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Remarks on the Hardy Inequality

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Let D be an open subset of $\mathbb{R}^n$ $(n \geq 2)$ with finite Lebesgue n-measure, let $d(x)$ be the distance from $x \in \mathbb{R}^n$ to the boundary ∂D of D, and let $1 < p < \infty$. We give a simple direct proof that if $\mathbb{R}^n \setminus D$ satisfies the plumpness condition of Martio and Väisälä [10], then the inequality of Hardy type,

$$\int_D (|u(x)|/d^{\alpha}(x))^p\,dx \leq C\int_D \left(|\nabla u(x)|/d^{\beta}(x)\right)^p dx, \quad u \in C_0^{\infty}(D),$$

holds whenever $\beta \geq \max\{0, \alpha - 1\}$. We also show that the plumpness condition may be replaced by ones which enable domains with lower-dimensional portions of their boundaries to be handled.

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1 INTRODUCTION

Let D be an open subset of $\mathbf{R}^n$, let $1 \leq p < \infty$, and given $x \in D$ let $d(x)$ be the distance from x to the boundary ∂D of D. It is well known (cf. [6], p. 223) that if u belongs to the Sobolev space $W_p^1(D)$ and $u/d \in L_p(D)$, then in fact u lies in $\overset{0}{W}{}_p^1(D)$, the closure of $C_0^{\infty}(D)$ in $W_p^1(D)$. This holds with no restrictions on ∂D. The result in the opposite direction, namely that if $u \in \overset{0}{W}{}_p^1(D)$ then $u/d \in L_p(D)$, would follow immediately if one knew that the Hardy inequality

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$$\int_D (|u(x)|/d(x))^p\, dx \le C \int_D |\nabla u(x)|^p\, dx,\ u \in \overset{0}{W}{}_p^1(D) \tag{1.1}$$

was true for the particular D and the particular p. The validity of (1.1) has been extensively investigated: for example, Davies [4] has shown that if $p = 2$ and D is bounded and satisfies a certain type of cone condition, then (1.1) holds; it is clear that his argument can be adapted to permit other values of p. Other work on the Hardy inequality (1.1), and weighted analogues of it, may be found in the paper by Ancona [2] and the book by Opic and Kufner [11]. Moreover, Lewis [9] has shown that if $1 < p \le n$ and $\mathbf{R}^n \setminus D$ is $(1, p)$ uniformly fat, then (1.1) holds; if $n < p < \infty$ he shows that (1.1) holds for all $D \ne \mathbf{R}^n$. The uniform fatness condition which he imposes when $1 < p \le n$ is that there is a positive constant λ such that for all $x \in \mathbf{R}^n \setminus D$ and all $r > 0$,

$$R_{1,p}\left(r^{-1}\left(B(x,r) \cap \left(\mathbf{R}^n \setminus D\right)\right)\right) \ge \lambda,$$

where $B(x, r)$ is the open ball in $\mathbf{R}^n$ with centre x and radius r, and $R_{1,p}$ is a certain Riesz capacity. Sufficient conditions for (1.1) to hold have also been given by Wannebo [14]; these are expressed in terms of a capacity introduced by Maz'ja and enable him to reproduce Lewis's results for $p > n$ and to show that (1.1) holds if $p > n - 1$ and D ($\ne \mathbf{R}^n$) is simply connected. In [15] other sufficient conditions for (1.1) to hold are given.

In the present paper we show that if D has finite volume and $\mathbf{R}^n \setminus D$ satisfies the plumpness condition of Martio and Väisälä [10], then not only does (1.1) hold but also more general inequalities of the form

$$\int_D \left(|u(x)|/d^\alpha(x)\right)^p dx \le C \int_D \left(|\nabla u(x)|/d^\beta(x)\right)^p dx,\, u \in C_0^\infty(D) \tag{1.2}$$

where $1 < p < \infty$ and $\beta \ge \max\{0, \alpha - 1\}$. This is established in Section 2 by comparatively straightforward procedures. When D has a Lipschitz boundary, our result agrees with Theorem 10.4 of Gurka and Opic [7], obtained by entirely different methods and under the additional assumption that $\beta > p/(p-1)$. If $\alpha = 1$ and $\beta = 0$, (1.2) coincides with (1.1) but does not then give anything new, as it can be shown that if $\mathbf{R}^n \setminus D$ is plump and unbounded, it is $(1, p)$ uniformly fat for every $p > 1$ so that Lewis's result applies. The plumpness condition, which will be explained in detail in Section 2, is a rather natural geometric condition on D which is easy to check and has nothing to do with p. Like Lewis, our arguments depend on a well-known lemma due to Carleson [3]; but we hope that our direct use of the plumpness condition may have some appeal for those who are less familiar with notions of capacity, and may stimulate further work. In Section 3, the plumpness condition is replaced by ones which enable us to handle domains with lower-dimensional portions of their boundaries, and here the range of possible p's for which, say, (1.1) holds is dependent upon the properties of D. While these results can in fact be obtained from capacity results, we hope that the direct method of proof will be of interest.

2 A WEIGHTED HARDY INEQUALITY

First we fix the notation and provide some basic definitions. Throughout the paper we shall assume (unless otherwise stated) that D is an open subset of $\mathbf{R}^n$ $(n \geq 2)$ with finite Lebesgue n-measure. Given any sets $A, B \subset \mathbf{R}^n$, the distance between A and B will denoted by $d(A, B)$ and the distance from $x \in \mathbf{R}^n$ to A by $d(x, A)$, writing $d(x) = d(x, \partial D)$ for shortness; if A has finite Lebesgue n-measure $|A|_n$, the average of a function u over A is defined to be

$$u_A = |A|_n^{-1} \int_A u(x)\, dx.$$

The open ball in $\mathbf{R}^n$ with centre x and radius $t > 0$ will be denoted by $B(x, t)$; when $m \in \mathbf{N} \cup \{\infty\}$, $C_0^m(D)$ will stand for the space of all m times continuously differentiable real-valued functions with compact support in D; we write $\|u\|_{p,D} = \left(\int_D |u(x)|^p\, dx\right)^{1/p}$ for all $p \in (1, \infty)$; k-dimensional Hausdorff measure on $\mathbf{R}^n$ will be denoted by $\mathcal{H}^k$ when $k < n$; $W_p^1(D)$ will stand for the Sobolev space of all functions which, together with their first-order distributional derivatives, are in $L_p(D)$. Given two non-negative expressions (that is, functions or functionals) R_1, R_2 we shall write $R_1 \preceq R_2$ as a shorthand for the statement that $R_1 \leq C R_2$ for some constant $C \in (0, \infty)$ independent of the variables in the expressions R_1, R_2; if $R_1 \preceq R_2$ and $R_2 \preceq R_1$ we write $R_1 \approx R_2$.

Definition 2.1 *Given any $b \in (0, 1]$, we say that $\mathbf{R}^n \setminus D$ is b-plump if there exists $\sigma > 0$ such that for all $y \in \partial D$ and all $t \in (0, \sigma]$, there is an $x \in (\mathbf{R}^n \setminus D) \cap \overline{B(y, t)}$ with $d(x) \geq bt$.*

This definition is due to Martio and Väisälä [10]; Jerison and Kenig [8] call the hypothesis of the definition a *corkscrew condition.* Moreover, there is a connection with the *exterior regular* domains of Triebel and Winkelvoss [13]: if D coincides with the interior of its closure, then D is exterior regular if, and only if, it is b-plump for some b.

Our first result is the following:

THEOREM 2.2. *Suppose that $\mathbf{R}^n \setminus D$ is b-plump for some $b \in (0, 1]$, let $1 < p < \infty$ and let $\alpha, \beta \in \mathbf{R}$ be such that*

$$\beta \geq \max\{0, \alpha - 1\}. \tag{2.1}$$

Then there is a constant $C > 0$ such that for all $u \in C_0^\infty(D)$,

$$\int_D \left(|u(x)| / d^\alpha(x)\right)^p dx \leq C \int_D \left(|\nabla u(x)| / d^\beta(x)\right)^p dx. \tag{2.2}$$

Proof Let $u \in C_0^\infty(D)$; we may and shall suppose that u is defined on all of $\mathbf{R}^n$ and is zero on $\mathbf{R}^n \setminus D$. Let $\mathcal{W}$ be a Whitney decomposition of D (see [12], p. 16); that is, $\mathcal{W}$ is a family of closed dyadic cubes Q, with pairwise disjoint interiors, such that $D = \bigcup_{Q \in \mathcal{W}} Q$,

$$1 \leq d(Q, \partial D) / \operatorname{diam}(Q) \leq 4 \ \text{ for all } Q \in \mathcal{W}$$

(diam (Q) being the diameter of Q) and

$$1/4 \leq \operatorname{diam}(Q_1) / \operatorname{diam}(Q_2) \leq 4 \text{ for all } Q_1, Q_2 \in \mathcal{W} \text{ with } Q_1 \cap Q_2 \neq \emptyset.$$

For each $Q \in \mathcal{W}$ we fix an $x_Q \in \partial D$ such that $d(\partial D, Q) = d(x_Q, Q)$ and choose a cube $\widetilde{Q} \supset Q$ with centre x_Q such that $\operatorname{diam}(\widetilde{Q}) \approx \operatorname{diam}(Q)$. Then

$$\begin{aligned}\int_D \left(|u(x)| / d^\alpha(x)\right)^p dx &\preceq \sum_{Q \in \mathcal{W}} \int_Q \left(|u(x)| / \operatorname{diam}^\alpha(Q)\right)^p dx \\ &\leq \sum_{Q \in \mathcal{W}} \int_{\widetilde{Q}} \left(|u(x)| / \operatorname{diam}^\alpha(Q)\right)^p dx. \end{aligned} \tag{2.3}$$

Since $\mathbf{R}^n \setminus D$ is b-plump, there exists $\sigma > 0$ such that for all $z \in \partial D$ and all $t \in (0, \sigma]$, there is a $y \in (\mathbf{R}^n \setminus D) \cap \overline{B(z, t)}$ with $d(y, \partial D) > bt$. We may assume that $\sigma \geq \operatorname{diam}(\widetilde{Q})$ for all $Q \in \mathcal{W}$ and so may choose $t = \operatorname{diam}(\widetilde{Q})$; for if there is a maximal $Q_0 \in \mathcal{W}$ such that $\operatorname{diam}(\widetilde{Q}_0) > \sigma$, we simply take $k > 0$ such that $\sigma \geq k \operatorname{diam}(\widetilde{Q}_0)$ and then work with

k diam $(\widetilde{Q})$ instead of diam $(\widetilde{Q})$. It follows that for each $Q \in \mathcal{W}$ there is a $y \in (\mathbf{R}^n \setminus D) \cap \overline{B(x_Q, \text{ diam } (\widetilde{Q}))}$ with $d(y) > b$ diam $(\widetilde{Q})$; we write

$$\widetilde{A} = Q\left(y, b \text{ diam } (\widetilde{Q})/n\right),$$

the open cube with centre y and sides of length b diam $(\widetilde{Q})/n$ parallel to the axes.

As $\widetilde{A} \subset \mathbf{R}^n \setminus D$, the mean value $u_{\widetilde{A}} = 0$. Thus from (2.3) we obtain

$$\int_D \left(|u(x)|/d^\alpha(x)\right)^p dx \preceq \sum_{Q \in \mathcal{W}} \int_{\widetilde{Q}} \left(|u(x) - u_{\widetilde{A}}|/\text{ diam}^\alpha (Q)\right)^p dx. \tag{2.4}$$

Use of Hölder's and Minkowski's inequalities now shows that for all $c \in \mathbf{R}$,

$$\begin{aligned}\int_{\widetilde{Q}} |u(x) - u_{\widetilde{A}}|^p dx &\le 2^p \left(|\widetilde{Q}|_n / |\widetilde{A}|_n\right) \int_{\widetilde{Q}} |u(x) - c|^p dx \\ &= 2^p (n/b)^n \int_{\widetilde{Q}} |u(x) - c|^p dx.\end{aligned}$$

The choice $c = u_{\widetilde{Q}}$ in this inequality, together with the Poincaré inequality in a cube (see [6], p. 243), gives

$$\int_{\widetilde{Q}} |u(x) - u_{\widetilde{A}}|^p dx \preceq |\widetilde{Q}|_n^{\frac{p}{n}+1-\frac{p}{q}} \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{p/q}, \tag{2.5}$$

where p and q are related by $1 \le q < p = nq/(n-q)$; the constant implicit in the inequality is independent of $\widetilde{Q}$. Since $\nabla u(x) = 0$ whenever $x \in \mathbf{R}^n \setminus D$, we see that if $\beta \ge 0$,

$$\begin{aligned}\int_{\widetilde{Q}} |\nabla u(x)|^q dx &= \sum_{Q_1 \in \mathcal{W}, Q_1 \cap \widetilde{Q} \neq \emptyset} \int_{Q_1 \cap \widetilde{Q}} |\nabla u(x)|^q dx \\ &\approx \sum_{Q_1 \in \mathcal{W}, Q_1 \cap \widetilde{Q} \neq \emptyset} \int_{Q_1 \cap \widetilde{Q}} \{\text{diam } (Q_1)/d(x)\}^{\beta q} |\nabla u(x)|^q dx \\ &\preceq \left(\text{diam } (\widetilde{Q})\right)^{\beta q} \int_{\widetilde{Q}} \left(|\nabla u(x)|/d^\beta(x)\right)^q dx. \end{aligned} \tag{2.6}$$

Hence from (2.4)–(2.6) we find

$$\int_D \left(|u(x)|/d^{\alpha}(x)\right)^p dx \preceq \sum_{Q\in\mathcal{W}} |\widetilde{Q}|_n^{\frac{p}{n}+1-\frac{p}{q}-\frac{p\alpha}{n}+\frac{p\beta}{n}}$$

$$\times\left(\int_{\widetilde{Q}} \left(|\nabla u(x)|/d^{\beta}(x)\right)^q dx\right)^{p/q}$$

$$\preceq \sum_{Q\in\mathcal{W}} \left(\int_{\widetilde{Q}} \left(|\nabla u(x)|/d^{\beta}(x)\right)^q dx\right)^{p/q} |\widetilde{Q}|_n^{1-p/q}, \tag{2.7}$$

the final inequality being a consequence of our assumption that $\beta-\alpha+1\geq 0$. To conclude the proof we use the following well-known lemma first proved by Carleson [3] when $p=2$ and $n=1$ (see [12] for the general case).

LEMMA 2.3 *Let Q_0 be a cube in $\mathbf{R}^n$ and suppose that $\{Q_i\}$ is a sequence of cubes such that each Q_i is contained in Q_0 and $\sum_i |Q_i|_n \leq const\, |Q_0|_n$; let $v\in L_p(Q_0)$ for some $p\in(1,\infty)$. Then there is a constant C, independent of v, such that*

$$\sum_i |Q_i|_n^{1-p}\left(\int_{Q_i} |v(x)|\,dx\right)^p \leq C\int_{Q_0} |v(x)|^p\,dx. \tag{2.8}$$

We apply this to the $\widetilde{Q}$, noting that the basic hypothesis of the lemma is satisfied since for a fixed cube B,

$$\sum_{\widetilde{Q}\subset B,\,Q\in\mathcal{W}} |\widetilde{Q}|_n \preceq \sum_{\widetilde{Q}\subset B,\,Q\in\mathcal{W}} \operatorname{diam}^n(Q) \preceq |B|_n\,.$$

Since $p/q>1$, Carleson's lemma shows that the right-hand side of (2.7) can be estimated from above by a multiple of

$$\int_D \left(|\nabla u(x)|/d^{\beta}(x)\right)^p dx,$$

and the theorem follows.

Remark 2.4 (i) When D has a Lipschitz boundary it is plain that $\mathbf{R}^n\setminus D$ satisfies the plumpness condition, and so inequality (2.2) holds. This result, under the additional assumption that $\beta>p/(p-1)$, was obtained by Gurka and Opic ([7], Theorem 10.4). Their paper also contains sufficient conditions for (2.2) to hold when ∂D is in the Hölder class $C^{0,\kappa}$ for some $\kappa\in(0,1]$; and it gives results concerning the inequality analogous to (2.2) but with the left-hand side replaced by

$$\int_D |u(x)|^q / d^{\alpha q}(x)\, dx$$

for suitable q.

(ii) When $\alpha = 1$ and $\beta = 0$, (2.1) reduces to the Hardy inequality

$$\int_D (|u(x)| / d(x))^p\, dx \le C \int_D |\nabla u(x)|^p\, dx, \quad u \in C_0^\infty(D), \tag{2.9}$$

mentioned in the Introduction. As explained there, the special case of our results, that (2.9) holds when $\mathbf{R}^n \setminus D$ is plump and unbounded and $1 < p < \infty$, is contained in those of Lewis [9]. Note, however, that inspection of our proof shows that the constant C in (2.9) may be taken to be

$$\omega_n^{1+a}(6a)^a b^{-n} n^{\frac{1}{2}p(n-3)+\frac{n}{3}},$$

where

$$a = p\left(1 - \frac{1}{n}\right) - \frac{1}{3}$$

and $\omega_n = |B(0, 1)|_n$; if D is *convex,* then we may choose $b = 1$. In this connection we are informed that when D is convex and has C^1 boundary, then P. Sobolevski and T. Matskewich have very recently shown that the best constant C in (2.9) is $\left(1 - \frac{1}{p}\right)^p$: see also 'On the best constant for Hardy's inequality', M. Marcus, V.J. Mizel, Y. Pinchover (to appear). When $p = n = 2$ and D is a sector of a circle the best constant C in (2.9) has been shown by Davies [5] to be 4 if the angle of the sector is less than $\beta_0 \simeq 4.856$.

If we use the classical variational capacity argument, Lemma 2.5 below, the Hardy inequality (1.1) follows easily.

As normal, for a compact subset E of a nonempty open set D in $\mathbf{R}^n$ we write

$$p - \operatorname{cap}(E, D) = \inf \left\{ \|\nabla v\|_{p,D}^p : \begin{array}{l} v \in C_0^\infty(D),\ 0 \le v \le 1 \text{ on } D \text{ and} \\ v = 1 \text{ in an open neighbourhood} \\ \text{of } E \text{ in } D. \end{array} \right\}$$

LEMMA 2.5 [6, Corollary 2.4/Chapter VIII] *Let Q be a cube in $\mathbf{R}^n$ and define any $u \in C_0^\infty(D)$ to be zero outside the domain D. Let $1 \le q \le p \le \frac{nq}{n-q}$, $p < n$. If $q - \operatorname{cap}\left(\overline{Q} \cap (\mathbf{R}^n \setminus D), 2Q\right) > 0$, then for any $u \in C_0^\infty(D)$,*

$$\int_Q |u(x)|^p\,dx \le \frac{c(n,q)\ \operatorname{diam}\ (Q)^n}{\left(q-\ \operatorname{cap}\ \left(\overline{Q}\cap(\mathbf{R}^n\setminus D)\,,2Q\right)\right)^{p/q}}\left(\int_Q |\nabla u(x)|^q\,dx\right)^{p/q}$$

Using Lemma 2.5 and the proof of Theorem 2.2 we obtain the following theorem.

THEOREM 2.6 *Suppose that D is a domain with constants $\lambda > 0$, $c_0 > 0$ such that*

$$q-\ \operatorname{cap}\ \left(\overline{Q}\cap\left(\mathbf{R}^n\setminus D\right),2Q\right)\ \operatorname{diam}\ (Q)^{q-n}\ge\lambda \tag{2.10}$$

for all cubes $Q = Q(y)$ with centre $y \in \partial D$ and $0 < \operatorname{diam}(Q) < c_0\,|D|^{1/n}$. Let $1 \le q < p \le \frac{nq}{n-q}$, $p < n$. Then there exists a constant $c > 0$ such that for all $u \in C_0^\infty(D)$,

$$\int_D \frac{|u(x)|^p}{d(x,\partial D)^p}dx \le c\int_D |\nabla u(x)|^p\,dx.$$

Proof We use the same notation as in the proof of Theorem 2.2. We need to verify only the inequality

$$\left(\int_{\widetilde{Q}} |u(x)|^p\,dx\right)\ \operatorname{diam}\ (\widetilde{Q})^{-p} \lesssim\ \operatorname{diam}\ (\widetilde{Q})^{n\left(1-\frac{p}{q}\right)}\left(\int_{\widetilde{Q}} |\nabla u(x)|^q\,dx\right)^{p/q} \tag{2.11}$$

where $1 \le q < p \le \frac{nq}{n-q}$, $p < n$; otherwise the proof is similar to the proof of Theorem 2.2. However, Lemma 2.5 and the assumption of Theorem 2.6 immediately yield (2.11):

$$\begin{aligned}&\left(\int_{\widetilde{Q}} |u(x)|^p\,dx\right)\ \operatorname{diam}\ (\widetilde{Q})^{-p}\\ &\le\left[\frac{c(n,q)\ \operatorname{diam}\ (\widetilde{Q})^{n-q}}{q-\ \operatorname{cap}\ \left(\widetilde{Q}\cap(\mathbf{R}^n\setminus D),\ \operatorname{int}\ (2\widetilde{Q})\right)}\right]^{p/q}\\ &\quad\times\ \operatorname{diam}\ (\widetilde{Q})^{n\left(1-\frac{p}{q}\right)}\left(\int_{\widetilde{Q}} |\nabla u(x)|^q\,dx\right)^{p/q}\\ &\le\left[\frac{c(n,q)}{\lambda}\right]^{p/q}\ \operatorname{diam}\ (\widetilde{Q})^{n\left(1-\frac{p}{q}\right)}\left(\int_{\widetilde{Q}} |\nabla u(x)|^q\,dx\right)^{p/q}.\end{aligned}$$

Remark 2.7 To obtain the general case (1.2) the condition (2.10) should be replaced by $q-$ cap $\left(\overline{Q} \cap (\mathbf{R}^n \setminus D), 2Q\right)$ diam $(Q)^{q(\alpha-\beta)-n} \geq \lambda$, where $\beta \geq 0$.

3 OTHER CONDITIONS ON THE BOUNDARY OF *D*

First we establish the following result:

THEOREM 3.1 *Let D be a domain in $\mathbf{R}^n$ $(n > 1)$ and suppose there are constants $s \in (0,1)$ and $T > 0$ such that for each $y \in \partial D$ and all $t \in (0,T)$, there is a k-dimensional cube $Q_{k,t}(y) \subset \partial D$, with $y \in \overline{Q}_{k,t}(y)$ and $\mathcal{H}^k\left(Q_{k,t}(y)\right) > st^k$; suppose also that $p \in (1,n)$ is such that for all these $k, n-p < k \leq n-1$. Then there is a constant $C > 0$ such that Hardy's inequality*

$$\int_D (|u(x)|/d(x))^p \, dx \leq C \int_D |\nabla u(x)|^p \, dx, \; u \in C_0^1(D) \tag{3.1}$$

holds.

Our proof of this theorem hinges upon the following two lemmas. In these all cubes are assumed to have edges parallel to the coordinate axes in $\mathbf{R}^n$, and the intersection of a cube Q in $\mathbf{R}^n$ with a k-dimensional plane is denoted by Q^k $(1 \leq k \leq n)$, with the understanding that $Q^n = Q$.

LEMMA 3.2 *Let Q be a cube in $\mathbf{R}^n$ and let $p \in (n,\infty)$, $q \in [1,\infty)$. Then there is a constant $c = c(n,p,q)$ such that for every $u \in W_p^1(Q)$,*

$$\left|u(x) - u_Q\right| \leq c\left(\left\|u - u_Q\right\|_{q,Q}^{q(p-n)/p} \|\nabla u\|_{p,Q}^n\right)^{p/\{np+(p-n)q\}} \tag{3.2}$$

for almost all x in Q.

Proof The result is simply the special case $m = 1$ of Lemma 5.18 of [1], applied to $u - u_Q$.

LEMMA 3.3 *Let Q be a cube in $\mathbf{R}^n$, let $1 \leq k \leq n$ and let $0 < n-p < k \leq n$. Then there is a constant $c = c(n,p,q)$ such that for every $u \in W_p^1(Q)$,*

$$\left(\int_{Q^k} \left|u(y) - u_Q\right|^q dy^k\right)^{1/q} \leq c\left(\int_Q |\nabla u(y)|^p \, dy\right)^{1/p}, \tag{3.3}$$

where $q = kp/(n-p)$ and dy^k denotes Lebesgue measure on $\mathbf{R}^k$.

Proof Exactly as in the proof of Lemma 5.19 of [1] we find that

$$\int_{Q^k} |u(y) - u_Q|^q \, dy^k \leq \left(\int_Q |u(x) - u_Q|^{q_0} \, dx \right)^{\mu(p-\nu)/(p\lambda)} \times \left(\int_Q |\nabla u(x)|^p \, dx \right)^{\nu\mu/(p\lambda)}, \qquad (3.4)$$

where ν is the largest integer less than p, $\mu = \binom{k}{n-\nu}$, $\lambda = \binom{k-1}{n-\nu-1}$ and $q_0 = np/(n-p)$. By Poincaré's inequality in the cube Q we can estimate the term $\int_Q |u(x) - u_Q|^{q_0} \, dx$ in (3.4) by means of $\int_Q |\nabla u(x)|^p \, dx$, and the result follows.

Proof of Theorem 3.1 It is enough to prove (3.1) for $u \in C_0^\infty(D)$. Let $\mathcal{W}$ be a Whitney decomposition of D. Given any $Q \in \mathcal{W}$, fix $x_Q \in \partial D$ such that $d(Q, \partial D) = d(Q, x_Q)$; fix a cube $\widetilde{Q}$ with x_Q as centre and such that $Q \subset \widetilde{Q}$ and diam $(\widetilde{Q}) = c(n)$ diam (Q). Then

$$\int_D (|u(x)| / d(x))^p \, dx = \sum_{Q \in \mathcal{W}} \int_Q (|u(x)| / d(x))^p \, dx \preceq \sum_{Q \in \mathcal{W}} \int_Q (|u(x)| / \operatorname{diam}(Q))^p \, dx. \qquad (3.4)$$

For each cube $\widetilde{Q}$ there is a set $\widetilde{S}_{k,d(Q)} := \widetilde{S} \subset \partial D$ such that $\mathcal{H}^k\left(\widetilde{S}_{k,d(Q)}\right) \approx s$ diam $\left(\widetilde{Q}\right)^k$, where $k \in (n-p, n-1]$. Since $u = 0$ on $\widetilde{S}$,

$$\int_{\widetilde{Q}} |u(x)|^p \, dx = \int_{\widetilde{Q}} |u(x) - u_{\widetilde{S}}|^p \, dx. \qquad (3.5)$$

Moreover, Minkowski's inequality and the Poincaré inequality in a cube yield

$$\begin{aligned} \int_{\widetilde{Q}} |u(x) - u_{\widetilde{S}}|^p \, dx &\preceq \int_{\widetilde{Q}} \left|u(x) - u_{\widetilde{Q}}\right|^p dx + \int_{\widetilde{Q}} \left|u_{\widetilde{S}} - u_{\widetilde{Q}}\right|^p dx \\ &\preceq |Q|^{p\left(\frac{1}{n} + \frac{1}{p} - \frac{1}{q}\right)} \left(\int_Q |\nabla u(x)|^q \, dx \right)^{p/q} \\ &\quad + \int_{\widetilde{Q}} \left|u_{\widetilde{S}} - u_{\widetilde{Q}}\right|^p dx, \end{aligned} \qquad (3.6)$$

where $q = np/(p+k)$. Use of Hölder's inequality gives

$$\begin{aligned}\int_{\widetilde{Q}} \left|u_{\widetilde{S}} - u_{\widetilde{Q}}\right|^p dx &= \int_{\widetilde{Q}} \left|\frac{1}{\mathcal{H}^k(\widetilde{S})} \int_{\widetilde{S}} u\,dx^k - u_{\widetilde{Q}}\right|^p dx \\ &\leq \int_{\widetilde{Q}} \frac{1}{\mathcal{H}^k(\widetilde{S})} \int_{\widetilde{Q}^k} \left|u(x) - u_{\widetilde{Q}}\right|^p dx^k dx, \end{aligned} \tag{3.7}$$

where $\widetilde{Q}^k$ is the intersection of $\widetilde{Q}$ and the k-dimensional plane containing the cube $Q_{k,t}(x_Q)$ for suitable t. From Lemma 3.3 we have

$$\left(\int_{\widetilde{Q}^k} \left|u(x) - u_{\widetilde{Q}}\right|^p dx^k\right)^{1/p} \leq c(k,n,p) \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{1/q}, \tag{3.8}$$

where $q = np/(p+k)$, $p < n$ and $n - p < k \leq n$. Combination of (3.4)–(3.8) now shows that

$$\begin{aligned}\int_D (|u(x)|/d(x))^p dx &\preceq \sum_{Q \in \mathcal{W}} s^{-1} \operatorname{diam}(Q)^{n-k-p} \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{p/q} \\ &\approx \sum_{Q \in \mathcal{W}} s^{-1} \operatorname{diam}(Q)^{\frac{np}{q}-k-p} \left|\widetilde{Q}\right|_n^{1-p/q} \\ &\quad \times \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{p/q} \\ &\preceq \sum_{Q \in \mathcal{W}} s^{-1} \left|\widetilde{Q}\right|_n^{1-p/q} \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{p/q}, \end{aligned} \tag{3.9}$$

since $\frac{np}{q} - k - p = 0$. As the $\widetilde{Q}$ form a sequence of cubes to which Carleson's lemma, Lemma 2.3, may be applied, it follows from (3.9) that

$$\int_D (|u(x)|/d(x))^p dx \leq C \int_D |\nabla u(x)|^p dx$$

for some $C = C(k,n,p)\, s^{-1}$. The proof is complete.

A variant of Theorem 3.1 along the lines of Theorem 2.2 can easily be given.

THEOREM 3.4 *Let $p \in (1, n)$ and $\alpha, \beta \in \mathbf{R}$; let D be a domain in $\mathbf{R}^n$ $(n > 1)$ and suppose that there are constants $s \in (0, 1)$ and $T > 0$ such that for each $y \in \partial D$ and all $t \in (0, T)$, there is a k-dimensional cube $Q_{k,t}(y) \subset \partial D$ with $y \in \overline{Q_{k,t}(y)}$, $\mathcal{H}^k(Q_{k,t}(y)) > st^k$, and*

$$\beta \geq \max\{0, \alpha - 1\}, \quad n - p < k \leq n - 1. \tag{3.10}$$

Then there is a constant $C > 0$ such that for all $u \in C_0^\infty(D)$,

$$\int_D \left(|u(x)| / d^\alpha(x)\right)^p dx \leq C \int_D \left(|\nabla u(x)| / d^\beta(x)\right)^p dx. \tag{3.11}$$

Proof This follows the pattern of that of Theorem 3.1; just as before and with the same notation, it follows that

$$\int_D \left(|u(x)| / d^\alpha(x)\right)^p dx \preceq \sum_{Q \in \mathcal{W}} \operatorname{diam}(Q)^{\frac{np}{q} - k - \alpha p} |\widetilde{Q}|_n^{1-p/q} \times \left(\int_{\widetilde{Q}} |\nabla u(x)|^q dx\right)^{p/q}, \tag{3.12}$$

where $q = np/(p + k)$; see the inequalities leading up to (3.9). Under conditions (3.11) the right-hand side of (3.12) can be estimated from above by a constant times

$$\sum_{Q \in \mathcal{W}} \operatorname{diam}(Q)^{\frac{np}{q} - k - \alpha p + \beta p} |\widetilde{Q}|_n^{1-p/q} \left(\int_{\widetilde{Q}} \left(|\nabla u(x)| / d^\beta(x)\right)^q dx\right)^{p/q}$$
$$\preceq \sum_{Q \in \mathcal{W}} |\widetilde{Q}|_n^{1-p/q} \left(\int_{\widetilde{Q}} \left(|\nabla u(x)| / d^\beta(x)\right)^q dx\right)^{p/q}.$$

The result now follows as before on application of Carleson's lemma.

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