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Weighted Lagrange and Hermite–Fejér Interpolation on the Real Line*

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For a wide class of weights, a systematic investigation of the convergence-divergence behavior of Lagrange interpolation is initiated. A system of nodes with optimal Lebesgue constant is found, and for Hermite weights an exact lower estimate of the norm of projection operators is given. In the same spirit, the case of Hermite–Fejér interpolation is also considered.

Keywords: Interpolation; weights; Lebesgue constant.

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1 INTRODUCTION

Pointwise estimates of the error of Lagrange interpolation on the real line are scarce and not quite satisfactory. Perhaps the most significant is Freud's [3] 1969 result on the roots of Hermite polynomials for functions being uniformly continuous on **R**. Results of Nevai [11, 12] should also be mentioned. The characteristic of these results is either the restriction of the range of error estimate to a finite interval, and/or severe growth restrictions at infinity on the function (like in Freud's theorem). As Nevai mentions in his survey paper [12] in 1986, "There have been no new developments regarding pointwise convergence of Lagrange interpolation taken at zeros of orthogonal polynomials associated with Hermite, Laguerre or possibly Freud-type weights in the past ten years...". Indeed, a systematic treatment of

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weighted Lagrange interpolation for a wide class of weights and systems of nodes is a basic problem here. Perhaps the lack of this type of results is due to the fact that the theory of weighted polynomial approximation and that of orthogonal polynomials for general weights has been developed only in the last 20 years.

It seems that now we have sufficient knowledge to offer the beginning of such a systematic treatment of weighted Lagrange interpolation. In this paper we introduce the analogue of the Lebesgue constant, and investigate its order of magnitude. Also, in case of a special weight we present a Faber type result for general systems of nodes. In the proofs we make heavy use of estimates of Lubinsky and his collaborates on orthogonal polynomials with respect to Freud weights.

To begin with, let us mention that if the weight $w(x) = e^{-Q(x)}$ satisfies

$$\lim_{|x|\to\infty} \frac{Q(x)}{\log|x|} = \infty \tag{1}$$

(i.e. w^{-1} tends to 0 faster than any polynomial when $|x| \to \infty$), as well as some mild regularity conditions and the Akhiezer–Babenko relation

$$\int_{-\infty}^{\infty} \frac{Q(x)}{1+x^2}\,dx = \infty \tag{2}$$

then for functions $f(x)$ from the class

$$C_w(\mathbf{R}) := \{f : f \quad \text{is continuous on } \mathbf{R} \text{ and } \lim_{|x|\to\infty} w(x)f(x) = 0\}$$

we have

$$E_n(f)_w := \inf_{p\in\Pi_n} ||w(f-p)|| \to 0 \quad \text{as} \quad n \to \infty,$$

where Π_n is the set of polynomials of degree at most n, and $||\cdot||$ denotes the supremum norm over $\mathbf{R}$.

Now let $\mathcal{X}_n := \{x_1, x_2, \dots, x_n\}$ be an arbitrary set of (pairwise different) nodes and

$$L(f, \mathcal{X}_n, x) := \sum_{k=1}^{n} f(x_k) l_k(\mathcal{X}_n, x) \tag{3}$$

the corresponding Lagrange interpolation polynomial of $f \in C_w(\mathbf{R})$, where $l_k(\mathcal{X}_n, x) \in \Pi_{n-1}$ are the so-called fundamental polynomials, i.e. $l_k(\mathcal{X}_n, x_j) = \delta_{kj},\ k, j = 1, \dots, n$. The classical way of estimating the error of Lagrange interpolation is the following: take a $p_{n-1} \in \Pi_{n-1}$ such that $E_{n-1}(f)_w = ||w(f - p_{n-1})||$ (the existence of such a p is obvious), and consider

$$\begin{aligned} ||w(x)[f(x) - L(f, \mathcal{X}_n, x)]|| &\le ||w(x)[f(x) - p(x)]|| \\ &\quad + ||w(x)L(p - f, \mathcal{X}_n, x)|| \\ &\le E_{n-1}(f)_w \left(1 + \left\| w(x) \sum_{k=1}^{n} \frac{|l_k(\mathcal{X}_n, x)|}{w(x_k)} \right\| \right). \end{aligned} \tag{4}$$

Thus if we introduce the Lebesgue function

$$\lambda(\mathcal{X}_n, x)_w := w(x) \sum_{k=1}^{n} \frac{|l_k(\mathcal{X}_n, x)|}{w(x_k)}$$

and Lebesgue constant

$$\lambda(\mathcal{X}_n)_w := ||\lambda(\mathcal{X}_n, x)_w||,$$

then (4) takes the form

$$||w(f - L(f, \mathcal{X}_n, x))|| \le E_{n-1}(f)_w(1 + \lambda(\mathcal{X}_n)_w).$$

Under different structural properties on the function, we have a considerable knowledge of the behavior of the quantity $E_{n-1}(f)_w$ (see e.g. the recent survey paper [8] of D. S. Lubinsky). Thus what remains to be investigated is the Lebesgue constant $\lambda(\mathcal{X}_n)_w$.

2 THE LEBESGUE CONSTANT OF SOME SPECIAL SYSTEMS OF NODES

In general, the weight w and nodes $\mathcal{X}_n$ are chosen independently. However, if we expect a reasonable upper estimate for the Lebesgue constant $\lambda(\mathcal{X}_n)_w$ then we have to assume some connection between the two data. The most natural assumption, as we shall see, is that $\mathcal{X}_n$ is the set of roots of the orthogonal polynomials with respect to the weight w^2. Quite surprisingly, it will turn out that for a wide class of weights, *this is not an optimal choice* with respect to the order of magnitude of $\lambda(\mathcal{X}_n)_w$. Therefore we shall construct another system of nodes to achieve this optimal order.

The class of weights $\mathcal{W}$ we are dealing with is defined as follows (see e.g. Criscuolo, Della Vecchia, Lubinsky and Mastroianni [1]). We say that $w(x) = e^{-Q(x)}$ if $Q : \mathbf{R} \to \mathbf{R}$ is even, continuous in $\mathbf{R}$, Q'' is continuous in $(0, \infty)$, $Q' > 0$ in $(0, \infty)$, and for some $A, B > 1$,

$$A \leq \frac{(xQ'(x))'}{Q'(x)} \leq B \qquad (x \in (0, \infty)).$$

Of course, if $w \in \mathcal{W}$, then $w^2 \in \mathcal{W}$. The model case is the Freud type weights $w(x) = e^{-|x|^\alpha}$, $\alpha > 1$, when $\alpha = A = B$. It is also easy to see that for any $w \in \mathcal{W}$ the conditions (1)-(2) are satisfied.

In Theorem 1 below we will be concerned with the orthonormal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with respect to a weight $w^2 \in \mathcal{W}$. We shall denote by $x_1 > x_2 > \cdots > x_n$ the roots of p_n, and let $\mathcal{U}_n := \{x_1, \ldots, x_n\}$. With a little effort, one can deduce from Theorem 1.1 of Matjila [9] that for any $w \in \mathcal{W}$ we have

$$\lambda(\mathcal{U}_n)_w = O(n^{1/6}).$$

On the other hand, Sklyarov [13, Theorem 2(b)] proved for the Hermite weight

$$w_0(x) := e^{-x^2/2} \tag{5}$$

that $\lambda(\mathcal{U}_n)_{w_0} \sim n^{1/6}$. (Here and in what follows $\sim$ means that the ratio of the quantities on the left and right hand sides remains between two positive bounds independent of n, but possibly depending on the weight.) As we shall see after the proof of Lemma 5, this can be easily extended to

$$\lambda(\mathcal{U}_n)_w \sim n^{1/6} \qquad (w \in \mathcal{W}). \tag{6}$$

Our purpose in this section is to define a system of nodes such that the corresponding Lebesgue constant is of smaller order. Let $x_0 \geq 0$ denote a point such that

$$|p_n(x_0)|w(x_0) = ||p_n w|| \tag{7}$$

(we will see that $x_0 \neq 0$).

THEOREM 1 *For any $w \in \mathcal{W}$, with the notation $\mathcal{V}_{n+2} := \mathcal{U}_n \cup \{x_0, -x_0\}$ we have*

$$\lambda(\mathcal{V}_{n+2})_w \sim \log n.$$

Theorem 1 shows that the system of nodes $\mathcal{V}_{n+2}$ is superior over the set of roots $\mathcal{U}_n$ of the orthogonal polynomials. An obvious consequence of Theorem 1 is the following

COROLLARY 1 *Let $w \in \mathcal{W}$ and $f \in C_w(\mathbf{R})$. If*

$$E_n(f) = o\left(\frac{1}{\log n}\right)$$

then

$$\lim_{n\to\infty} ||w(x)[f(x) - L(f, \mathcal{V}_{n+2}, x)]|| = 0.$$

The proof of Theorem 1 is divided into a series of lemmas. First we recall (cf. e.g. Mhaskar and Saff [10]) that to any $w \in \mathcal{W}$ and $n \in \mathbf{N}_0$, there exists a positive real number $a_n = a_n(w)$ (called the Mhaskar–Rahmanov–Saff number associated with w) such that

$$\begin{cases} ||pw|| = \max_{|x|\le a_n} |p(x)|w(x) & \text{for all } p \in \Pi_n, \text{ and} \\ ||pw|| > |p(x)|w(x) & \text{for } |x| > a_n. \end{cases} \tag{8}$$

(In this section we will use the shorthand notation a_n, since the weight w will be the same everywhere.)

LEMMA 1 *If $w \in \mathcal{W}$, $m, n \in \mathbf{N}$ and $q_k \in \Pi_n$, $k = 1, \ldots, m$ are arbitrary polynomials then*

$$\left\| w \sum_{k=1}^{m} |q_k| \right\| = \max_{|x|\le a_n} \left[w(x) \sum_{k=1}^{m} |q_k(x)| \right].$$

Proof Let y be such that

$$\left\| w \sum_{k=1}^{m} |q_k| \right\| = w(y) \sum_{k=1}^{m} |q_k(y)|,$$

and consider the polynomial $q(x) := \sum_{k=1}^{m} q_k(x) \operatorname{sgn} q_k(y) \in \Pi_n$. Evidently,

$$\left\| w \sum_{k=1}^{m} |q_k| \right\| \ge ||wp|| \ge w(y)|q(y)| = \left\| w \sum_{k=1}^{m} |q_k| \right\|,$$

i.e. "$\ge$" here can be replaced by equality. But then by the above definition (7) of the Mhaskar–Rhamanov–Saff number $|y| \le a_n$. The lemma is proved.

LEMMA 2 *For any $w \in \mathcal{W}$ we have*

$$a_n\left(1-\frac{c_1}{n^{2/3}}\right) \le x_0 \le a_n \qquad (n \in \mathbf{N}_0) \tag{9}$$

with some constant $c_1 > 1$ independent of n.

(In what follows, $c_1, c_2, \ldots$ will denote constants independent of n, but possibly depending on w.)

Proof By Corollary 1.4 in Levin–Lubinsky [5] we have for all $n \in \mathbf{N}_0$

$$||p_n w|| \ge c_2 a_n^{-1/2} n^{1/6} \tag{10}$$

and

$$|p_n(x)|w(x) \le c_3 a_n^{-1/2}\psi_n(x)^{-1/4} \le c_3 a_n^{-1/2} n^{1/6} \qquad (x \in \mathbf{R}) \tag{11}$$

where

$$\psi_n(x) := \max\left\{n^{-2/3}, 1-\frac{|x|}{a_n}\right\}. \tag{12}$$

(These imply that $c_2 \le c_3$.) Hence by the definition (7) of x_0 we have

$$c_2 a_n^{-1/2} n^{1/6} \le |p_n(x_0)|w(x_0) \le c_3 a_n^{-1/2}\psi_n(x_0)^{-1/4},$$

whence

$$\psi_n(x_0) \le \left(\frac{c_3}{c_2}\right)^4 n^{-2/3}. \tag{13}$$

Thus if $x_0 < a_n(1-n^{-2/3})$ then by (12) $\psi_n(x_0) = 1 - \frac{x_0}{a_n}$, which together with (13) implies the left hand side inequality in (9) with $c_1 := \left(\frac{c_3}{c_2}\right)^4$. The right hand side inequality in (9) follows from (7) and (8). Lemma 2 is proved.

Denote $l_0(\mathcal{V}_{n+2}, x)$ and $l_{n+2}(\mathcal{V}_{n+2}, x)$ the fundamental polynomials of Lagrange interpolation with respect to the nodes x_0 and $-x_0$, respectively.

LEMMA 3 *We have*

$$\left\|w(x)\frac{l_0(\mathcal{V}_{n+2}, x)}{w(x_0)}\right\| = \left\|w(x)\frac{l_{n+2}(\mathcal{V}_{n+2}, x)}{w(-x_0)}\right\| = O(1). \tag{14}$$

Proof Since Q is even, the norms in (14) are evidently equal. Consider

$$w(x)\frac{|l_0(\mathcal{V}_{n+2}, x)|}{w(x_0)} = \frac{w(x)|p_n(x)||x_0+x|}{2x_0|p_n(x_0)|w(x_0)}.$$

By (7), it is sufficient to estimate this quantity for $|x| \le a_{n-1} < a_n$. Thus using (11), (7), (10) and Lemma 2 we obtain

$$w(x)\frac{|l_0(\mathcal{V}_{n+2},x)|}{w(x_0)} \le \frac{c_3 a_n^{-1/2} n^{1/6} 2a_n}{2a_n\left(1-\frac{c_1}{n^{2/3}}\right) c_2 a_n^{-1/2} n^{1/6}} = O(1).$$

This proves Lemma 3.

LEMMA 4 *We have*

$$|x_0 - |x_k|| \sim a_n \psi_n(x_k) \qquad (k = 1, \dots, n). \tag{15}$$

Proof In proving the lower estimate in (15), we distinguish two cases.

CASE 1: $1 - |x_k|/a_n \le 2c_1/n^{2/3}$ (c_1 is the constant appearing in Lemma 2). Using the inequality

$$\left|\frac{p_n(x)w(x)}{x - x_k}\right| \le c_4 n a_n^{-3/2} \psi_n(x)^{1/4} \qquad (k = 1, \dots, n,\ x \in \mathbf{R})$$

(cf. [5, Theorem 12.3(a)]) we obtain by (7), (10) and (13)

$$\begin{aligned} |x_0 - |x_k|| &\ge \frac{|p_n(x_0)|w(x_0)}{c_4 n a_n^{-3/2} \psi_n(x_0)^{1/4}} \ge \frac{c_2 a_n^{-1/2} n^{1/6}}{c_4 n a_n^{-3/2} (c_3/c_2) n^{-1/6}} \\ &\ge \frac{c_2^2}{c_3 c_4} \frac{a_n}{n^{2/3}} \ge \frac{c_2^2}{2c_1 c_3 c_4} a_n \left(1 - \frac{|x_k|}{a_n}\right), \end{aligned}$$

i.e.

$$|x_0 - |x_k|| \ge \frac{c_2^2}{2c_1 c_3 c_4} a_n \psi_n(x_k).$$

CASE 2: $1 - |x_k|/a_n \ge 2c_1/n^{2/3} > n^{-2/3}$. Then by (12) $\psi_n(x_k) = 1 - |x_k|/a_n$, whence and by Lemma 2

$$\begin{aligned} |x_0 - |x_k|| &\ge (a_n - |x_k|) - (a_n - x_0) \ge a_n \psi_n(x_k) - \frac{c_1 a_n}{n^{2/3}} \\ &\ge a_n \psi_n(x_k) - \frac{1}{2} a_n \psi_n(x_k) = \frac{1}{2} a_n \psi_n(x_k). \end{aligned}$$

In order to prove the upper estimate in (15), again we distinguish two cases.

CASE 1: $|x_k| \le a_n$. Then by Lemma 2 and (12)

$$|x_0 - |x_k|| \le |a_n - x_0| + a_n - |x_k| \le \frac{c_1 a_n}{n^{2/3}} + a_n \psi_n(x_k) \le (c_1 + 1) a_n \psi_n(x_k). \tag{16}$$

CASE 2: $|x_k| > a_n$. Then, using the inequality

$$\left|1 - \frac{x_1}{a_n}\right| \le \frac{c_5}{n^{2/3}}$$

(cf. [1, Lemma 4.4]) we get

$$\begin{aligned}|x_0 - |x_k|| \le |a_n - x_0| + x_k - a_n &\le \frac{c_1 a_n}{n^{2/3}} + x_1 - a_n \le (c_1 + c_5)\frac{a_n}{n^{2/3}} \\ &\le (c_1 + c_5) a_n \psi_n(x_k). \end{aligned} \tag{17}$$

Lemma 4 is completely proved.

Let

$$\Delta x_k := x_k - x_{k+1} \qquad (k = 1, \dots, n-1).$$

LEMMA 5 *We have*

$$\frac{w(x)|l_k(\mathcal{V}_{n+2}, x)|}{w(x_k)} = O\left[\left(\frac{\psi_n(x)}{\psi_n(x_k)}\right)^{3/4} \frac{\Delta x_k}{|x - x_k|}\right]$$
$$(k = 1, \dots, n-1;\ k \ne j;\ |x| \le a_{n+1}).$$

Proof Obviously,

$$l_k(\mathcal{V}_{n+2}, x) = \frac{x_0^2 - x^2}{x_0^2 - x_k^2} l_k(\mathcal{U}_n, x) \qquad (k = 1, \dots, n). \tag{18}$$

Here $|x_0 - x|$ can be esimated as $|x_0 - x_k|$ in (16), and we get

$$|x_0 - x| \le (c_1 + 1) a_n \psi_n(x)$$

for $|x| \le a_n$. Now if $a_n < |x| \le a_{n+1}$, then using the relation

$$a_{n+1} \le a_n \left(1 + \frac{c_6}{n}\right) \tag{19}$$

(cf. [1, Lemma 4.5(c)]) we obtain, like in (16)

$$|x_0 - x| \le |a_n - x| + |x_0 - a_n| \le \frac{c_6 a_n}{n} + \frac{c_1 a_n}{n^{2/3}} \le (c_1 + c_6) a_n \psi_n(x).$$

Thus

$$|x_0 - x| = O(a_n \psi_n(x)) \qquad (|x| \le a_{n+1}),$$

and we obtain from (18), (19), Lemmas 2 and 4

$$l_k(\mathcal{V}_{n+2}, x) = O\left(\frac{a_n \psi_n(x) a_n}{a_n \psi_n(x_k) a_n}\right) |l_k(\mathcal{U}_n, x)| = O\left(\frac{\psi_n(x)}{\psi_n(x_k)}\right) |l_k(\mathcal{U}_n, x)|$$
$$(k = 1, \dots, n;\ |x| \le a_{n+1},\ \varepsilon = 0 \ \text{ or } \ 1). \tag{20}$$

Now here by (11) and the relations

$$|p_n'(x_k)| w(x_k) \sim n a_n^{-3/2} \psi_n(x_k)^{1/4} \qquad (k = 1, \dots, n) \tag{21}$$

(cf. [5, Corollary 1.3]) and

$$\Delta x_k \sim \frac{a_n}{n \psi_n(x_k)^{1/2}} \qquad (k = 1, \dots, n-1) \tag{22}$$

(cf. [1, Lemma 4.4]) we get

$$\begin{aligned} \frac{w(x)|l_k(\mathcal{U}_n, x)|}{w(x_k)} &= \frac{|p_n(x)| w(x)}{|p_n'(x_k)| w(x_k) |x - x_k|} \\ &= O\left(\frac{a_n}{n \psi_n(x)^{1/4} \psi_n(x_k)^{1/4} |x - x_k|}\right) \\ &= O\left(\left(\frac{\psi_n(x_k)}{\psi_n(x)}\right)^{1/4} \frac{\Delta x_k}{|x - x_k|}\right) \\ &\quad (k = 1, \dots, n-1;\ |x| \le a_{n+1}). \end{aligned} \tag{23}$$

This together with (20) proves the lemma.

Now we are in the position to prove the lower estimate in (6). We have by Lemma 2, (7), (10) and (21)

$$\begin{aligned} \lambda(\mathcal{U}_n, x_0)_w &\ge \sum_{0 \le x_k \le a_n/2} \frac{|p_n(x_0)| w(x_0)}{|p_n'(x_k)| w(x_k) x_0} \ge c_7 \sum_{0 \le x_k \le a_n/2} \frac{a_n^{-1/2} n^{1/6}}{n a_n^{-3/2} a_n} \\ &\ge c_7 n^{-5/6} \sum_{0 \le x_k \le a_n/2} 1 \ge c_8 n^{1/6}, \end{aligned}$$

since by (22), there are at least $c_9 n$ terms in the last sum.

Let us introduce the notation

$$|x - x_j| = \min_{1 \le k \le n} |x - x_k| \qquad (x \in \mathbf{R}) \tag{24}$$

LEMMA 6 *We have*

$$\sum_{\substack{k=1 \\ k \neq j}}^{n-1} \frac{\Delta x_k}{|x - x_k|^\alpha} = \begin{cases} O(a_n^{1-\alpha}) & \text{if } 0 < \alpha < 1, \\ O(\log n) & \text{if } \alpha = 1, \\ O\left(\frac{n\psi_n(x)^{1/2}}{a_n}\right)^{\alpha-1} & \text{if } \alpha > 1 \end{cases} \qquad (x \in \mathbf{R}).$$

Proof Evidently, it suffices to prove the lemma for $|x| \le x_1$. It is easily seen that (cf. [5], relation (11.10))

$$\frac{\Delta x_k}{|x - x_k|} = O(1) \qquad (k = 1, \dots, n-1,\ k \neq j,\ x \in \mathbf{R})$$

and

$$\frac{\Delta x_k}{\Delta x_{k-1}} = O(1) \qquad (k = 2, \dots, n-1).$$

Thus

$$\sum_{\substack{k=1 \\ k \neq j}}^{n-1} \frac{\Delta x_k}{|x - x_k|^\alpha} = \sum_{k=1}^{j-2} \frac{\Delta x_k}{|x - x_k|^\alpha} + O\left(\sum_{k=j+2}^{n-1} \frac{\Delta x_{k-1}}{|x - x_k|^\alpha}\right) + O(1)$$

$$= O\left(\left(\int_{x_n}^{x_{j+1}} + \int_{x_{j-1}}^{x_1}\right) \frac{dy}{|x - y|^\alpha}\right) + O(1).$$

(One of the two sums and integrals here may be empty, depending on the position of x.) Hence performing the integrations for $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$ separately, and using that

$$a_n = O(n^{1/A})$$

(cf. [1, Lemma 4.5(b)]), as well as $|x|,\ |x_1| = O(a_n)$ we get the lemma.

Proof of Theorem 1. We may assume that $|x| \le a_{n+1}$, by Lemma 1. Using the relation

$$\left\| \frac{w(x) l_k(\mathcal{U}_n, x)}{w(x_k)} \right\| = O(1) \qquad (k = 1, \dots, n) \tag{25}$$

(cf. Lubinsky–Móricz [7, Lemma 2.6(b)]) we obtain from Lemmas 3 and 5

$$\lambda(\mathcal{V}_{n+2}, x)_w = w(x) \sum_{\substack{k=1 \\ k \neq j}}^{n-1} \frac{|l_k(\mathcal{Y}_n, x)|}{w(x_k)} + O(1) =$$

$$O\left[\sum_{\substack{k=1 \\ k \neq j}}^{n-1} \left(\frac{\psi_n(x_k)}{\psi_n(x)}\right)^{1/4-\varepsilon} \frac{\Delta x_k}{|x - x_k|}\right] + O(1).$$

Evidently

$$\frac{\psi_n(x)}{\psi_n(x_k)} \leq 2 + \frac{|x - x_k|}{a_n \psi_n(x_k)},$$

whence by arithmetic-geometric means inequality and Lemma 6 (used with $\alpha = 1$)

$$\sum_{\substack{k=1 \\ k \neq j}}^{n-1} \left(\frac{\psi_n(x)}{\psi_n(x_k)}\right)^{3/4} \frac{\Delta x_k}{|x - x_k|} =$$

$$O\left(\sum_{\substack{k=1 \\ k \neq j}}^{n-1} \frac{\Delta x_k}{|x - x_k|} + \sum_{\substack{|x_k| \leq a_n(1-n^{-2/3}) \\ k \neq j,n}} \frac{\Delta x_k}{(a_n - x_k)^{3/4} |x - x_k|^{1/4}} \right.$$

$$\left. + \frac{n^{1/2}}{a_n^{3/4}} \sum_{\substack{|x_k| > a_n(1-n^{-2/3}) \\ k \neq j,n}} \frac{\Delta x_k}{|x - x_k|^{1/4}}\right)$$

$$= O\left(\log n + \sum_{\substack{|x_k| \leq a_n(1-n^{-2/3}) \\ k \neq j,n}} \frac{\Delta x_k}{a_n - x_k} + \sum_{\substack{k=1 \\ k \neq j}}^{n-1} \frac{\Delta x_k}{|x - x_k|} \right.$$

$$\left. + \frac{n^{1/2}}{a_n^{3/4}} \sum_{\substack{|x_k| > a_n(1-n^{-2/3}) \\ k \neq j,n}} \frac{a_n n^{-2/3}}{a_n^{1/4} n^{-1/6}}\right) = O(\log n),$$

since in the last sum there are only $O(1)$ terms by (22). Thus the upper estimates of Theorem 1 are proved.

Now we prove the lower estimate in Theorem 1. According to Lemma 4.2(d) in [1], there exist constants $c_{10}, c_{11} > 0$ such that for $y := c_{10}a_n/n$ we have

$$|p_n(y)|w(y) \geq c_{11}a_n^{-1/2}y = c_{12}a_n^{-1/2}.$$

Hence and by (21), (22)

$$\begin{aligned}\lambda(\mathcal{V}_{n+2}, y) &\geq \sum_{a_n/4\leq x_k\leq a_n/2} \frac{(x_0^2 - y^2)|p_n(y)|w(y)}{(x_0^2 - x_k^2)|p_n'(x_k)|w(x_k)(x_k - y)} \\ &\geq c_{13}\frac{a_n}{n} \sum_{a_n/4\leq x_k\leq a_n/2} \frac{1}{x_k} \\ &\geq c_{14} \sum_{k=c_{15}}^{c_{15}'n} \frac{1}{k} \sim \log n,\end{aligned}$$

and Theorem 1 is completely proved.

3 LOWER ESTIMATE OF THE NORM OF PROJECTION OPERATORS FOR HERMITE WEIGHTS

The problem of stating a Faber-type theorem for general systems of nodes is a natural question. In this respect, unfortunately, we have to restrict ourselves to the special Hermite weight (5). As we will see, the main reason for this restriction is the lack of proper asymptotics for orthogonal polynomials with respect to weights $w \in \mathcal{W}$.

On the other hand, the result we are going to state is more general from another point of view. Namely, instead of Lagrange interpolation, we will consider linear projection operators $L_n : C_{w_0}(\mathbf{R}) \to \Pi_n$ endowed with the finite weighted norm

$$|||L_n|||_{w_0} := \sup_{0\not\equiv f\in C_{w_0}(\mathcal{R})} \frac{||w_0(x)L_n(f, x)||}{||w_0(x)f(x)||}. \tag{26}$$

THEOREM 2 *For any projection operator L_n we have*

$$|||L_n|||_{w_0} \geq c_{16}\log n.$$

In particular,

COROLLARY 2 *For any system of nodes $\mathcal{X}_n$ we have*

$$\lambda(\mathcal{X}_n)_{w_0} = \left\| e^{-x^2/2} \sum_{k=1}^{n} e^{x_k^2/2} |l_k(\mathcal{X}_n, x)| \right\| \geq c_{16} \log n. \tag{27}$$

COROLLARY 3 *Suppose the weight $w(x) = e^{-Q(x)}$ satisfies*

$$\liminf_{x \to \infty} \frac{Q(x)}{x^2} > \frac{1}{2} \tag{28}$$

and $f \in C_w(\mathbf{R})$ is such that the n-th partial sum $S_n(f, x)$ of the expansion of f in terms of the orthogonal polynomials $\{p_k(x)\}_{k=0}^{\infty}$ with respect to $w(x)$ exists. Then

$$|||S_n|||_{w_0} = \left\| e^{-x^2/2} \int_{-\infty}^{\infty} e^{y^2/2} w(y) \left| \sum_{k=0}^{n} p_k(x) p_k(y) \right| dy \right\| \geq c_{16} \log n. \tag{29}$$

In order to prove these corollaries, we have to show that the stated expressions (27) and (29) are indeed the norms of the corresponding operators defined in the sense of (26). First, note that by (28) the integral in (29) exists. Then, on the one hand, the expressions in the middle of (27) and (29) are easily seen to be majorized by the corresponding norms on the left hand sides, by using (3) and

$$S_n(f, x) = \int_{-\infty}^{\infty} w(y) f(y) \sum_{k=0}^{n} p_k(x) p_k(y)\, dy,$$

respectively. On the other hand, denoting by $\bar{x}$ a point where the norms in (27) and (29) are attained, we can define an $f_0 \in C_w(\mathbf{R})$ by

$$f_0(x_k) = e^{x_k^2/2} sgn\, l_k(\mathcal{X}_n, \bar{x}) \qquad (k = 1, \ldots, n),$$

$f_0(x)$ be linear and continuous between these nodes, constant in $\mathbf{R} \setminus (x_n, x_1)$; and

$$f_0(x) = e^{x^2/2} sgn \sum_{k=0}^{n} p_k(\bar{x}) p_k(x),$$

respectively. Applying the operators for these functions we can see the opposite inequalities in (27) and (29), whence the equalities.

To prove Theorem 2 we need some lemmas. Let $h_n(x)$ be the nth degree orthonormal Hermite polynomial (with respect to the weight e^{-x^2}), and define

$$\omega_n(x) = \pi^{1/2}\left(\frac{n}{2}\right)^{1/4}\left(1-\frac{x^2}{2n+1}\right)h_n(x). \tag{30}$$

Lemma 7 *We have*

$$\int_{-\infty}^{\infty} e^{-x^2}\omega_n^2(x)\,dx \geq c_{17}\sqrt{n}.$$

Proof We get

$$\int_{-\infty}^{\infty} e^{-x^2}\omega_n^2(x)\,dx = \pi\sqrt{\frac{n}{2}}\left[1-\frac{2}{2n+1}\int_{-\infty}^{\infty} e^{-x^2}x^2h_n^2(x)\,dx \right. \\ \left. +\frac{1}{(2n+1)^2}\int_{-\infty}^{\infty} e^{-x^2}x^4h_n(x)^2\,dx\right]. \tag{31}$$

Using the recurrence relation

$$x\sqrt{2}h_n(x) = \sqrt{n}h_{n-1}(x) + \sqrt{n+1}h_{n+1}(x)$$

(cf. Szegö [14], formula (5.5.8) rewritten for the orthonormal polynomials), we get

$$x^2h_n^2(x) = \frac{n}{2}h_{n-1}^2(x) + \frac{n+1}{2}h_{n+1}^2(x) + \sqrt{n(n+1)}h_{n-1}(x)h_{n+1}(x),$$

i.e.

$$\int_{-\infty}^{\infty} e^{-x^2}x^2h_n^2(x)\,dx = \frac{2n+1}{2}.$$

Iterating the above recurrence relations we get

$$x^4h_n^2(x) = \frac{n(n-1)}{4}h_{n-2}^2(x) + \dots,$$

where the dots mean either the square of Hermite polynomials with positive coefficients, or the product of two Hermite polynomials of different degrees. Hence

$$\int_{-\infty}^{\infty} e^{-x^2}x^4h_n^2(x)\,dx > \frac{n(n-1)}{4}.$$

Substituting these values in (31) we get the lemma.

For $|x| \le \sqrt{2n+1}$, introduce the notation

$$x = \sqrt{2n+1}\cos\phi_n. \tag{32}$$

LEMMA 8 *We have*

$$e^{-x^2/2}\omega_n(x) = \begin{cases} \sin^{3/2}\phi_n \sin\left[\frac{2n+1}{4}(\sin 2\phi_n - 2\phi_n) + \frac{3\pi}{4}\right] + O(n^{-1/2})\,' \\ \quad \text{if } |x| \le \sqrt{2n+1} - \frac{c_{18}}{n^{1/6}}, \\ O(n^{-1/2}) \\ \quad \text{if } ||x| - \sqrt{2n+1}| \le \frac{c_{18}}{n^{1/6}}, \end{cases}$$

where $c_{18} > 0$ *is an arbitrary constant.*

Proof We make use of the following asymptotic formula:

$$e^{-x^2/2}h_n(x) = \pi^{-1/2}2^{1/4}n^{-1/4}\sin^{-1/2}\phi_n \sin\left[\frac{2n+1}{4}(\sin 2\phi_n - 2\phi_n) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n\sin^3\phi_n}\right) \qquad (0 < \phi_n < \pi)$$

(cf. Erdélyi [2], formula (6.12) transformed to the orthonormal Hermite polynomials). Hence and from (30)

$$e^{-x^2/2}\omega_n(x) = \sin^{3/2}\phi_n \sin\left[\frac{2n+1}{4}(\sin 2\phi_n - 2\phi_n) + \frac{3\pi}{4}\right] + O\left(\frac{1}{n\sin^{3/2}\phi_n}\right) \qquad (0 < \phi_n < \pi).$$

By symmetry, it is sufficient to prove the lemma for $x \ge 0$. Now if $0 \le x \le \sqrt{2n+1} - \frac{c_{18}}{n^{1/6}}$, then it is easily seen that

$$\sin\phi_n \ge \frac{\sqrt{c_{18}}}{n^{1/3}}, \tag{33}$$

whence the first statement of the lemma is obtained. To prove the second statement we use the estimate

$$e^{-x^2/2}|h_n(x)| = O(n^{-1/12}) \qquad \left(|x - \sqrt{2n+1}| \le \frac{c_{18}}{n^{1/6}}\right)$$

(see Szegő [14], formula (8.22.14), transformed again to the orthonormal Hermite polynomials). Hence and from (30)

$$e^{-x^2/2}|\omega_n(x)| = O(n^{1/4}n^{-2/3}n^{-1/12}) = O(n^{-1/2})$$
$$\left(|x-\sqrt{2n+1}| \le \frac{c_{18}}{n^{1/6}}\right).$$

LEMMA 9 *We have*

$$e^{-\frac{x^2+y^2}{2}}\left|\sum_{k=3}^{[n^{1/4}]} \frac{\omega_{n-k}(x)\omega_{n-k}(y) - \omega_{n+k}(x)\omega_{n+k}(y)}{k}\right| = O(1) \qquad (x, y \in \mathbf{R}).$$

Proof We distinguish two cases.

CASE 1: $|x|, |y| \le \sqrt{2(n-[n^{1/4}])+1} - \frac{1}{(n-[n^{1/4}])^{1/6}}$. Then we can use the first asymptotic formula in Lemma 8 for all $\omega_{n\pm k}$ in question. From (32)

$$x = \sqrt{2n+1}\cos\phi_n = \sqrt{2(n\pm k)+1}\cos\phi_{n\pm k}$$
$$\left(k = 3, \dots, [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

Consider the function

$$\begin{aligned} g(k) &:= \sin 2\phi_{n+k} - 2\phi_{n+k} \\ &= 2\cos\phi_n\sqrt{\frac{2n+1}{2n+2k+1}\left(1 - \frac{2n+1}{2n+2k+1}\cos^2\phi_n\right)} \\ &\quad - 2\arccos\left(\sqrt{\frac{2n+1}{2n+2k+1}}\cos\phi_n\right) \\ &\quad \left(|k| \le [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right). \end{aligned}$$

We obtain

$$g'(k) = -4\phi'_{n+k}\sin^2\phi_{n+k} = \frac{4\sqrt{2n+1}\cos\phi_n}{(2n+2k+1)^{3/2}}\sqrt{1 - \frac{2n+1}{2n+2k+1}\cos^2\phi_n},$$

whence

$$g'(0) = \frac{2\sin 2\phi_n}{2n+1},$$

and

$$|g''(k)| = O\left(\frac{1}{n^2} + \frac{1}{n^2\sqrt{1 - \frac{2n+1}{2n-2[n^{1/4}]+1}\cos^2\phi_n}}\right) = O(n^{-5/3})$$
$$\left(|k| \le [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

Thus a second degree Taylor expansion of $g(k)$ about $k = 0$ yields

$$\sin 2\phi_{n+k} - 2\phi_{n+k} = \sin 2\phi_n - 2\phi_n + \frac{2\sin 2\phi_n}{2n+1}k + O(n^{-7/6})$$
$$\left(|k| \le [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

Hence

$$\frac{2n+2k+1}{4}(\sin 2\phi_{n+k} - 2\phi_{n+k})$$
$$= \frac{2n+1}{4}(\sin 2\phi_n - 2\phi_n) + (\sin 2\phi_n - \phi_n)k + O(n^{-1/6})$$
$$\left(|k| \le [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

Similarly, we can easily check that

$$\sin\phi_{n+k} = \sin\phi_n + O\left(\frac{1}{n\sin\phi_n}\right) = \sin\phi_n + O(n^{-2/3})$$
$$\left(|k| \le [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

Thus the first part of Lemma 8 applied for $n \pm k$ instead of n gives

$$e^{-x^2/2}\omega_{n\pm k}(x) = \sin^{3/2}\phi_n \sin(\pm\alpha_n k + \beta_n) + O(n^{-1/6})$$
$$\left(k = 3, \dots, [n^{1/4}],\ \sin\phi_n \ge \frac{1}{2n^{1/3}}\right).$$

where

$$\alpha_n := \sin 2\phi_n - \phi_n, \quad \beta_n := \frac{2n+1}{4}(\sin 2\phi_n - 2\phi_n).$$

Similarly to (32), let

$$y = \sqrt{2n+1}\cos\psi_n = \sqrt{2(n\pm k)+1}\cos\psi_{n\pm k}$$
$$\left(k = 3,\dots,[n^{1/4}],\ \sin\psi_n \ge \frac{1}{2n^{1/3}}\right);$$

then

$$e^{-y^2/2}\omega_{n\pm k}(y) = \sin^{3/2}\psi_n \sin(\pm\gamma_n k + \delta_n) + O(n^{-1/6}),$$

where

$$\gamma_n := \sin 2\psi_n - \psi_n, \quad \delta_n := \frac{2n+1}{4}(\sin 2\psi_n - 2\psi_n).$$

Hence we obtain

$$e^{-\frac{x^2+y^2}{2}}\left|\sum_{k=3}^{[n^{1/4}]}\frac{\omega_{n-k}(x)\omega_{n-k}(y) - \omega_{n+k}(x)\omega_{n+k}(y)}{k}\right|$$
$$= (\sin\phi_n \sin\psi_n)^{3/2}\left|\sum_{k=3}^{[n^{1/4}]}\frac{\sin(\alpha_n k+\beta_n)\sin(\gamma_n k+\delta_n) - \sin(\alpha_n k-\beta_n)\sin(\gamma_n k-\delta_n)}{k}\right| + O\left(\frac{\log n}{n^{1/6}}\right)$$
$$= O\left(\left|\sum_{k=3}^{[n^{1/4}]}\frac{\sin(\alpha_n-\gamma_n)k}{k}\right| + \left|\sum_{k=3}^{[n^{1/4}]}\frac{\sin(\alpha_n+\gamma_n)k}{k}\right| + \frac{\log n}{n^{1/6}}\right) = O(1).$$

Case 2: $\sqrt{2(n+[n^{1/4}])+1} - \frac{2}{(n+[n^{1/4}])^{1/6}} \le \max(|x|,|y|) \le \sqrt{2(n-[n^{1/4}])+1} + \frac{2}{(n-[n^{1/4}])^{1/6}}$. Then at least for one of x and y, the second part of Lemma 8 applies. Since the first part automatically yields the bound $O(1)$, we get in this case

$$e^{-\frac{x^2+y^2}{2}}\left|\sum_{k=3}^{[n^{1/4}]}\frac{\omega_{n-k}(x)\omega_{n-k}(y) - \omega_{n+k}(x)\omega_{n+k}(y)}{k}\right|$$
$$= O\left(\sum_{k=3}^{[n^{1/4}]}\frac{1}{k\sqrt{n}}\right) = O\left(\frac{\log n}{\sqrt{n}}\right).$$

Since for sufficiently large n's

$$\sqrt{2(n+[n^{1/4}])+1} - \frac{2}{(n+[n^{1/4}])^{1/6}} \le$$
$$\sqrt{2(n-[n^{1/4}])+1} - \frac{1}{(n-[n^{1/4}])^{1/6}},$$

the lemma is proved so far for all $|x|, |y| \le \sqrt{2(n-[n^{1/4}])+1}+\frac{1}{(n-[n^{1/4}])^{1/6}}$. But it is easily seen that the Mhaskar–Rahmanov–Saff result (8) extends to polynomials of two variables in an obvious way. (One just has to fix one variable, and apply the one variable result to the other variable.) In our case, for the weights $e^{-x^2/2}$, $e^{-y^2/2}$ and polynomials of degree at most $n+[n^{1/4}]+2$ in both variables, the M-R-S number is

$$\sqrt{2(n+[n^{1/4}]+2)} < \sqrt{2(n-[n^{1/4}])+1}+\frac{2}{(n-[n^{1/4}])^{1/6}}$$

for sufficiently large n's, which means that the lemma is proved for all $x, y \in \mathbf{R}$.

Proof of Theorem 2. Consider the polynomial

$$Q_n(x,y)=\sum_{k=3}^{[n^{1/4}]}\frac{\omega_{n+k}(x)\omega_{n+k}(y)-\omega_{n-k}(x)\omega_{n-k}(y)}{k}$$

of two variables, and apply the projection operator L_n for it with respect to x. Using the reproducing property we obtain

$$L_n(Q_n(\cdot,y),x)=\sum_{k=3}^{[n^{1/4}]}\frac{\omega_{n+k}(y)L_n(\omega_{n+k},x)-\omega_{n-k}(x)\omega_{n-k}(y)}{k},$$

whence

$$L_n(Q_n(\cdot,y),y)=\sum_{k=3}^{[n^{1/4}]}\frac{\omega_{n+k}(y)L_n(\omega_{n+k},y)-\omega_{n-k}^2(y)}{k}.$$

Multiplying both sides by e^{-y^2}, integrating over $\mathbf{R}$ and using the orthogonality of the Hermite polynomials (see (30)), as well as Lemma 7 we get

$$\int_{-\infty}^{\infty}e^{-y^2}L_n(Q_n(\cdot,y),y)\,dy=\sum_{k=3}^{[n^{1/4}]}\frac{1}{k}\int_{-\infty}^{\infty}e^{-y^2}\omega_{n-k}^2(y)\,dy \tag{34}$$

$$\ge c_{17}\sqrt{n}\sum_{k=3}^{[n^{1/4}]}\frac{1}{k}\ge c_{19}\sqrt{n}\log n.$$

By a result of Freud [4], here with suitable $c_{20}, c_{21} > 0$

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} e^{-y^2} L_n(Q_n(\cdot, y), y)\, dy\right| &\le c_{20} \int_{-c_{21}\sqrt{n}}^{c_{21}\sqrt{n}} e^{-y^2} |L_n(Q_n(\cdot, y), y)|\, dy \\
&\le 2c_{20}c_{21}\sqrt{n} e^{-y_0^2} |L_n(Q_n(\cdot, y_0), y_0)| \\
&\le 2c_{20}c_{21}\sqrt{n} e^{-y_0^2/2} ||e^{-x^2/2} L_n(Q_n(\cdot, y_0), x)|| \qquad (35)
\end{aligned}
$$

with a properly chosen $|y_0| \le c_{21}\sqrt{n}$. Since by Lemma 9

$$||e^{-x^2/2} Q_n(x, y_0)|| = O\left(e^{y_0^2/2}\right),$$

we finally obtain from (34) and (35)

$$|||L_n|||_{w_0} \ge \frac{||e^{-x^2/2} L_n(Q(\cdot, y_0), x)||}{||e^{-x^2/2} Q_n(x, y_0)||} \ge c_{22} \log n.$$

Remark It is clear from the proof that if in the definition of the norm of projection operator, instead of $f \in C_{w_0}$, we restrict ourselves to $f \in \Pi_{n+[n^{1/4}]}$, then the statement remains true.

4 HERMITE–FEJÉR INTERPOLATION

Take two arbitrary weights $w_1(x) \le w_2(x)$ $(x \in \mathbf{R})$ both satisfying the conditions (1)–(2), and for an arbitrary $f \in C_{w_2}(\mathbf{R})$ and system of nodes $\mathcal{X}_n$ consider the uniquely determined Hermite–Fejér interpolating polynomial $H(f, \mathcal{X}_n, x) \in \Pi_{2n-1}$ subject to the conditions

$$H(f, \mathcal{X}_n, x_k) = f(x_k), \quad H'(f, \mathcal{X}_n, x_k) = 0 \qquad (k = 1, \dots, n).$$

In analogy with the finite interval case, we are interested in the following problem: under what conditions on $w_1, w_2, \mathcal{X}_n$ will the relation

$$\lim_{n\to\infty} ||w_1(x)[f(x) - H(f, \mathcal{X}_n, x)]|| = 0 \quad \text{for all } f \in C_{w_2}(\mathbf{R}) \qquad (36)$$

hold.

The reader will at once notice that, compared to the Lagrange interpolation, there is an essential difference in posing this question, namely we have two weights instead of one. Of course, the ideal situation would be $w_1(x) \equiv w_2(x)$, but at present we are unable to state any result in this respect.

Let us mention that Lubinsky [6, Corollary 2.4(a)] proved (36) when

$$\tilde{w}_1(x) = \frac{w^2(x)}{(1+|Q'(x)|)^\alpha(1+|x|)},$$
$$\tilde{w}_2(x) = w^2(x)(1+|Q'(x)|)^\beta(1+|x|)^2,$$

and $\mathcal{X}_n = \mathcal{U}_n$ are the roots of the orthogonal polynomials $p_n(x)$ with respect to any weight $w \in \mathcal{W}$. Here $\alpha > 1,\ \beta > 2$ depend on the weight w and on the orthogonal polynomials p_n in a complicated way.

Our purpose in this section is to improve this result, and also to give an error estimate similar to the case of finite interval.

THEOREM 3 *Let $w \in \mathcal{W}$,*

$$w_1(x) = \frac{w^2(x)}{(1+|x|)^{1/3}}, \quad w_2(x) = w^2(x)(1+|Q'(x)|).$$

Then for an arbitrary $f \in C_{w_2}(\mathbf{R})$ we have

$$\|w_1(x)[f(x) - H(f, \mathcal{U}_n, x)]\| = O\left(E_m(f)_{w_2} + \left(\frac{a_n}{n}\right)^{2/3} \log n \sum_{k=0}^{m} \frac{E_k(f)_{w_2}}{a_k}\right) \qquad (m \le 2n-1) \tag{37}$$

where $a_0 = 1$ and $a_k,\ k \ge 1$ are the M-R-S numbers associated with w.

Remarks 1. The parameter $m \le 2n-1$ is arbitrary, and has to be chosen optimally. It is readily seen that if m tends to infinity sufficiently slowly then we have convergence in (37). In the particular case

$$w(x) = e^{-|x|^\alpha}, \quad E_n(f)_{w_2} = O(n^{-\beta}) \qquad (\alpha > 1,\ \beta > 0)$$

we obtain (with a proper choice of m)

$$\|w_1(x)[f(x) - H(f, \mathcal{U}_n, x)]\| = \begin{cases} O\left(\frac{\log^{\frac{\alpha\beta}{\alpha-1}} n}{n^{2\beta/3}}\right) & \text{if } 1/\alpha + \beta < 1, \\ O\left(\frac{\log^2 n}{n^{2\beta/3}}\right) & \text{if } 1/\alpha + \beta = 1, \\ O\left(\frac{\log n}{n^{\frac{2}{3}(1-\frac{1}{\alpha})}}\right) & \text{if } 1/\alpha + \beta > 1. \end{cases}$$

2. Since

$$\tilde{w}_1(x) \le w_1(x) \le w_2(x) \le \tilde{w}_2(x) \qquad (x \in \mathbf{R}),$$

our theorem is stronger than the quoted result of Lubinsky [6] (namely, it holds for a wider class of functions, with a larger weight in the error estimate).

Proof of Theorem 3. Let $m \le 2n-1$, and let $q_m \in \Pi_m$ be a polynomial such that

$$\begin{aligned} ||w_1(x)[f(x) - q_m(x)|| &\le ||w^2(x)[f(x) - q_m(x)|| \\ &\le ||w_2(f - q_m)|| = E_m(f)_{w_2}. \end{aligned}$$

Using well-known formulas for the Hermite–Fejér interpolation we get

$$\begin{aligned} &||w_1(x)[f(x) - H(f, \mathcal{U}_n, x)|| \\ &\quad \le ||w_1(x)[f(x) - q_m(x)]|| + ||w_1(x)[q_m(x) - H(q_m, \mathcal{U}_n, x)]|| \\ &\quad\quad + ||w_1(x)H(q_m - f, \mathcal{U}_n, x)|| \\ &\quad \le E_m(f)_{w_2}\left[1 + \left\| \frac{1}{(1+|x|)^{1/3}} \sum_{k=1}^n |v_k(x)| \frac{w^2(x) l_k^2(\mathcal{U}_n, x)}{w^2(x_k)(1 + |Q'(x_k)|)} \right\| \right] \\ &\quad\quad + \left\| \frac{1}{(1+|x|)^{1/3}} \sum_{k=1}^n |q_m'(x_k)||x - x_k| w^2(x) l_k^2(\mathcal{U}_n, x) \right\|, \end{aligned}$$

where

$$v_k(x) := 1 - \frac{p_n''(x_k)}{p_n'(x_k)}(x - x_k) \qquad (k = 1, \dots, n).$$

Since

$$|v_k(x)| = O(1 + (1 + |Q'(x_k)|)|x - x_k|) \qquad (k = 1, \dots, n)$$

(cf. Lubinsky [6], Lemma 4.3(iii)), we get

$$\begin{aligned} ||w_1(x)[f(x) - H(f, \mathcal{U}_n, x)]|| &\le E_m(f)_{w_2}\left[1 + \left\| \sum_{k=1}^n \frac{w^2(x) l_k^2(\mathcal{U}_n, x)}{w^2(x_k)} \right\| \right] \\ + [E_m(f)_{w_2} + ||w^2 q_m'||] &\left\| \frac{1}{(1+|x|)^{1/3}} \sum_{k=1}^n \frac{|x - x_k| w^2(x) l_k^2(\mathcal{U}_n, x)}{w^2(x_k)} \right\|. \end{aligned} \tag{38}$$

Here using the notation (24), as well as (25), (11), (21) and Lemma 6 with $\alpha = 2$,

$$\sum_{k=1}^{n} \frac{w^2(x) l_k^2(\mathcal{U}_n, x)}{w^2(x_k)} = O(1) + O\left(\sum_{k \neq j} \frac{a_n^{-1} \psi_n^{-1/2}(x)}{n^2 a_n^{-3} \psi_n(x_k)^{1/2} (x - x_k)^2} \right)$$
$$= O(1) + O\left(\frac{a_n}{n \psi_n(x)^{1/2}} \right) \sum_{k \neq j,n} \frac{\Delta x_k}{(x - x_k)^2} = O(1) \qquad (x \in \mathbf{R}). \tag{39}$$

Similarly, we obtain for the other sum

$$\frac{1}{(1 + |x|)^{1/3}} \sum_{k=1}^{n} \frac{|x - x_k| w^2(x) l_k^2(\mathcal{U}_n, x)}{w^2(x_k)}$$
$$= O\left(\frac{a_n^{-1/2} \psi_n^{-1/4}(x)}{n a_n^{-3/2} \psi_n^{1/4}(x_j)(1 + |x|)^{1/3}} \right)$$
$$+ O\left(\frac{a_n}{n \psi_n^{1/2}(x)(1 + |x|)^{1/3}} \right) \sum_{k \neq j,n} \frac{\Delta x_k}{|x - x_k|}$$
$$= O\left(\frac{a_n \log n}{n \psi_n^{1/2}(x)(1 + |x|)^{1/3}} \right). \tag{40}$$

Here, considering the definition (12) of $\psi_n(x)$ we get

$$\frac{1}{\psi_n^{1/2}(x)(1 + |x|)^{1/3}} = O(n^{1/3} a_n^{-1/3}) \qquad (x \in \mathbf{R}),$$

whence and from (38), (39) and (40)

$$||w_1(x)[f(x) - H(f, \mathcal{U}_n, x)]|| = O(E_m(f)_{w_2})$$
$$+ ||w^2 q_m'|| O\left(\left(\frac{a_n}{n} \right)^{2/3} \log n \right). \tag{41}$$

Thus it remains to estimate $||w^2 q_m'||$. Let $2^s \leq m < 2^{s+1}$, and consider

$$q_m = q_0 + \sum_{k=1}^{s} (q_{2^k - 1} - q_{2^{k-1} - 1}) + (q_m - q_{2^s - 1}). \tag{42}$$

Here

$$\|w^2(q_{2^k-1} - q_{2^{k-1}-1})\| \le \|w_2(q_{2^k-1} - f)\| + \|w_2(q_{2^{k-1}-1} - f)\|$$
$$\le 2E_{2^{k-1}-1}(f)_{w_2} \qquad (k = 1, \dots, s)$$

and similarly

$$\|w^2(q_m - q_{2^s-1})\| \le 2E_{2^s-1}(f)_{w_2}.$$

Hence by the Markov type estimate for the weighted norm of the derivative of a polynomial (see Levin and Lubinsky [5])

$$\|w^2(q'_{2^k-1} - q'_{2^{k-1}-1})\| = O\left(\frac{2^k E_{2^{k-1}-1}(f)_{w_2}}{a_{2^k-1}}\right) = O\left(\sum_{i=2^{k-2}}^{2^{k-1}-1} \frac{E_i(f)_{w_2}}{a_i}\right)$$
$$(k = 2, \dots, s+1),$$

and similarly

$$\|w^2(q'_m - q'_{2^s-1})\| = O\left(\frac{mE_{2^s-1}(f)_{w_2}}{a_m}\right) = O\left(\sum_{i=1}^{2^s} \frac{E_i(f)_{w_2}}{a_i}\right).$$

(Here we used the obvious relation $a_{2^k-1}(w^2) \le a_{2^k}(w^2) = a_{2^{k-1}}(w)$.) Thus by differentiating (42)

$$\|w^2 q'_m\| = O\left(E_0(f)_{w_2} + \sum_{i=1}^{2^s} \frac{E_i(f)_{w_2}}{a_i}\right) = O\left(\sum_{i=0}^{m} \frac{E_i(f)_{w_2}}{a_i}\right).$$

Substituting this into (41), the theorem is completely proved.

Remark It is interesting to note that the use of the system of nodes $\mathcal{V}_{n+2}$ instead of $\mathcal{U}_n$ does not improve the result (in sharp contrast to Lagrange interpolation).

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