J. of Inequal. & Appl., 1997, Vol. 1, pp. 239–252
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Weighted Inequalities for Hilbert Transforms and Multiplicators of Fourier Transforms

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(Received 12 August 1996)

As is well known, invariant operators with a shift can be bounded from L^p into L^q only if $1 < p \leq q < \infty$. We show that the case $q < p$ might also hold for weighted spaces. We derive the sufficient conditions for the validity of strong (weak) (p, q) type inequalities for the Hilbert transform when $1 < q < p < \infty$ $(q = 1, 1 < p < \infty)$.

The examples of couple of weights which guarantee the fulfillness of two-weighted strong (weak) type inequalities for singular integrals are presented. The method of proof of the main results allows us to generalize the results of this paper to the singular integrals which are defined on homogeneous groups.

The Fourier multiplier theorem is also proved.

Keywords: Hilbert transform; two-weight inequality; weak and strong type estimates; Fourier transform; multiplier.

1991 Mathematics Subject Classification: Primary 42B25; Secondary 46E30.

1 INTRODUCTION

The paper deals with two-weighted estimates of strong type (p, q), $1 < q < p < \infty$, for Hilbert transforms and multiplicators of Fourier transforms. As is well known, invariant operators with a shift can be bounded from L^p into L^q only if $1 < p \leq q < \infty$. Here we show that the case $q < p$ might also hold for Lebesgue weighted spaces.

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For $1 < p < \infty$ two-weighed inequalities of strong type (p, p) were established for a Hilbert transform in the case of monotone weights in [1]. A similar problem for singular integrals defined on the Euclidean spaces was considered in [2]. Optimal conditions for a pair of weights, ensuring the validity of two-weighted inequalities of strong type (p, p), $1 < p < \infty$, were found in [3] for Calderon–Zygmund singular integrals in the case of more general weights. In [4] two-weighted estimates of strong type (p, q), $1 < p \leq q < \infty$, were established for multiplicators of Fourier transforms.

2 SOME DEFINITIONS AND THE FAMILIAR RESULTS

An almost everywhere positive, locally integrable function $w : \mathbb{R}^1 \to \mathbb{R}^1$ will be called a weight.

In what follows $L^p_w(\mathbb{R}^1)$, $1 \leq p < \infty$, will denote the set of all measurable functions for which

$$\|f\|_{L^p_w(\mathbb{R}^1)} = \left(\int_{-\infty}^{+\infty} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

For functions $f \in L^p(\mathbb{R}^1)$, $1 \leq p < \infty$, we obtain a Hilbert transform by the formula

$$(Hf)(x) = \text{p.v.} \int_{-\infty}^{+\infty} \big(f(y)/(x-y)\big)\, dy \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \big(f(y)/(x-y)\big)\, dy.$$

We have the following theorems on one-weighted inequalities for Hilbert transforms.

THEOREM 1.1 [5] *Let $1 < p < \infty$. For the inequality*

$$\int_{-\infty}^{+\infty} |(Hf)(x)|^p w(x)\, dx \leq c \int_{-\infty}^{+\infty} |f(x)|^p w(x)\, dx,$$

where the positive constant c does not depend on $f \in L^p_w(\mathbb{R}^1)$, it is necessary and sufficient that $w \in A_p$, i.e.,

$$\sup_I \left(1/|I| \int_I w(x)\, dx \right) \left(1/|I| \int_I w^{1-p'}(x)\, dx \right)^{p-1} < \infty$$

($p' = p/(p-1)$ and the least upper bound is taken over all intervals $I \subset \mathbb{R}^1$).

THEOREM 1.2 [5] *The inequality*

$$\int_{\{x:|(Hf)(x)|>\lambda\}} w(x)\,dx \le c\lambda^{-1}\int_{-\infty}^{+\infty} |f(x)|w(x)\,dx,$$

where the positive constant c does not depend on $f \in L^1_w(\mathbb{R}^1)$, holds iff $w \in A_1$, i.e.,

$$1/|I|\int_I w(x)\,dx \le b \operatorname*{ess\,inf}_{x\in I} w(x)$$

with the positive constant b not depending on $I \subset \mathbb{R}^1$.

The main theorems will be proved by using the following theorems.

THEOREM 1.3 [6] *Let $1 \le q < p < \infty$, v and w be weight functions on $(-\infty, +\infty)$. For the inequality*

$$\left(\int_{-\infty}^{+\infty} |f(x)|^q v(x)\,dx\right)^{1/q} \le c\left(\int_{-\infty}^{+\infty} |f(x)|^p w(x)\,dx\right)^{1/p},$$

where the positive constant c does not depend on f, it is necessary and sufficient that

$$\int_{-\infty}^{+\infty} v^{p/(p-q)}(x)w^{-q/(p-q)}(x)\,dx < \infty.$$

THEOREM 1.4 [7] *Let $1 < q < p < \infty$, $p' = p/(p-1)$, $\alpha(t)$ and $\beta(t)$ be positive functions on $(0, \infty)$.*

(1) The inequality

$$\left(\int_0^\infty \left|\int_0^t F(\tau)\,d\tau\right|^q \alpha(t)\,dt\right)^{1/q} \le c_1\left(\int_0^\infty |F(t)|^p\beta(t)\,dt\right)^{1/p},$$

where the constant $c_1 > 0$ does not depend on F, holds iff

$$\int_0^\infty \left[\left(\int_t^\infty \alpha(\tau)\,d\tau\right)\left(\int_0^t \beta^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{p/(p-q)} \beta^{1-p'}(\tau)\,d\tau < \infty.$$

(2) For the inequality

$$\left(\int_0^\infty \left|\int_t^\infty F(\tau)\,d\tau\right|^q \alpha(t)\,dt\right)^{1/q} \le c_2\left(\int_0^\infty |F(t)|^p\beta(t)\,dt\right)^{1/p},$$

where the constant $c_2 > 0$ does not depend on F, it is necessary and sufficient that

$$\int_0^\infty \left[\left(\int_0^t \alpha(\tau)\,d\tau\right)\left(\int_t^\infty \beta^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{p/(p-q)} \beta^{1-p'}(\tau)\,d\tau < \infty.$$

LEMMA 1.1 *Let $p > 1$, $\alpha(t)$ and $\beta(t)$ be positive functions on $(0,\infty)$ satisfying the condition*

$$\int_0^\infty \left(\int_\tau^\infty \alpha(t)\,dt\right)^{p'} \beta^{1-p'}(\tau)\,d\tau < \infty.$$

Then there exists a positive constant $c > 0$ such that the inequality

$$\int_0^\infty \alpha(t)\left|\int_0^t F(\tau)\,d\tau\right|dt \le c\left(\int_0^\infty |F(t)|^p \beta(t)\,dt\right)^{1/p}$$

is fulfilled for any F.

Proof Changing the integration order and using the Hölder inequality we obtain

$$\begin{aligned}
\int_0^\infty \alpha(t)\left|\int_0^t F(\tau)\,d\tau\right|dt &\le \int_0^\infty \alpha(t)\left(\int_0^t |F(\tau)|\,d\tau\right)dt \\
&= \int_0^\infty |F(\tau)|\left(\int_\tau^\infty \alpha(t)dt\right)\frac{\beta^{1/p}(\tau)}{\beta^{1/p}(\tau)}\,d\tau \\
&\le \left(\int_0^\infty |F(\tau)|^p\beta(\tau)\,d\tau\right)^{1/p} \cdot \left(\int_0^\infty \left(\int_\tau^\infty \alpha(t)\,dt\right)^{p'} \beta^{-p'/p}(\tau)\,d\tau\right)^{1/p'} \\
&\le c\left(\int_0^\infty |F(\tau)|^p\beta(\tau)\,d\tau\right)^{1/p}.
\end{aligned}$$

□

The next lemma is proved similarly.

LEMMA 1.2 *Let $p > 1$, $\alpha(t)$ and $\beta(t)$ be positive functions on $(0,\infty)$ satisfying the condition*

$$\int_0^\infty \left(\int_0^\tau \alpha(t)\,dt\right)^{p'} \beta^{1-p'}(\tau)\,d\tau < \infty.$$

Then the inequality

$$\int_0^\infty \alpha(t)\left|\int_t^\infty F(\tau)\,d\tau\right|dt \le c\left(\int_0^\infty |F(t)|^p \beta(t)\,dt\right)^{1/p}$$

holds, where the constant $c > 0$ does not depend on F.

DEFINITION Let $1 \leq q < p < \infty$, $p' = p/(p-1)$.

(1) A pair (h_1, h) of positive measurable functions will be said to belong to the class a_{pq} if

$$\int_0^\infty \left[\left(\int_t^\infty h_1(\tau)\tau^{-q}\,d\tau\right)\left(\int_0^{t/2} h^{1-p'}(\tau)\,d\tau\right)^{q-1}\right]^{p/(p-q)} \times h^{1-p'}(t/2)\,dt < \infty$$

for $1 < q < p < \infty$, and

$$\int_0^\infty \left(\int_t^\infty h(\tau)\tau^{-1}d\tau\right)^{p'} h_1^{1-p'}(t/2)\,dt < \infty$$

for $1 = q < p < \infty$.

(2) Denote by b_{pq} a class of all pairs (h_1, h) of positive measurable functions which satisfy the condition

$$\int_0^\infty \left[\left(\int_0^{t/2} h_1(\tau)\,d\tau\right)\left(\int_t^\infty h^{1-p'}(\tau)\tau^{-p'}\,d\tau\right)^{q-1}\right]^{p/(p-q)} \times h^{1-p'}(t)t^{-p'}\,dt < \infty$$

for $1 < q < p < \infty$, and

$$\int_0^\infty \left(\int_0^{t/2} h_1(\tau)\,d\tau\right)^{p'} h^{1-p'}(t)t^{-p'}\,dt < \infty$$

for $1 = q < p < \infty$.

It is easy to prove the following lemmas:

LEMMA 1.3 *Let* $1 < q < p < \infty$, $v(t)$ *and* $w(t)$ *be positive monotone functions on* $(0, \infty)$.

(1) If $v(t)$ *and* $w(t)$ *are increasing functions and* $(v, w) \in a_{pq}$, *then we have*

$$\int_0^\infty v^{p/(p-q)}(t)w^{-q/(p-q)}(t/2)\,dt < \infty.$$

(2) If $v(t)$ *and* $w(t)$ *are decreasing functions and* $(v, w) \in b_{pq}$, *then*

$$\int_0^\infty v^{p/(p-q)}(t/2)w^{-q/(p-q)}(t)\,dt < \infty.$$

LEMMA 1.4 *Let $p > 1$, $v(t)$ and $w(t)$ be positive functions on* $(0, \infty)$.
(1) If $v(t)$ is an increasing function and $(v, w) \in a_{p1}$, then

$$\int_0^\infty v^{p'}(t) w^{1-p'}(t/2)\, dt < \infty.$$

(2) If $v(t)$ is a decreasing function and $(v, w) \in b_{p1}$, then

$$\int_0^\infty v^{p'}(t/2) w^{1-p'}(t)\, dt < \infty.$$

3 TWO-WEIGHTED ESTIMATES FOR HILBERT TRANSFORMS

Now we shall state the main results of this paper.

THEOREM 2.1 *Let $1 < q < p < \infty$, $v(t)$ and $w(t)$ be positive increasing functions on* $(0, \infty)$ *and $(v, w) \in a_{pq}$. Then there exists a postive constant c such that the inequality*

$$\left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q v(|x|)\, dx\right)^{1/q} \leq c\left(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|)\, dx\right)^{1/p} \quad (1)$$

holds for any $f \in L^p_{w(|x|)}(\mathbb{R}^1)$.

Proof Without loss of generality it can be assumed that the function $v(t)$ can be written as

$$v(t) = v(0) + \int_0^t \varphi(u)\, du,$$

where $v(0) = \lim\limits_{t\to 0} v(t)$ and $\varphi(u)$ is a positive function on $(0, \infty)$. The left-hand side of inequality (1) can be estimated as follows:

$$\begin{aligned}
&\left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q v(|x|)\, dx\right)^{1/q} \\
&\quad = \left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q \left(v(0) + \int_0^{|x|} \varphi(t)\, dt\right) dx\right)^{1/q} \\
&\quad \leq \left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q v(0)\, dx\right)^{1/q} \\
&\qquad + \left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q \left(\int_0^{|x|} \varphi(t)\, dt\right) dx\right)^{1/q} = I_1 + I_2.
\end{aligned}$$

For $v(0) = 0$ we have $I_1 = 0$, but for $v(0) \neq 0$ we have, by virtue of M. Riesz' theorem, Theorem 1.3 and Lemma 1.3 (the first part),

$$\begin{aligned}
I_1 &= v^{1/q}(0) \cdot \left(\int_{-\infty}^{+\infty} |(Hf)(x)|^q \, dx \right)^{1/q} \\
&\le c_1 v^{1/q}(0) \left(\int_{-\infty}^{+\infty} |f(x)|^q \, dx \right)^{1/q} = c_1 \left(\int_{-\infty}^{+\infty} |f(x)|^q v(0) \, dx \right)^{1/q} \\
&\le c_1 \left(\int_{-\infty}^{+\infty} |f(x)|^q v(|x|) \, dx \right)^{1/q} \le c_2 \left(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|) \, dx \right)^{1/p}.
\end{aligned}$$

Let us estimate the integral I_2:

$$\begin{aligned}
I_2 &= \left(\int_0^{\infty} \varphi(t) \left(\int_{|x|>t} |(Hf)(x)|^q \, dx \right) dt \right)^{1/q} \\
&\le c_3 \left(\int_0^{\infty} \varphi(t) \left(\int_{|x|>t} \left| \int_{|y|>t/2} [f(y)/(x-y)] \, dy \right|^q dx \right) dt \right)^{1/q} \\
&\quad + c_3 \left(\int_0^{\infty} \varphi(t) \left(\int_{|x|>t} \left| \int_{|y|\le t/2} [f(y)/(x-y)] \, dy \right|^q dx \right) dt \right)^{1/q} \\
&= I_{21} + I_{22}.
\end{aligned}$$

Again using M. Riesz' theorem, Theorem 1.3 and Lemma 1.3 (the first part), we obtain

$$\begin{aligned}
I_{21} &\le c_3 \left(\int_0^{\infty} \varphi(t) \left(\int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} [f(y) \cdot \chi_{\{z:|z|>t/2\}}(y)(x-y)^{-1}] \, dy \right|^q dx \right) dt \right)^{1/q} \\
&\le c_4 \left(\int_0^{\infty} \varphi(t) \left(\int_{|x|>t/2} |f(x)|^q \, dx \right) dt \right)^{1/q} \\
&= c_4 \left(\int_{-\infty}^{+\infty} |f(x)|^q \left(\int_0^{2|x|} \varphi(t) \, dt \right) dx \right)^{1/q} \\
&\le c_5 \left(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|) \, dx \right)^{1/p}.
\end{aligned}$$

Now we shall estimate I_{22}. It is easy to verify that if $|x| > t$ and $|y| \le t/2$, then $|x - y| \ge |x|/2$. We have

$$
\begin{aligned}
I_{22} &\leq \bigg(\int_0^\infty \varphi(t)\bigg(\int_{|x|>t}\bigg(\int_{|y|\leq t/2}\big(|f(y)|/|x-y|\big)\,dy dx\bigg)dt\bigg)^{1/q} \\
&\leq c_6\bigg(\int_0^\infty \varphi(t)\bigg(\int_{|x|>t}|x|^{-q}dx\bigg)\bigg(\int_{|y|\leq t/2}|f(y)|\,dy\bigg)^q dt\bigg)^{1/q} \\
&= c_7\bigg(\int_0^\infty \varphi(2s)\bigg(\int_{2s}^\infty \gamma^{-q}d\gamma\bigg)\bigg(\int_0^s\big(|f(y)|+|f(-y)|\big)\,dy\bigg)^q ds\bigg)^{1/q}
\end{aligned}
$$

Taking into account

$$
\begin{aligned}
\int_t^\infty \varphi(2s)\bigg(\int_{2s}^\infty \tau^{-q}d\tau\bigg)ds &= (1/2)\int_{2t}^\infty \tau^{-q}\bigg(\int_{2t}^\tau \varphi(s)\,ds\bigg)d\tau \\
&\leq (1/2)\int_{2t}^\infty v(\tau)\tau^{-q}d\tau,
\end{aligned}
$$

we find by virtue of Theorem 1.4 that

$$
\begin{aligned}
&\bigg(\int_0^\infty \varphi(2s)\bigg(\int_{2s}^\infty \gamma^{-q}d\gamma\bigg)\bigg(\int_0^s\big(|f(y)|+|f(-y)|\big)dy\bigg)^q ds\bigg)^{1/q} \\
&\leq c_8\bigg(\int_0^\infty w(t)\big(|f(t)|+|f(-t)|\big)^p dt\bigg)^{1/p} \\
&\leq c_9\bigg(\int_0^\infty w(t)|f(t)|^p dt+\int_0^\infty w(t)|f(-t)|^p dt\bigg)^{1/p} \\
&= c_9\bigg(\int_0^\infty w(t)|f(t)|^p dt+\int_{-\infty}^0 w(-t)|f(t)|^p dt\bigg)^{1/p} \\
&= c_9\bigg(\int_{-\infty}^{+\infty}|f(x)|^p w(|x|)dx\bigg)^{1/p}.
\end{aligned}
$$

□

If we write the function $v(t)$ as

$$
v(t) = v(+\infty) + \int_t^\infty \psi(u)\,du,
$$

where $v(+\infty) = \lim_{t\to+\infty} v(t)$ and $\psi(u)$ is a positive function on $(0,\infty)$, then using M. Riesz' theorem, Theorem 1.3, Theorem 1.4, Lemma 1.3 (the second part) and repeating the arguments used in proving Theorem 2.1, we shall obtain

THEOREM 2.2 *Let* $1 < q < p < \infty$, $v(t)$ *and* $w(t)$ *be positive decreasing functions on* $(0, \infty)$ *and* $(v, w) \in b_{pq}$. *Then inequality* (1) *is valid.*

EXAMPLE 1 Let

$$w(t) = \begin{cases} t^{p-1} \log^{\gamma}(1/t) & \text{for } t < 1/c, \\ (c^{\alpha-p+1} \log^{\gamma} c)t^{\alpha} & \text{for } t \geq 1/c, \end{cases}$$

$$v(t) = \begin{cases} t^{q-1} \log^{\beta}(1/t) & \text{for } t < 1/c, \\ (c^{\lambda-q+1} \log^{\beta} c)t^{\lambda} & \text{for } t \geq 1/c, \end{cases}$$

where $c = \exp(p/(p-q))$, $p - 1 < \gamma < p$, $\beta < q\gamma/p - q + q/p + 1$, $0 \leq \lambda < \alpha q/p + q/p - 1$, $p/q - 1 < \alpha < p - 1$. Then $w(t)$ and $v(t)$ are increasing functions and the pair $(v, w) \in a_{pq}$.

THEOREM 2.3 *Let* $p > 1$, $v(t)$ *and* $w(t)$ *be positive functions on* $(0, \infty)$, $v(t)$ *be an increasing function and the pair* $(v, w) \in a_{p1}$. *Then there exists a positive constant c such that the inequality*

$$\int_{\{x:|(Hf)(x)|>\lambda\}} v(|x|)\, dx \leq c\lambda^{-1}\left(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|)\, dx\right)^{1/p} \tag{2}$$

holds for any $f \in L^p_{w(|x|)}(\mathbb{R}^1)$ *and* $\lambda > 0$.

Proof Without loss of generality it can be assumed that the function v has the form

$$v(t) = v(0) + \int_0^t \varphi(u)\, du,$$

where $v(0) = \lim\limits_{t\to 0} v(t)$, $\varphi(u)$ is a positive function on $(0, \infty)$. We have

$$\int_{\{x:|(Hf)(x)|>\lambda\}} v(|x|)\, dx = \int_{\{x:|(Hf)(x)|>\lambda\}} v(0)\, dx + \int_{\{x:|(Hf)(x)|>\lambda\}} \left(\int_0^{|x|} \varphi(t)\, dt\right) dx = I_1 + I_2.$$

For $v(0) = 0$ we have $I_1 = 0$, but for $v(0) \neq 0$ we obtain by virtue of Kolmogorov's theorem, Theorem 1.3 and Lemma 1.4 (the first part)

$$I_1 = v(0)\big|\{x : |(Hf)(x)| > \lambda\}\big| \leq c\lambda^{-1} v(0) \int_{-\infty}^{+\infty} |f(x)|\, dx$$
$$\leq c_1\lambda^{-1} \int_{-\infty}^{+\infty} |f(x)| v(|x|)\, dx \leq c_2\lambda^{-1}\left(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|)\, dx\right)^{1/p}$$

Let us estimate I_2 as follows:

$$
\begin{aligned}
I_2 &= \int_0^\infty \varphi(t)\Big(\int_{|x|>t} \chi\{x : |(Hf)(x)| > \lambda\}\,dx\Big)\,dt \\
&\le \int_0^\infty \varphi(t)\Big(\int_{|x|>t} \chi\Big\{x : \Big|\int_{|y|>t/2} f(y)(x-y)^{-1}dy\Big| > \lambda/2\Big\}\,dx\Big)\,dt \\
&\quad + \int_0^\infty \varphi(t)\Big(\int_{|x|>t} \chi\Big\{x : \Big|\int_{|y|\le t/2} f(y)(x-y)^{-1}dy\Big| > \lambda/2\Big\}\,dx\Big)\,dt \\
&= I_{21} + I_{22}.
\end{aligned}
$$

To estimate I_{21} we again use Kolmogorov's theorem, Theorem 1.3 and Lemma 1.4 (the first part) and obtain

$$
\begin{aligned}
I_{21} &\le c_3\lambda^{-1}\int_0^\infty \varphi(t)\Big(\int_{|x|>t/2} |f(x)|\,dx\Big)\,dt \\
&= c_3\lambda^{-1}\int_{-\infty}^{+\infty} |f(x)|\Big(\int_0^{2|x|} \varphi(t)\,dt\Big)\,dx \\
&\le c_3\lambda^{-1}\int_{-\infty}^{+\infty} |f(x)|v(2|x|)\,dx \\
&\le c_4\lambda^{-1}\Big(\int_{-\infty}^{+\infty} |f(x)|^p w(|x|)\,dx\Big)^{1/p}.
\end{aligned}
$$

It is easy to verify that if $|x| > t$ and $|y| \le t/2$, then $|x-y| \ge |x|/2$. For I_{22} we have

$$
\begin{aligned}
I_{22} &\le 2\cdot\lambda^{-1}\int_0^\infty \varphi(t)\Big(\int_{|x|>t}\Big|\int_{|y|\le t/2} f(y)(x-y)^{-1}dy\Big|\,dx\Big)\,dt \\
&\le 2\cdot\lambda^{-1}\int_0^\infty \varphi(t)\Big(\int_{|x|>t} |x|^{-1}dx\Big)\Big(\int_{|y|\le t/2} |f(y)|\,dy\Big)\,dt \\
&= c_5\lambda^{-1}\int_0^\infty \varphi(2s)\Big(\int_{2s}^\infty \gamma^{-1}d\gamma\Big)\Big(\int_{|y|\le s} |f(y)|\,dy\Big)\,ds \\
&= c_5\lambda^{-1}\int_0^\infty \varphi(2s)\Big(\int_{2s}^\infty \gamma^{-1}d\gamma\Big)\Big(\int_0^s \big(|f(y)| + |f(-y)|\big)\,dy\Big)\,ds.
\end{aligned}
$$

In that case

$$
\int_t^\infty \varphi(2s)\Big(\int_{2s}^\infty \tau^{-1}d\tau\Big)\,ds \le (1/2)\int_{2t}^\infty v(\tau)\tau^{-1}d\tau
$$

and by Lemma 1.1 we obtain

$$
\begin{aligned}
&c_5\lambda^{-1}\int_0^\infty \varphi(2s)\left(\int_{2s}^\infty \gamma^{-1}d\gamma\right)\left(\int_0^s \left(|f(y)|+|f(-y)|\right)dy\right)ds\\
&\le c_6\lambda^{-1}\left(\int_0^\infty w(y)\left(|f(y)|+|f(-y)|\right)^p dy\right)^{1/p}\\
&\le c_7\lambda^{-1}\left(\int_0^\infty w(y)|f(y)|^p dy+\int_0^\infty w(y)|f(-y)|^p dy\right)^{1/p}\\
&= c_7\lambda^{-1}\left(\int_{-\infty}^{+\infty} |f(y)|^p w(y)\,dy\right)^{1/p}.
\end{aligned}
$$

□

If we write the function $v(t)$ as

$$v(t)=v(+\infty)+\int_t^\infty \psi(t)\,dt,$$

where $v(+\infty)=\lim_{t\to+\infty} v(t)$ and $\psi(t)>0$ on $(0,\infty)$, use Kolmogorov's theorem, Theorem 1.3, Lemma 1.2, Lemma 1.4 (the second part) and repeat the arguments used in proving Theorem 2.3, then we shall have

THEOREM 2.4 *Let $p>1$, $v(t)$ and $w(t)$ be positive functions on $(0,\infty)$, $v(t)$ be a decreasing function and the pair $(v,w)\in b_{p1}$. Then inequality (2) is valid.*

EXAMPLE 2.2 Let

$$
w(t)=\begin{cases} t^{-1}\log^\gamma(1/t) & \text{for } t<c,\\ (c^{-\alpha-1}\log^\gamma(1/c))t^\alpha & \text{for } t\ge c,\end{cases}
$$

$$
v(t)=\begin{cases} t^{-1}\log^\beta 1/t & \text{for } t<c,\\ (c^{-\lambda-1}\log^\beta(1/c))t^\lambda & \text{for } t\ge c,\end{cases}
$$

where $\beta<-1$, $\gamma>p(\beta+2)+1$, $-1<\lambda<0$, $\alpha>(\lambda+1)p-1$, $c=\exp(2\beta)$. Then $w(t)$ and $v(t)$ are decreasing functions on $(0,\infty)$ and $(v,w)\in b_{p1}$.

4 TWO-WEIGHTED ESTIMATES OF STRONG TYPE (p, q), $1 < q < p < \infty$, FOR MULTIPLICATORS OF FOURIER TRANSFORMS

In what follows S will denote a space of basic L. Schwartz functions. S consists of infinitely differentiable functions φ vanishing at infinity more rapidly than any power $|x|$. For $\varphi \in S$ we denote by $\widetilde{\varphi}$ the Fourier transform of φ:

$$\widetilde{\varphi}(\lambda) = (2\pi)^{-1/2} \int_{\mathbb{R}^1} \varphi(x) \exp(-i\lambda x)\, dx.$$

Conversely,

$$\varphi(x) = (2\pi)^{-1/2} \int_{\mathbb{R}^1} \widetilde{\varphi}(\lambda) \exp(i\lambda x)\, d\lambda.$$

Since weight functions w are locally summable, it can easily be shown that for any w the space S will be dense everywhere in $L^p_w(\mathbb{R}^1)$.

We have

THEOREM 3.1 *Let a measurable function $\Phi(\lambda)$ be defined by the formula*

$$\Phi(\lambda) = \int_{-\infty}^{\lambda} d\mu(t),$$

where $\mu(t)$ is the finite measure in $\mathbb{R}^1$.

Moverover, let $1 < q < p < \infty$, v and w be positive increasing functions on $(0, \infty)$ and $(v, w) \in a_{pq}$. Then the transform $\mathcal{K}$ given in terms of Fourier images by the formula

$$\widetilde{\mathcal{K}_{\varphi}}(\lambda) = \Phi(\lambda)\widetilde{\varphi}(\lambda), \quad \varphi \in S,$$

will define the bounded mapping of $L^p_{w(|x|)}(\mathbb{R}^1)$ into $L^q_{v(|x|)}(\mathbb{R}^1)$.

Proof We consider the Heaviside function

$$\widetilde{\chi}(\lambda) = \begin{cases} 1 & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda < 0. \end{cases}$$

The transform $T_{\lambda_0} : \varphi \to \psi$ given on S by the relation

$$\widetilde{g}(\lambda) = \widetilde{\chi}(\lambda - \lambda_0)\widetilde{\varphi}(\lambda) \tag{3}$$

generates the bounded transform from $L^p_w(\mathbb{R}^1)$ into $L^q_v(\mathbb{R}^1)$.

Assume first that $\lambda_0 = 0$. Equality (3) corresponds to the convolution in the initial space

$$g = \chi * \varphi,$$

where the generalized function χ has the form

$$\chi(x) = (2\pi)^{1/2}\big(\delta(x)/2 + 1/(2\pi i x)\big).$$

Now by Theorem 2.1 we obtain the boundedness of the operator T_0 from $L^p_w(\mathbb{R}^1)$ into $L^q_v(\mathbb{R}^1)$ for $\lambda_0 = 0$. Next, the function $\widetilde{\chi}$ has a shift of the argument by λ_0, which corresponds to multiplication of χ by $\exp(i\lambda_0 x)$ and which does not alter the estimate. Obviously, the norm of the operator T_{λ_0} coincides with the norm T_0. By virtue of the definition of the operator $\mathcal{K}$ and the function Φ we have

$$\mathcal{K}\varphi(x) = (2\pi)^{-1/2}\int_{\mathbb{R}^1} \Phi(\lambda)\widetilde{\varphi}(\lambda)\exp(i\lambda x)\,d\lambda = (2\pi)^{-1/2}\int_{\mathbb{R}^1} T_t\varphi(t)\,d\mu(t)$$

for any $\varphi \in S$.

Now we shall estimate the integral of the vector-valued function by using the boundedness of the opertor T_t from $L^p_w(\mathbb{R}^1)$ into $L^q_v(\mathbb{R}^1)$ with the norm not depending on t. We obtain

$$\begin{aligned}\|\mathcal{K}\varphi\|_{L^q_v(\mathbb{R}^1)} &\le (2\pi)^{-1/2}\int_{\mathbb{R}^1} \|T_t\varphi\|_{L^q_v(\mathbb{R}^1)}|d\mu(t)| \\ &\le c(2\pi)^{-1/2}\int_{\mathbb{R}^1} |d\mu|\cdot\|\varphi\|_{L^p_w(\mathbb{R}^1)}.\end{aligned}$$

Finally, using the density of the space S in $L^p_w(\mathbb{R}^1)$, we find that $\mathcal{K}$ has a bounded continuation in $L^q_v(\mathbb{R}^1)$. □

THEOREM 3.2 *Let* $1 < q < p < \infty$, v *and* w *be positive decreasing functions on* $(0, \infty)$ *and* $(v, w) \in b_{pq}$. *Then the statement of Theorem 3.1 is true.*

The proof is similar to the preceding one, we only have to use Theorem 2.2.

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