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Hermite Interpolation and an Inequality for Entire Functions of Exponential Type

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Let $c \in [0, 1), p > 0$. It is shown that if f is an entire function of exponential type $cm\pi$ and $\sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p < \infty$, where $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers satisfying $|\lambda_n - n| \le \Delta < \infty$, $|\lambda_{n+u} - \lambda_n| \ge \delta > 0$ for $u \neq 0$, then $\int_{-\infty}^{\infty} |f(x)|^p dx < B \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p$, where B depends only on c, p, Δ and δ. A sampling theorem for irregularly spaced sample points is obtained as a corollary. Our proof of the main result contains ideas which help us to obtain an extension of a theorem of R.J. Duffin and A.C. Schaeffer concerning entire functions of exponential type bounded at the points of the above sequence $\{\lambda_n\}_{n\in\mathbb{Z}}$.

Keywords: Entire functions of exponential type; Hermite interpolation; L^p inequalities; nonuniform sampling theorems; Carlson's theorem.

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1 INTRODUCTION AND STATEMENT OF RESULTS

According to a famous theorem of Carlson [12, Theorem 5.81] if f is an entire function of exponential type $< \pi$ which vanishes at $n = 0, \pm 1, \pm 2, \ldots$, then

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it must be identically zero. An extension of this result due to Plancherel and Pólya [10, Section 33] reads as follows.

THEOREM A *Let* $p > 0$ *and* $c \in [0, 1)$. *If* f *is an entire function of exponential type such that*

$$\limsup_{y\to\infty} y^{-1}\log\{|f(iy)| + |f(-iy)|\} = c\pi\ , \tag{1.1}$$

then there exists a constant B *depending only on* p *and* c *such that*

$$\int_{-\infty}^{\infty} |f(x)|^p dx\ <\ B\sum_{n=-\infty}^{\infty} |f(n)|^p\ . \tag{1.2}$$

It was shown by Boas [1] that the sampling points in (1.2) do not have to be integers. The following theorem is covered by his generalization of Theorem A.

THEOREM B *Let* $\lambda := \{\lambda_n\}$ *be a sequence of real numbers such that*

$$|\lambda_n - n| \le \Delta < \infty\ ,\ |\lambda_{n+u} - \lambda_n| \ge \delta > 0\ ,\ (u \ne 0)\ . \tag{1.3}$$

If p, c *and* f *are as in Theorem A, then there exists a constant* B *depending on* p, c, Δ *and* δ *such that*

$$\int_{-\infty}^{\infty} |f(x)|^p dx\ <\ B\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^p\ . \tag{1.4}$$

With $\lambda := \{\lambda_n\}$ as above let

$$G(z)\ :=\ (z-\lambda_0)\prod_{n=-\infty}^{\infty}\left(1-\frac{z}{\lambda_n}\right)\left(1-\frac{z}{\lambda_{-n}}\right)\ . \tag{1.5}$$

The proof of Theorem B makes essential use of the fact that for certain positive constants c_1, c_2 depending only on Δ, δ we have [8]

$$|G(z)|\ <\ c_1(|z|+1)^{4\Delta}\exp(\pi|\Im m\, z|) \qquad \text{for all} \quad z \in \mathbb{C}\ , \tag{1.6}$$

$$|G'(\lambda_n)|\ >\ c_2(1+|\lambda_n|)^{-4\Delta-1} \tag{1.7}$$

and for each $\varepsilon > 0$ holds [9, pp. 92–93]

$$\frac{\exp(\pi|\Im m\, z|)}{|G(z)|}\ =\ O\left(\exp(\varepsilon|z|)\right) \quad \text{if} \quad |z-\lambda_n| \ge \delta/2\ . \tag{1.8}$$

These inequalities extend certain very important properties of the function $\sin \pi z$ to which $G(z)$ reduces when $\lambda_n = n$ for all $n \in \mathbb{Z}$. From (1.6) it can be concluded that for some constant c_3 depending only on Δ and δ we have [11, see (3.3″)]

$$\frac{|G(z)|}{|z-\lambda_n|} < c_3(|z|+1)^{4\Delta}\exp(\pi|\Im m\, z|) \quad \text{for all} \quad z \in \mathbb{C}, \tag{1.9}$$

where the function on the left is assumed to have its singularity at $z = \lambda_n$ removed. Hereafter we will use y to denote $\Im m\, z$.

Here is another extension of Theorem A which was obtained only a few years ago.

THEOREM C [4, Theorem 3] *Let $m \in \mathbb{N}$, $p > 0$, $c \in [0, 1)$. If f is an entire function of exponential type such that*

$$\limsup_{y\to\infty} y^{-1}\log\{|f(iy)|+|f(-iy)|\} = cm\pi\,, \tag{1.10}$$

then there exists a constant B depending only on m, p and c such that

$$\int_{-\infty}^{\infty}|f(x)|^p dx < B\sum_{n=-\infty}^{\infty}\sum_{\mu=0}^{m-1}|f^{(\mu)}(n)|^p\,. \tag{1.11}$$

One might wonder why we restricted ourselves to the sequence $\{n\}_{n\in\mathbb{Z}}$; but consideration of an arbitrary sequence $\{\lambda_n\}$ satisfying (1.3) would have required an additional property of the function $G(z)$ which was not available to us at that time. According to it, for each $k \geq 2$, there exists a constant $c_{4,k}$ depending only on Δ and δ such that [5, see Theorem 1 and Remark 6]

$$\frac{|G^{(k)}(\lambda_n)|}{|G'(\lambda_n)|} < c_{4,k} \qquad \text{for all} \quad n \in \mathbb{Z}\,. \tag{1.12}$$

The details of the proof of this crucial inequality were given in [5] in the case $\Delta \leq 1/4$. In Remark 6 of that paper it was stated that the inequality remains true for arbitrary Δ but the details were left out because, there the case $\Delta > 1/4$ was of little importance. Here it is important to let Δ be *any* positive number and so we give below some hints which the reader might find helpful in verifying the inequality in the case $\Delta > 1/4$.

From (1.6) it follows that $|G(z)| < c_1\exp(\pi)(|\lambda_n|+2)^{4\Delta}$ in the disk $|z-\lambda_n| \leq 1$ and so by the Cauchy's integral formula for the kth derivative, we have

$$|G^{(k)}(\lambda_n)| < k!c_1\exp(\pi)(|\lambda_n|+2)^{4\Delta}\,.$$

This is in conjunction with (1.7) implies that

$$\frac{|G^{(k)}(\lambda_n)|}{|G'(\lambda_n)|} < k!(c_1/c_2)\exp(\pi)(|\lambda_n|+2)^{4\Delta}(|\lambda_n|+1)^{4\Delta+1},$$

from which the desired estimate for $|G^{(k)}(\lambda_n)|/|G'(\lambda_n)|$ follows trivially if n is bounded. So we may suppose $|n| > 4\Delta$.

The proof of (1.12) in the case $\Delta \leq 1/4$ was based on the fact that for each $n \in \mathbb{Z}$,

$$\varphi_n(N) := \left| \sum_{\substack{\nu=-N \\ \nu \notin \{-n,0,n\}}}^{N} \frac{1}{\lambda_\nu - \lambda_n} \right| < 10$$

if $N \geq N_n$, where N_n is an integer depending on n, and the estimates

$$\sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{(\lambda_\nu - \lambda_n)^2} \leq \pi^2, \quad \sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^k} < \pi^2 + 2^{k+1} \text{ for } k = 3, 4, \ldots$$

hold for all $N \in \mathbb{N}$. We note that, for $\Delta > 1/4$, this remains true in the sense that the quantities

$$\varphi_n(N), \quad \sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{(\lambda_\nu - \lambda_n)^2}, \quad \sum_{\substack{\nu=-N \\ \nu\neq n}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^k} \quad \text{where } k = 3, 4, \ldots,$$

are bounded by constants depending only on Δ and δ. To see this assume $n > 4\Delta$ and for sufficiently large N write

$$\varphi_n(N) = |B(n) - A(n) + E(n)| \leq |B(n) - A(n)| + |E(n)|$$

where

$$A(n) := \sum_{\nu=1}^{[n-2\Delta]-1} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{(n + \nu + (\delta_n - \delta_{-\nu}))(n - \nu + (\delta_n - \delta_\nu))},$$

$$B(n) := \sum_{\nu=[n+6\Delta]+3}^{N} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{(n + \nu + (\delta_n - \delta_{-\nu}))(\nu - n + (\delta_\nu - \delta_n))},$$

$$E(n) := \sum_{\substack{\nu=[n-2\Delta] \\ \nu\neq n}}^{[n+6\Delta]+2} \frac{2n + 2\delta_n - \delta_\nu - \delta_{-\nu}}{(n + \nu + (\delta_n - \delta_{-\nu}))(\nu - n + (\delta_\nu - \delta_n))}.$$

The quantities $A(n)$, $B(n)$ can be estimated from below and above as in the case $\Delta \leq 1/4$. Besides, we easily see that

$$|E(n)| < \frac{24(1+2\Delta)}{\delta} .$$

The desired property of $\varphi_n(N)$ can then be proved in essentially the same way as before.

The quantities

$$\sum_{\substack{\nu=-N \\ \nu \neq n}}^{N} \frac{1}{(\lambda_\nu - \lambda_n)^2} , \quad \sum_{\substack{\nu=-N \\ \nu \neq n}}^{N} \frac{1}{|\lambda_\nu - \lambda_n|^k} ,$$

where $k = 3, 4, \ldots$ present no new problems.

We are now able to prove our main result.

THEOREM 1 *Let $m \in \mathbb{N}$, $p > 0$, $c \in [0, 1)$ and $\lambda := \{\lambda_n\}$ be a sequence of real numbers satisfying (1.3). If f is as in Theorem C, then there exists a constant B depending only on m, p, c, Δ and δ such that*

$$\int_{-\infty}^{\infty} |f(x)|^p dx < B \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p . \tag{1.13}$$

Remark 1 Theorem 1 implies, in particular, that if f is an entire function of exponential type satisfying (1.10) for some $c \in [0, 1)$ and vanishes along with its derivatives of order $1, \ldots, m-1$ at points λ_n for which (1.3) holds, then it must be identically zero. This is an extension of the theorem of Carlson mentioned above.

Let $\lambda := \{\lambda_n\}$ be an arbitrary sequence satisfying (1.3), G as in (1.5), m a positive integer and

$$\Psi_{m,n}(z) = \Psi_{m,n}(\lambda; z) := \left(\frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}\right)^m \quad (n \in \mathbb{Z}) .$$

For $0 \leq \mu \leq m-1$ we consider the function

$$\Phi_{m,n,\mu}(z) = \Phi_{m,n,\mu}(\lambda; z)$$
$$:= (1/\mu!)(z-\lambda_n)^{\mu}\Psi_{m,n}(z) \sum_{j=0}^{m-1-\mu} (1/j!)a_{m,n,j}(z-\lambda_n)^j ,$$

where $a_{m,n,0} := 1$, $a_{m,n,1} := -\Psi'_{m,n}(\lambda_n)$ and for $j \geq 2$,

$$a_{m,n,j} := (-1)^j \begin{vmatrix} \binom{j}{1}\Psi'_{m,n}(\lambda_n) & \binom{j}{2}\Psi''_{m,n}(\lambda_n) & \cdots & \binom{j}{j}\Psi^{(j)}_{m,n}(\lambda_n) \\ 1 & \binom{j-1}{1}\Psi'_{m,n}(\lambda_n) & \cdots & \binom{j-1}{j-1}\Psi^{(j-1)}_{m,n}(\lambda_n) \\ 0 & 1 & \cdots & \binom{j-2}{j-2}\Psi^{(j-2)}_{m,n}(\lambda_n) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{1}{1}\Psi'_{m,n}(\lambda_n) \end{vmatrix} .$$

It is not hard to verify that

$$\begin{cases} \Phi^{(k)}_{m,n,\mu}(\lambda_n) = \delta_{\mu,k}, \\ \Phi^{(k)}_{m,n,\mu}(\lambda_\nu) = 0, \qquad \text{for } k = 0, \dots, m-1 \text{ and } \nu \neq n. \end{cases} \tag{1.14}$$

According to a formula for the j-th derivative of the reciprocal of a j times differentiable function [5, Lemma 3]

$$a_{m,n,j} = \frac{d^j}{dz^j}\left(\frac{1}{\Psi_{m,n}(z)}\right)\bigg|_{z=\lambda_n} . \tag{1.15}$$

Given $m \in \mathbb{N}$ and a sequence $\lambda := \{\lambda_n\}$ satisfying (1.3), we associate with any function $f : \mathbb{R} \to \mathbb{C}$ belonging to $C^{m-1}(\mathbb{R})$ the formal series

$$L_{m,\lambda}(f; z) := \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} f^{(\mu)}(\lambda_n)\Phi_{m,n,\mu}(\lambda; z) . \tag{1.16}$$

Although $L_{m,\lambda}(f; z)$ may not be defined for $z \notin \{\lambda_n\}$ it follows from (1.14) that $L^{(\mu)}_{m,\lambda}(f; \lambda_n) = f^{(\mu)}(\lambda_n)$ for all $n \in \mathbb{Z}$ and $\mu = 0, \dots, m-1$. Considerably more can be said if f in (1.16) is an entire function of exponential type belonging to $L^p(\mathbb{R})$ for some $p > 0$.

THEOREM D [5,7] *Let $m \in \mathbb{N}$, $0 < p < \infty$ and $\lambda := \{\lambda_n\}$ a sequence satisfying (1.3) with*

$$\Delta \leq \begin{cases} \frac{1}{4m}, & \text{if } 0 < p \leq 2 \\ \frac{1}{2pm}, & \text{if } 2 \leq p < \infty. \end{cases} \tag{1.17}$$

If f is an entire function of exponential type $m\pi$ belonging to $L^p(\mathbb{R})$, then $f(z) = L_{m,\lambda}(f; z)$ for all $z \in \mathbb{C}$.

Now from Theorem 1 we readily obtain

COROLLARY 1 *Let $m \in \mathbb{N}, 0 < p < \infty, \lambda := \{\lambda_n\}$ a sequence satisfying (1.3) with Δ restricted as in (1.17). If f is an entire function of exponential type less then $m\pi$ satisfying $\sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p < \infty$, then $f(z) = L_{m,\lambda}(f; z)$ for all $z \in \mathbb{C}$.*

2 AUXILIARY RESULTS

Using the generalized Leibnitz's formula [3, p. 219] it can be shown that [5, Lemma 2]

$$\Psi_{m,n}^{(s)}(\lambda_n) = \sum_{\substack{s_1+\cdots+s_m=s \\ 0\le s_1,\ldots,s_m\le s}} \frac{s!}{(s_1+1)!\cdots(s_m+1)!} \prod_{j=1}^{m} \frac{G^{(s_j+1)}(\lambda_n)}{G'(\lambda_n)} .$$

From (1.12) it then follows that if $c_{4,1} = 1$ and $\mathcal{M}_s := \max_{1\le k\le s+1} c_{4,k}$, then for all $n \in \mathbb{Z}$ we have [5, Remark 4]

$$|\Psi_{m,n}^{(s)}(\lambda_n)| \le \frac{(\mathcal{M}_s)^m s! m^{s+m}}{(s+m)!} . \tag{2.1}$$

Since $a_{m,n,j}$ is a polynomial in $\Psi'_{m,n}(\lambda_n), \ldots, \Psi_{m,n}^{(j)}(\lambda_n)$ there exists a constant c_5 depending only on Δ, δ and m such that

$$|a_{m,n,j}| \le c_5 , \quad \text{where } 0 \le j \le m-1, n \in \mathbb{Z}. \tag{2.2}$$

Hence using (1.7) and (1.9) we conclude that for all $z \in \mathbb{C}$ we have

$$|\Phi_{m,n,\mu}(z)| < c_6(|z|+1)^{4m\Delta}(\exp(\pi m|y|))(|z|+1+|\lambda_n|)^{m-1}(1+|\lambda_n|)^{(4\Delta+1)m},$$

where $c_6 \le (m+1)c_5(c_3/c_1)^m$. Since $(|z|+1+|\lambda_n|)^{m-1} \le (|z|+1)^{m-1}(1+|\lambda_n|)^{m-1}$, we get

$$|\Phi_{m,n,\mu}(z)| < c_6(|z|+1)^{(4\Delta+1)m-1}(\exp(\pi m|y|))(1+|\lambda_n|)^{(4\Delta+2)m-1} . \tag{2.3}$$

Using this estimate we can easily show that if $f : \mathbb{R} \to \mathbb{C}$ is a function belonging to $C^{m-1}(\mathbb{R})$ such that for some $M > 0$ and some $\alpha > (4\Delta+2)m$,

$$|f^{(\mu)}(\lambda_n)| \le \frac{M}{1+|\lambda_n|^\alpha} , \quad (n \in \mathbb{Z},\ \mu = 0, \ldots, m-1) , \tag{2.4}$$

then on each given compact set $E \subset \mathbb{C}$ the series $\sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} f^{(\mu)}(\lambda_n)$ $\Phi_{m,n,\mu}(z)$ converges absolutely and uniformly, i.e. $L_{m,n,\mu}(f; \cdot)$ is an entire function. Further,

$$|L_{m,\lambda}(f; z)| = O\left((|z|+1)^{(4\Delta+1)m-1} \exp(\pi m|y|)\right) . \tag{2.5}$$

Hence, we have

LEMMA 1 *If (2.4) holds for some* $\alpha > (4\Delta + 1)m$, *then* $L_{m,\lambda}(f;\cdot)$ *is an entire function of exponential type* $m\pi$.

It is interesting and useful for us to know that more can be said when f is an entire function of exponential type satisfying (1.10).

LEMMA 2 *Let* f *be an entire function of exponential type satisfying (1.10). If (2.4) holds for some* $\alpha > (4\Delta + 2)m$, *then* $f(z) \equiv L_{m,\lambda}(f;z)$.

Proof Since $L_{m,\lambda}^{(\mu)}(f;\lambda_n) = f^{(\mu)}(\lambda_n)$ for all $n \in \mathbb{Z}$ and $\mu = 0, \ldots, m-1$, the entire function $g(z) := f(z) - L_{m,\lambda}(f;z)$ has zeros of multiplicity at least m at each of the points λ_n of the sequence λ. Hence $H(z) := \frac{g(z)}{(G(z))^m}$ is entire. Since g is of exponential type, say τ, we may use (1.8) to conclude that for z lying outside the union of disks $D_n := \{z : |z - \lambda_n| < \delta/2\}$ we have

$$|H(z)| < K\exp((\tau + 1)|z|), \tag{2.6}$$

where K is a constant. If $z \in D_n$, then by the maximum modulus principle

$$|H(z)| < K\exp((\tau + 1)(|\lambda_n| + \delta/2)) < K\exp\left(\frac{(\tau + 1)(2|\lambda_n| + \delta)|z|}{2|\lambda_n| - \delta}\right),$$

whence

$$|H(z)| < K\exp\left(\frac{(\tau + 1)(2\Delta + \delta)|z|}{2\Delta - \delta}\right) \tag{2.7}$$

if $|\lambda_n| > \Delta$. In view of (2.6) the preceding estimate holds for all z with $|z| > \Delta$. If $K_1 := \max_{|z|\le\delta} |H(z)|$, then clearly

$$|H(z)| < \max\{K, K_1\}\exp\left(\frac{(\tau + 1)(2\Delta + \delta)|z|}{2\Delta - \delta}\right) \quad \text{for all } z \in \mathbb{C},$$

i.e. H is of exponential type.

We next estimate $H(re^{i\theta})$ more precisely for large r and θ near $\pm\pi/2$. Our hypothesis about f implies that for all θ,

$$|f(r\exp(i\theta))| = O\left(\exp(c'm\pi|\sin\theta| + d|\cos\theta|)r\right),$$

where $c' < 1$ and d is finite. So by (1.8)

$$\left|\frac{f(r\exp(i\theta))}{(G(r\exp(i\theta)))^m}\right| = O\left(\exp(-(1 - c')m\pi|\sin\theta| + d|\cos\theta| + m\varepsilon)r\right),$$

where ε is arbitrarily small; thus $\frac{f(z)}{(G(z))^m}$ is bounded on $\arg z = \theta$ if θ is so near $\pm\pi/2$ that $-(1-c')m\pi|\sin\theta| + d|\cos\theta| + m\varepsilon < 0$. Next, we note that

$$\begin{aligned}
\left|\frac{\Phi_{m,n,\mu}(z)}{(G(z))^m}\right| &\leq \frac{1}{\mu!|G'(\lambda_n)|^m}\sum_{j=0}^{m-1-\mu}\frac{|a_{m,n,j}||z-\lambda_n|^{\mu+j-m}}{j!} \\
&< (1/c_2)^m c_5(1+|\lambda_n|)^{(4\Delta+1)m}\sum_{j=0}^{m-1-\mu}|z-\lambda_n|^{\mu+j-m} \\
&\qquad \text{by (1.7) and (2.2)} \\
&\leq \frac{mc_5(1+|\lambda_n|)^{(4\Delta+1)m}}{c_2^m|z-\lambda_n|} \qquad \text{if} \quad |z-\lambda_n| \geq 1\,.
\end{aligned}$$

Hence, for $|y| \geq 1$,

$$\begin{aligned}
\left|\frac{L_{m,\lambda}(f;z)}{(G(z))^m}\right| &\leq \sum_{n=-\infty}^{\infty}\sum_{\mu=0}^{m-1}|f^{(\mu)}(\lambda_n)|\left|\frac{\Phi_{m,n,\mu}(z)}{(G(z))^m}\right| \\
&< \frac{mc_5M}{c_2^m|y|}\sum_{n=-\infty}^{\infty}\frac{(1+|\lambda_n|)^{(4\Delta+1)m}}{1+|\lambda_n|^\alpha} \qquad \text{by (2.4)} \\
&= O\left(\frac{1}{|y|}\right) \qquad \text{since } \alpha > (4\Delta+2)m\,.
\end{aligned}$$

In particular,

$$\left|\frac{L_{m,\lambda}(f;r\exp(i\theta))}{(G(r\exp(i\theta)))^m}\right|$$

is bounded on $\arg z = \theta$ if $0 < \theta < \pi$. Thus,

$$\left|\frac{H(z)}{(G(z))^m}\right| \leq \left|\frac{f(z)}{(G(z))^m}\right| + \left|\frac{L_{m,\lambda}(f;z)}{(G(z))^m}\right|$$

is bounded on $\arg z = \theta$ if θ is sufficiently close to $\pm\pi/2$. Hence H is bounded on four rays any two consecutive ones of which make an angle of less than π. Since H is an entire function of exponential type it must be bounded everywhere by a Phragmén–Lindelöf theorem [2, Theorem 1.4.2] and so is a constant. Finally, this constant must be zero since $H(iy) \to 0$ as $y \to \infty$. Consequently, $g(z) \equiv 0$, i.e. $f(z) \equiv L_{m,\lambda}(f;z)$. □

For the proof of Theorem 1 we shall also need the following.

LEMMA 2 *For any* η *in* $(0, \pi - c\pi)$ *let* $\alpha_1(\eta) < \alpha_2(\eta) < \cdots$ *be the positive zeros of* $\sin \eta z$ *arranged in increasing order. Given any sequence* $\{\lambda_n\}$ *satisfying (1.3) and a positive integer k, we can find in each subinterval* $I := [\eta', \eta'']$ *of* $(0, \pi - c\pi)$ *with* $\alpha_1(\eta') - \alpha_1(\eta'') = \delta$, *a point* η_k *such that* $|\alpha_j(\eta_k) - \lambda_n| \geq \delta/2^k$ *for all* $n \in \mathbb{Z}$ *and* $j = 1, \ldots, k$.

Proof Choose η in I such that $|\alpha_1(\eta) - \lambda_n| \geq \delta/2$ for all $n \in \mathbb{Z}$ and call it η_1. We can change this value of η to a new value η_2 contained in I such that $|\alpha_2(\eta_2) - \lambda_n| > \delta/2^2$ for all $n \in \mathbb{Z}$. Since $\alpha_j(\eta) = j\pi/\eta$ this can be achieved without changing $\alpha_1(\eta)$ by more than $\delta/2^3$. This new value η_2 of η can be changed (if necessary) to another value η_3 contained in I such that $|\alpha_3(\eta_3) - \lambda_n| \geq \delta/2^3$ for all $n \in \mathbb{Z}$. This can be done without causing $\alpha_1(\eta)$ to move by more than $(1/3)(\delta/2^3) < \delta/2^4$; the value of $\alpha_2(\eta)$ changes by less than $\delta/2^3$. We can continue this process of moving η and obtain at the k-th stage a point η_k in I such that $|\alpha_j(\eta_k) - \lambda_n| \geq \delta/2^k$ for all $n \in \mathbb{Z}$ and $j = 1, \ldots, k$. □

3 PROOF OF THEOREM 1

We assume the right-hand side of (1.13) to be finite, since otherwise there is nothing to prove. In particular, $f, \ldots, f^{(m-1)}$ are bounded at the points λ_n. Let

$$M_1 := \sup_{n \in \mathbb{Z}} \max_{0 \leq \mu \leq m-1} |f^{(\mu)}(\lambda_n)| .$$

Let N be an integer and put $\lambda_n^{(N)} := \lambda_{n+N} - \lambda_N$, so that $\lambda_0^{(N)} = 0$, $|\lambda_n^{(N)} - n| \leq |\lambda_{n+N} - (n+N)| + |\lambda_N - N| \leq 2\Delta$, $|\lambda_{n+u}^{(N)} - \lambda_n^{(N)}| = |\lambda_{n+N+u} - \lambda_{n+N}| \geq \delta$ if $u \neq 0$. Hence

$$G(N; z) := z \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n^{(N)}}\right)\left(1 - \frac{z}{\lambda_{-n}^{(N)}}\right)$$

satisfies (1.6), (1.7) and (1.9) with Δ replaced by 2Δ. It also satisfies (1.8) and (1.12); the constants c_1, c_2, c_3 and $c_{4,k}$ are all independent of N.

Let

$$\sigma := \min\{\pi - c\pi, 1/(2\Delta)\}, \ \eta' := \pi\sigma/(2\pi + \delta\sigma), \ \eta'' := \sigma/2 \quad (3.1)$$

and k be an integer larger than $(8\Delta+2)m$ or $(8\Delta+2)m-1+1/p$ according as $p \geq 1$ or $0 < p < 1$, respectively. Refer to Lemma 2 and find an η_k in $[\eta', \eta'']$ such that $|\alpha_j(\eta_k) - \lambda_n^{(N)}| > \delta/2^k$ for all $n \in \mathbb{Z}$ and $j = 1, \ldots, k$. We recall that $\alpha_1(\eta) < \alpha_2(\eta) < \cdots$ are the positive zeros of $\sin(\eta z)$ arranged in increasing order. Consider the function

$$F(N; z) := f(z+\lambda_N)\frac{(\sin(\eta_k z))^m}{\prod_{j=1}^{k}(z-\alpha_j)}, \quad (\alpha_j = \alpha_j(\eta_k)). \tag{3.2}$$

We claim that

$$F(N; z) \equiv L_{m,\lambda^{(N)}}(F(N; \cdot); z), \quad (\lambda^{(N)} := \{\lambda_n^{(N)}\}). \tag{3.3}$$

In order to prove it we use Lemma 1. Let us estimate $|F^{(\mu)}(N; \lambda_n^{(N)})|$ for $0 \leq \mu \leq m-1$. Writing

$$F = f_1 \cdot f_2 \cdots f_{m+1} \cdot f_{m+2} \cdots f_{m+k+1},$$

where $f_1(z) := f(z+N)$, $f_2(z) = \cdots = f_{m+1}(z) := \sin(\eta_k z)$ and $f_{m+j+1}(z) := 1/(z-\alpha_j)$ for $j = 1, \ldots, k$ and applying the generalized Leibnitz's formula for the μth derivative of the product of several functions, we obtain

$$\begin{aligned} F^{(\mu)}(N; \lambda_n^{(N)}) = & \sum_{\substack{\mu_1+\cdots+\mu_{m+k+1}=\mu \\ 0\leq\mu_1,\ldots,\mu_{m+k+1}\leq\mu}} \frac{\mu!}{\mu_1!\cdots\mu_{m+k+1}!}\Bigg[f^{(\mu_1)}(x+\lambda_N) \\ & \times \prod_{\nu=2}^{m+1}\frac{d^{\mu_\nu}}{dx^{\mu_\nu}}(\sin(\eta_k x))\prod_{j=1}^{k}\frac{d^{\mu_{m+j+1}}}{dx^{\mu_{m+j+1}}}\left(\frac{1}{x-\alpha_j}\right)\Bigg]_{x=\lambda_n^{(N)}} \\ = & \frac{1}{\prod_{j=1}^{k}(\lambda_n^{(N)}-\alpha_j)}\sum_{l=0}^{\mu}\frac{f^{(l)}(\lambda_{n+N})}{l!} \\ & \times \sum_{\substack{\mu_2+\cdots+\mu_{m+k+1}=\mu-l \\ 0\leq\mu_2,\ldots,\mu_{m+k+1}\leq\mu-l}} \frac{\mu!}{\mu_2!\cdots\mu_{m+k+1}!} \\ & \times \prod_{\nu=2}^{m+1}\left[\frac{d^{\mu_\nu}}{dx^{\mu_\nu}}(\sin(\eta_k x))\right]_{x=\lambda_n^{(N)}}\prod_{j=1}^{k}\frac{(-1)^{\mu_{m+j+1}}\mu_{m+j+1}!}{(\lambda_n^{(N)}-\alpha_j)^{\mu_{m+j+1}}}. \end{aligned}$$

So

$$|F^{(\mu)}(N;\lambda_n^{(N)})| \le \frac{\max\{\eta^{m-1},1\}}{\prod_{j=1}^{k}|\lambda_n^{(N)}-\alpha_j|}\left(\frac{2^k}{\delta}\right)^{\mu}\prod_{j=1}^{k}(\mu_{m+j+1}!)$$
$$\times\sum_{l=0}^{\mu}\binom{\mu}{l}|f^{(l)}(\lambda_{n+N})|\sum_{\substack{\mu_2+\cdots+\mu_{m+k+1}=\mu-l\\ 0\le\mu_2,\ldots,\mu_{m+k+1}\le\mu-l}}\frac{(\mu-l)!}{\mu_2!\cdots\mu_{m+k+1}!}\,.$$

Note that the last sum is equal to $(m+k)^{\mu-l}$. Setting $M_2 := \max\{\eta_k^{m-1},1\}$ $(2^k/\delta)^m(m+k)^m\prod_{j=1}^{k}\mu_{m+j+1}!$, which depends only on Δ,δ and m, we obtain

$$|F^{(\mu)}(N;\lambda_n^{(N)})| \le \frac{M_2}{\prod_{j=1}^{k}|\lambda_n^{(N)}-\alpha_j|}\sum_{l=0}^{\mu}\binom{\mu}{l}|f^{(l)}(\lambda_{n+N})|\,. \tag{3.4}$$

Since $|F^{(\mu)}(N;\lambda_n^{(N)})| \le \frac{2^{\mu}M_1M_2}{\prod_{j=1}^{k}|\lambda_n^{(N)}-\alpha_j|}$, the function $F(N;\cdot)$ satisfies the condition (2.4) at the points $\lambda_n^{(N)}$ with $\alpha = k > (8\Delta+2)m$. So (3.3) holds by Lemma 2.

We may suppose $\Delta \ge 1/2$. Let Γ be the boundary of the square of side 4Δ with centre at the origin and sides parallel to the coordinate axes. Then by the maximum modulus principle

$$v_N := \max_{|x-\lambda_N|\le 2\Delta}|f(x)| = \max_{-2\Delta\le x\le 2\Delta}|f(x+\lambda_N)| \le \max_{z\in\Gamma}|f(z+\lambda_N)|\,.$$

Using (3.2), (3.3) and (1.16) we get

$$f(z+\lambda_N) = \left(\prod_{j=1}^{k}(z-\alpha_j)\right)(1/\sin(\eta_k z))^m$$
$$\times\sum_{n=-\infty}^{\infty}\sum_{\mu=0}^{m-1}F^{(\mu)}(N;\lambda_n^{(N)})\Phi_{m,n,\mu}(\lambda^{(N)};z)\,.$$

Since $\min\{(1-c)\pi/(2+\delta(1-c)),\ \pi/(4\pi\Delta+\delta)\} \le \eta_k \le 1/(4\Delta)$ and $2\Delta \le |z| \le 2\sqrt{2}\Delta$ for $z\in\Gamma$ it follows that $|1/(\sin(\eta_k z))|$ is bounded above on Γ by a constant M_3 depending only on c, Δ and δ. Besides, from (2.3) it follows that for $z\in\Gamma$,

$$|\Phi_{m,n,\mu}(\lambda^{(N)};z)| \le M_4(1+|\lambda_n^{(N)}|)^{(8\Delta+2)m-1}\,,$$

where M_4 depends only on Δ, δ and m. It is clear that $\max_{z\in\Gamma}\prod_{j=1}^k |z-\alpha_j| \leq M_5$ where M_5 depends only on c, Δ, δ and m. Hence, using (3.4) we obtain

$$\begin{aligned} v_N \;&\leq\; M_5(M_3)^m M_4 M_2 \sum_{n=-\infty}^{\infty} \frac{(1+|\lambda_n^{(N)}|)^{(8\Delta+2)m-1}}{\prod_{j=1}^k |\lambda_n^{(N)}-\alpha_j|} \\ &\quad\times \sum_{\mu=0}^{m-1}\sum_{l=0}^{\mu}\binom{\mu}{l}|f^{(l)}(\lambda_{n+N})| \\ &\leq \gamma \sum_{n=-\infty}^{\infty} d_n^{(N)} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_{n+N})| \,, \end{aligned}$$

where $\gamma := M_5(M_3)^m M_4 M_2 m\binom{m-1}{\left[\frac{m-1}{2}\right]}$ and

$$d_n^{(N)} \;:=\; \frac{(1+|\lambda_n^{(N)}|)^{(8\Delta+2)m-1}}{\prod_{j=1}^k |\lambda_n^{(N)}-\alpha_j|} \;\leq\; \frac{(1+|n|+2\Delta)^{(8\Delta+2)m-1}}{\prod_{j=1}^k |\lambda_n^{(N)}-\alpha_j|} \,.$$

Clearly

$$\prod_{j=1}^k |\lambda_n^{(N)}-\alpha_j| \geq L(n,k) := \begin{cases} ||n+2\Delta|+\alpha_1|^k, & \text{if } n < -2\Delta \\ n-2\Delta-\alpha_k, & \text{if } n > \alpha_k + 4\Delta \\ \left(\frac{\delta}{2^k}\right)^k, & \text{if } -2\Delta \leq n \leq \alpha_k + 4\Delta \,. \end{cases}$$

Note that $\alpha_1 \geq 2\pi/\sigma$, $\alpha_k \leq k(2\pi+\delta\sigma)/\sigma$ where σ is as in (3.1). Hence $d_n^{(N)} \leq \tilde{d}_n$ where

$$\tilde{d}_n \;:=\; \frac{(1+|n|+2\Delta)^{(8\Delta+2)m-1}}{L(n,k)} \tag{3.5}$$

which means, in particular, that $\tilde{d}_n$ does not depend on N. Now we distinguish two cases.

Case (i). $1 \leq p < \infty$.

By the choice of k the series $\sum_{n\in\mathbb{Z}} \tilde{d}_n$ converges. Denote its sum by S. Having assumed Δ to be $\geq 1/2$ we clearly have

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|^p dx &\leq \sum_{N=-\infty}^{\infty} \int_{-\Delta}^{\Delta} |f(x+N)|^p dx \\ &\leq 2\Delta \sum_{N=-\infty}^{\infty} \max_{-\Delta\leq x\leq\Delta} |f(x+N)|^p \,. \end{aligned}$$

Since $|x+N| = |x+\lambda_N+(N-\lambda_N)|$ and $|N-\lambda_N| \le \Delta$ it follows that

$$\max_{-\Delta\le x\le\Delta} |f(x+N)| \le \max_{-2\Delta\le x\le 2\Delta} |f(x+\lambda_N)|$$

and so

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^p dx &\le 2\Delta \sum_{N=-\infty}^{\infty} (v_N)^p \\
&\le 2\Delta S^p \sum_{N=-\infty}^{\infty} \gamma^p \left(\sum_{n=-\infty}^{\infty} \frac{\tilde{d}_n}{S} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_{n+N})| \right)^p \\
&= 2\delta S^p \gamma^p \sum_{N=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\tilde{d}_{n-N}}{S} \left(\sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)| \right)^p \\
&\le 2^m \Delta S^{p-1} \gamma^p \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p \sum_{N=-\infty}^{\infty} \tilde{d}_N
\end{aligned}$$

by the properties of convex functions [6, p. 72]. Hence

$$\int_{-\infty}^{\infty} |f(x)|^p dx \le 2^m \Delta S^p \gamma^p \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p$$

which proves Theorem 1 in the case $p \ge 1$.

CASE (ii). $0 < p < 1$.

By the choice of k the series $\sum_{n\in\mathbb{Z}} (\tilde{d}_n)^p$ converges to a finite sum say, S_p. As above

$$\int_{-\infty}^{\infty} |f(x)|^p dx \le 2\Delta\gamma^p \sum_{N=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} \tilde{d}_n \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_{n+N})| \right)^p .$$

Set $\mathfrak{S}_s(a) := \left(\sum_{n=-\infty}^{\infty} (a_n)^s\right)^{1/s}$ where $a_n := \tilde{d}_n \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_{n+N})|^p$ and apply inequality (2.10.3) from [6] with $s = 1, r = 1$ to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)|^p dx &\le 2\Delta\gamma^p \sum_{N=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (\tilde{d}_{n-N})^p \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p \\
&= 2\Delta\gamma^p \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p \sum_{N=-\infty}^{\infty} (\tilde{d}_N)^p \\
&= 2\Delta S_p \gamma^p \sum_{n=-\infty}^{\infty} \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_n)|^p
\end{aligned}$$

and so Theorem 1 holds also in the case $0 < p < 1$. □

Remark 2 Let $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers for which (1.3) holds. From above it follows that if f is an entire function of exponential type satisfying (1.10) for some $c \in [0, 1)$ and

$$|f^{(\mu)}(\lambda_n)| \le M_1 \quad \text{for } \mu = 0, \dots, m-1 \text{ and all } n \in \mathbb{Z}\,,$$

then for all $N \in \mathbb{Z}$,

$$\begin{aligned}\max_{-\Delta\le x\le\Delta} |f(x+N)| &\le \max_{-2\Delta\le x\le 2\Delta} |f(x+\lambda_N)| \\ &\le \gamma \sum_{n=-\infty}^{\infty} \tilde{d}_n \sum_{\mu=0}^{m-1} |f^{(\mu)}(\lambda_{n+N})| \\ &\le \gamma m M_1 \sum_{n=-\infty}^{\infty} \tilde{d}_n \\ &= \gamma m S M_1\,,\end{aligned}$$

i.e. $|f(x)|$ is bounded on the real line by a constant depending only on M_1, c, Δ, δ and m. This extends a result of R.J. Duffin and A.C. Schaeffer for which we refer the reader to [2, Theorem 10.5.1].

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