* $[a, b_i) \subset I$ $(a \geq 0)$ *and* $u_i(b_i) = \infty$. *If* (A) *holds and*

$$u_1(a+) < u_2(a+) \quad and \quad u'_1(a) \leq u'_2(a),$$

then $b_1 > b_2$.

Proof If follows from the comparison theorem that the inequalities $u_1 < u_2$, $u'_1 \leq u'_2$ hold in (a, b_2) and that $b_1 \geq b_2$; furthermore, since $u_2 = u_1 + c$ would imply that g is constant in s, we have $u'_1 < u'_2$ at a point r_0 and then also in $[r_0, b_2)$.

For the proof by contradiction, we assume that $a < b_1 = b_2 < b$, and that $a > b/p$ and that strict inequalities $u_1 < u_2$, $u'_1 < u'_2$ hold in $[a, b_1)$. The proof is based on the following idea. We consider the function

$$v(r) = u_1(\phi(r)), \quad \text{where} \quad \phi(r) = (1 - \epsilon)r + b\epsilon,$$

and show that for small $\epsilon > 0$ the function v is a subsolution to the differential equation (11). Because of the strict inequalities at $r = a$, we have $v(a) < u_2(a)$, $v'(a) < u'_2(a)$. Hence $v < u_2$ and $v(b') = \infty$, where $\phi(b') = b_1$ and therefore $b_2 \leq b' < b_1$, which is the desired contradiction.

We want to use (7) and consider the expression

$$Dv = (p-1)|u_1'|^{p-2}\left\{u_1''((1-\epsilon)^p - 1) + \alpha' u_1' \frac{N(r)}{r\phi(r)}\right\}$$

with

$$N(r) = (1-\epsilon)^{p-1}[(1-\epsilon)r + b\epsilon] - r.$$

For ϵ small, $(1-\epsilon)^p \approx 1 - p\epsilon$ and $(1-\epsilon)^{p-1} \approx 1-(p-1)\epsilon$, hence

$$N(r) \approx -p\epsilon r + (1-(p-1)\epsilon)b\epsilon < \epsilon(b - pr) < 0$$

because $r \geq a > b/p$. Thus the second term of Dv is negative. Next we show that u_1'' is positive for u_1 large. The derivative u_1' is positive, so u_1 has an inverse function $r(u)$ with $r' = 1/u'$. The function $z(u) = u_1'(r(u))$ satisfies (with u as independent variable and $r = r(u)$)

$$z'(u) = \frac{u_1''}{u_1'} = \frac{g(r,u)}{(p-1)z^{p-1}} - \frac{\alpha'}{r} < \frac{c_2 f(u)}{(p-1)z^{p-1}}, \quad z(u_0) = u_0',$$

where $u_0 = u_1(a)$, $u_0' = u_1'(a)$. Solving the corresponding initial value problem for $y(u)$,

$$y' = \frac{c_2 f(u)}{(p-1)y^{p-1}}, \quad y(u_0) = u_0',$$

we obtain

$$y(u) = \left(\frac{pc_2}{p-1}(F(u) - F(u_0)) + u_0'^p\right)^{1/p} > z(u),$$

which implies

$$z' > \frac{g(r,u)}{(p-1)y^{p-1}} - \frac{\alpha'}{r} > \frac{c_1 f(u)}{(p-1)y^{p-1}} - \frac{\alpha'}{a}.$$

By the lemma, $f(u)/y^{p-1} \to \infty$ as $u \to \infty$, which implies $u_1''/u_1' \to \infty$ as $r \to b_1$. Hence we may assume (by moving the point a to the right, if necessary) that u'' is positive and therefore $Dv < 0$ in $[a, b_1)$. According to (7) we have to show that

$$Dv \leq g(r,v) - g(\phi(r), v).$$

This is obviously true if $g(r, s)$ is decreasing in r. In this case, which covers the case where $g(r, s) = f(s)$, the theorem is proved. In the general case, we take ϵ small and u large, which implies $(1-\epsilon)^p - 1 \approx -p\epsilon$ and $u'/u'' \approx 0$, hence

$$\begin{aligned} Dv &\le -\frac{1}{2}p\epsilon(p-1)|u_1'|^{p-2}u_1'' \le -\frac{1}{3}(L_p^\alpha u_1)(\phi(r))p\epsilon \\ &\le -L(b-a)\epsilon g(\phi(r), v) \le g(r, v) - g(\phi(r), v). \end{aligned}$$

It was used that $\phi(r) - r \le b - a$ and $L(b-a) < p/3$ (a and b can be chosen close to b_1). These inequalities show that v is indeed a lower solution. ■

COROLLARY *We consider solutions $u(r; r_0, u_0, u_0')$ of* (3) *under the assumption* (A).

(a) *Case $r_0 = 0$. Assume that the maximal solution $\bar{u}(r; 0, u_0, 0)$ ($u_0 \ge s_0$) blows up at $r = b_0$. Then the initial value problem* (3) *has, for $u_0 < \lambda < \infty$ and $u_0' = 0$, a unique solution $u(r; 0, \lambda, 0)$ which blows up at b_λ. The function b_λ is continuous and strictly decreasing in $\lambda \in (u_0, \infty)$ and $b_\lambda \to 0$ as $\lambda \to \infty$, $b_\lambda \to b_0$ as $\lambda \to u_0$.*

(b) *Case $r_0 = a > 0$. Assume that the maximal solution $\bar{u}(r; a, u_0, u_0')$ ($u_0 \ge s_0$, $u_0' \ge 0$) blows up at b_0. Then the solution $u(r; a, \lambda, \mu)$ is unique for $\lambda \ge u_0$, $\mu \ge u_0'$, $(\lambda, \mu) \ne (u_0, u_0')$, and it blows up at a point $b_{\lambda\mu}$. The function $b_{\lambda\mu}$ is continuous and strictly decreasing in λ and μ, and it tends to a as $\lambda \to \infty$ or $\mu \to \infty$ and to b_0 as $(\lambda, \mu) \to (u_0, u_0')$.*

(c) *Uniqueness of blow-up solution. Under the assumption of* (a) *the blow-up problem $L_p^\alpha u = g(r, u)$ in $J = (0, R)$, $u'(0) = 0, u(R) = \infty$ has for given $R \in (0, b_0)$ one and only one solution.*

If $g(r, u_0) = 0$, $g(r, s) > 0$ for $s > u_0$, $r \ge 0$ and if the maximal solution $\bar{u}(r; 0, u_0, 0)$ is $u \equiv u_0$, then the blow-up problem has for every $R > 0$ a unique solution.

(d) *The statements in* (a)–(c) *remain true if $g(r, s) = l(r)\tilde{g}(r, s)$, where $\tilde{g}(r, s)$ satisfies* (A) *and the assumptions in* (a)–(c), *l is continuous, increasing and $l(r_0+) > 0$.*

Remark In contrast to earlier work [1, 2, 4, 7, 8] on the general N-dimensional blow-up problem, the above uniqueness result is obtained merely by monotonicity, Keller's condition and (in the nonautonomous case) by a Lipschitz condition with respect to r. Without this last condition (A5), the theorem fails. For a counterexample take a blow-up solution u of $L_p^\alpha u = f(u)$ and define g by

$$g(r, s) = \begin{cases} f(s) & \text{for } s \le u(r), \\ f(s-1) & \text{for } s \ge u(r) + 1, \\ \text{const. in } s & \text{between } u(r) \text{ and } u(r) + 1. \end{cases}$$

The functions $w(r) = u(r) + \lambda$, $0 \le \lambda \le 1$ are blow-up solutions of $L_p^\alpha w = g(r, w)$. If one takes, e.g., $f(s) = e^s$, then (A1)–(A4) hold, and (A5) is violated.

Proof The assumption that $f = 0$ in $[u_0, u_0 + \epsilon]$ implies $\bar{u} \equiv u_0$ because of (A4). Hence $f(s) > 0$ for $s > u_0$. (a) We write $u(r, \lambda)$ for $u(r; 0, \lambda, 0)$. Uniqueness follows from Corollary 3.(f) since $g(0, \lambda) > 0$ for $\lambda > s_0$, and strict monotonicity of b_λ from Theorem 6. As $\lambda \downarrow \lambda_0 > u_0$, the solution $u(r, \lambda)$ tends to $u(r, \lambda_0)$ uniformly in compact subsets of $[0, b_{\lambda_0})$, which together with $b_\lambda < b_{\lambda_0}$ implies $b_\lambda \to b_{\lambda_0}$. This remains true for $\lambda_0 = u_0$ since $\bar{u}(r; 0, u_0, 0)$ is the maximal solution. Since $g(r, s) \ge c_1 f(s)$ for $s \ge s_1$, $u(r, \lambda)$ is for $\lambda > s_1$ a supersolution for the problem with the right hand side $c_1 f(s)$. Since according to Lemma 1.(c) in [9] the solutions w of the latter problem with $w_0 = \lambda$, $w_0' = 0$ are supersolutions to $L_p^0 v = (\alpha + 1) c_1 f(v)$, $v_0 = \lambda$, $v_0' = 0$, and since the asymptote of $v(r, \lambda)$ can be computed and tends $\to 0$ as $\lambda \to \infty$ (see [10], Satz 1.1), the same is true for the asymptotes b_λ of $u(r, \lambda)$.

(b) The proof is similar to the proof of (a) and will therefore be omitted.

(c) The first part follows readily from (a). In the second part we have $u(r, \lambda) \to u_0$ as $\lambda \to u_0$ uniformly on compact intervals. Since $u'(r, \lambda)$ is strictly increasing, u assumes large values, and for these $g(r, s) \ge c_1 f(s)$ (see (a)), i.e., $u(r, \lambda)$ is a blow-up solution and $b_\lambda \to \infty$ as $\lambda \to u_0$.

(d) The proofs of (a)–(c) remain almost unchanged: the uniqueness of the initial value problem now follows from the corollary to Theorem 4. Furthermore one has to observe that the solutions w of $L_p^\alpha w = c_1 l(r) f(w)$ with $w_0 = \lambda$ and $w_0' = 0$ are, due to the monotonicity of $l(r)$, supersolutions to the problem $L_p^0 v = (\alpha + 1) c_1 l(r) f(v)$ with $v_0 = \lambda$, $v_0 = 0$, for which the asymptote can be computed explicitly, is strictly decreasing in λ and has the same asymptotic behaviour as in (a)–(c). ∎

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