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The Best Possibility of the Bound for the Kantrovich Inequality and Some Remarks

M. TSUKADA[a] and SIN-EI TAKAHASI[b]

[a] *Department of Information Sciences, Toho University, Funabashi, Chiba 274, Japan*
[b] *Department of Basic Technology, Applied Mathematics and Physics, Yamagata University, Yonezawa, Yamagata 992, Japan*

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The necessary and sufficient condition for equality to be attaind in the Kantrovich inequality is given and applied to an inequality of normal operators on Hilbert spaces.

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1 INTRODUCTION

Kantrovich [3] established the following inequality: for any $0 < m \leq x_1, x_2, \ldots, x_n \leq M$ and $p_1, p_2, \ldots, p_n \geq 0$ with $\sum_{i=1}^n p_i = 1$,

$$\sum_{i=1}^n x_i p_i \sum_{i=1}^n \frac{1}{x_i} p_i \leq \frac{(m+M)^2}{4mM}.$$

(Note that the left hand side is greater than or equal to 1, which is a direct consequence of the Schwarz inequality.) Henrich [2] showed that the equality in the Kantrovich inequality holds if and only if

$$\sum_{\lambda_i = m} \lambda_i = \sum_{\lambda_i = M} \lambda_i = \frac{1}{2},$$

when $m \neq M$. Similar to the case for the Schwarz inequality, Kantrovich's also has its integral forms. First, Schweitzer [9] showed it for Riemann integrals, and later Nakamura [5] for Lebesgue-Stieltjes integrals on a compact space and he also commented out the condition for attaining the equality.

We here present the Kantrovich inequality for conditional expectations on probability spaces.

THEOREM 1 *Let $(\Omega, \mathcal{F}, p)$ be a probability measure space, $\mathcal{G}$ a $\sigma-$subfield of $\mathcal{F}$. For any random variables X, Y and Z such that both Y and Z are $\mathcal{G}-$measurable and $0 < Y \leq X \leq Z$, a.s., the following inequality holds:*

$$E(X|\mathcal{G})E(\frac{1}{X}|\mathcal{G}) \leq \frac{(Y+Z)^2}{4YZ}, \text{ a.s.},$$

where $E(\cdot|\mathcal{G})$ denotes the conditional expectation with respect to the $\sigma-$subfield $\mathcal{G}$.

Note that when $\mathcal{G} = \{\emptyset, \Omega\}$, this is the Kantrovich inequality of integral forms. The proof is inspired by the recent work due to Pták [7]. There are two essential points in the proof. One is the inequality between the arithmetic mean and the geometric mean. The other is the fact that, for any real number a and x with $0 < a^{-1} \leq x \leq a$, we have

$$x + x^{-1} \leq a + a^{-1}.$$

This is got from the shape of the graph of $y = x + x^{-1}$. If $0 < Y \leq X \leq Z$, a.s. , then

$$0 < \sqrt{\frac{Y}{Z}} \leq \frac{X}{\sqrt{YZ}} \leq \sqrt{\frac{Z}{Y}}, \text{ a.s.},$$

and hence we have, using the inequality mentioned above, that

$$\frac{X}{\sqrt{YZ}} + \frac{\sqrt{YZ}}{X} \leq \sqrt{\frac{Z}{Y}} + \sqrt{\frac{Y}{Z}}, \text{ a.s..}$$

Therefore

$$
\begin{aligned}
E(X|\mathcal{G})E(\frac{1}{X}|\mathcal{G}) &= E(\frac{X}{\sqrt{YZ}}|\mathcal{G})E(\frac{\sqrt{YZ}}{X}|\mathcal{G}) \\
&\leq \frac{1}{4}\left(E(\frac{X}{\sqrt{YZ}}|\mathcal{G}) + E(\frac{\sqrt{YZ}}{X}|\mathcal{G})\right)^2 \\
&= \frac{1}{4}E(\frac{X}{\sqrt{YZ}} + \frac{\sqrt{YZ}}{X}|\mathcal{G})^2 \\
&\leq \frac{1}{4}E(\sqrt{\frac{Z}{Y}} + \sqrt{\frac{Y}{Z}}|\mathcal{G})^2 \\
&= \frac{1}{4}\left(\sqrt{\frac{Z}{Y}} + \sqrt{\frac{Y}{Z}}\right)^2 \\
&= \frac{(Y+Z)^2}{4YZ}
\end{aligned}
$$

holds a.s., where the first and fifth equalities are from the $\mathcal{G}-$measurability of Y and Z, the second inequality is the arithmetic-geometric mean, the fourth equality is the linearity of conditional expectations, and the fifth inequality is from the above mentioned.

2 A NECESSARY AND SUFFICIENT CONDITIONS FOR THE EQUALITY

THEOREM 2 *Let $(\Omega, \mathcal{F}, p)$ be a probability measure space. For any complex-valued random variable X with $0 < m \leq |X| \leq M$, a.s. , the following inequality holds*

$$\left|\int Xdp \int \frac{1}{X}dp\right| \leq \frac{(m+M)^2}{4mM}.$$

The equality is attained if and only if there exists $0 \leq \theta < 2\pi$ such that

$$p(\{X = me^{i\theta}\}) = p(\{X = Me^{i\theta}\}) = \begin{cases} \frac{1}{2}, & (\textit{if } m < M) \\ 1, & (\textit{if } m = M) \end{cases}.$$

The first part of the theorem is a collorary of Theorem 1. Sufficiency of the second part is obvious, so we prove here necessity. Refering the proof of Theorem 1 for $\mathcal{G} = \{\emptyset, \Omega\}$, if the equality holds, it must be

$$\int \frac{|X|}{\sqrt{mM}}dp \int \frac{\sqrt{mM}}{|X|}dp = \frac{1}{4}\left(\int \left(\frac{|X|}{\sqrt{mM}} + \frac{\sqrt{mM}}{|X|}\right) dp\right)^2$$
$$= \frac{(m+M)^2}{4mM}.$$

Therefore

$$\int \frac{|X|}{\sqrt{mM}}dp = \int \frac{\sqrt{mM}}{|X|}dp = \frac{m+M}{2\sqrt{mM}}$$

and hence we have

$$\int |X|dp = \frac{m+M}{2}$$
$$\int \frac{1}{|X|}dp = \frac{1}{2}\left(\frac{1}{m} + \frac{1}{M}\right).$$

On the other hand because

$$\frac{|X|}{\sqrt{mM}} + \frac{\sqrt{mM}}{|X|} \leq \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}}, \text{ a.s.}$$

and

$$\int \left(\frac{|X|}{\sqrt{mM}} + \frac{\sqrt{mM}}{|X|}\right) dp = \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}},$$

we have

$$\frac{|X|}{\sqrt{mM}} + \frac{\sqrt{mM}}{|X|} = \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}}, \text{ a.s..}$$

Hence

$$(|X| - M)(|X| - m) = 0, \text{ a.s.},$$

that is, $|X|$ must be equal to m or M with probability one. It is easy to see that

$$p(\{|X| = m\}) = p(\{|X| = M\}).$$

Since the case for $m = M$ is trivial, we assume here $m < M$. Define

$$X(\omega) = \begin{cases} me^{i\theta_m(\omega)}, & (\text{if } |X(\omega)| = m) \\ Me^{i\theta_M(\omega)}, & (\text{if } |X(\omega)| = M) \end{cases} \quad \text{a.s.},$$

and put

$$\alpha = \int_{|X|=m} e^{i\theta_m} dp, \quad \beta = \int_{|X|=M} e^{i\theta_M} dp.$$

Note that $|\alpha|, |\beta| \leq 1/2$. Since

$$\begin{aligned}
\frac{(m+M)^2}{4mM} &= \left| \int X dp \int \frac{1}{X} dp \right| \\
&= \left| \left(\int_{|X|=m} m e^{i\theta_m} dp + \int_{|X|=M} M e^{i\theta_M} dp \right) \left(\int_{|X|=m} \frac{e^{-i\theta_m}}{m} dp + \int_{|X|=M} \frac{e^{-i\theta_M}}{M} dp \right) \right| \\
&= \left| (m\alpha + M\beta) \left(\frac{\overline{\alpha}}{m} + \frac{\overline{\beta}}{M} \right) \right| \\
&= \left| |\alpha|^2 + |\beta|^2 + \frac{m}{M} \alpha \overline{\beta} + \frac{M}{m} \overline{\alpha} \beta \right| \\
&\leq \frac{1}{4} \left| 2 + \frac{m}{M} + \frac{M}{m} \right| \\
&= \frac{(m+M)^2}{4mM},
\end{aligned}$$

it must be $|\alpha| = |\beta| = 1/2$. Thus

$$\left| \int_{|X|=m} e^{i\theta_m} dp \right| = \frac{1}{2},$$

$$\left| \int_{|X|=M} e^{i\theta_M} dp \right| = \frac{1}{2}.$$

We note that these equalities are the special cases of the Schwarz inequality. Because of the possibility of the bound of the inequality, both $e^{i\theta_m}$ and $e^{i\theta_M}$ are constant on the sets $\{|X| = m\}$ and $\{|X| = M\}$ respectively. It is easily seen that these constants must be equal. This completes the proof of the theorem.

By the spectral integral the Kantrovich inequality is generalized to the inequality for normal operators on Hilbert spaces (cf. [1, 5]).

COROLLARY 3 *Let X be a normal operator on a complex Hilbert space H with $0 < m \leq |X| \leq M$ for some real numbers m, M, where $|X| = (X^*X)^{1/2}$. Then for any unit vector ϕ*

$$|\langle X\phi|\phi\rangle\langle X^{-1}\phi|\phi\rangle| \leq \frac{(m+M)^2}{4mM}.$$

The equality is attained if and only if there exists $0 \leq \theta < 2\pi$ such that $me^{i\theta} \in \sigma(X)$, $Me^{i\theta} \in \sigma(X)$ and

$$\omega_\phi(\{me^{i\theta}\}) = \omega_\phi(\{Me^{i\theta}\}) = \begin{cases} \frac{1}{2}, & (\text{if } m < M) \\ 1, & (\text{if } m = M) \end{cases},$$

where ω_ϕ is the measure on the spectrum $\sigma(X)$ of X corresponding to the vector state $\langle \cdot\, \phi|\phi\rangle$ on the commutative C-algebra generated by $\{1, X\}$.*

3 $0 < A^{-1} \leq X \leq A \Rightarrow X + X^{-1} \leq A + A^{-1}$?

Let $(\mathcal{N}, \varphi)$ be a noncommutative probability space (i.e., $\mathcal{N}$ is a von Neuman algebra and φ is a faithful normal state on $\mathcal{N}$), and $E(\cdot|\mathcal{M})$ the conditional expectation onto a von Neumann subalgebra $\mathcal{M}$ of $\mathcal{N}$. It is interesting whether or not

$$E(X|\mathcal{M})E(X^{-1}|\mathcal{M}) \leq (Y+Z)^2Y^{-1}Z^{-1}/4.$$

for $X \in \mathcal{N}$ and $Y, Z \in \mathcal{M}$ with $0 < Y \leq |X| \leq Z$. It is not said in general. In the proof of Theorem 1 an elementary inequality

$$0 < a^{-1} \leq x \leq a \Rightarrow x + x^{-1} \leq a + a^{-1}$$

was essential, where a and x are real numbers. In this section we give an example such that $0 < A^{-1} \leq X \leq A$ but $X + X^{-1} \not\leq A + A^{-1}$, where A and X are operators on a Hilbert space.

Put

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$

and

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, U^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Then because

$$A^{-1} < \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \leq \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \leq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} < A,$$

we have

$$A^{-1} < \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \leq U^* \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} U \leq \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} < A.$$

For any $\lambda \geq 1$, put

$$X_\lambda = U^* \begin{pmatrix} 2\lambda & 0 \\ 0 & \frac{1}{2\lambda} \end{pmatrix} U,$$

and define

$$f_\lambda(\phi) = \langle (A - X_\lambda)\phi | \phi \rangle,$$
$$g_\lambda(\phi) = \langle (X_\lambda - A^{-1})\phi | \phi \rangle.$$

Because X_λ is a Hermitian matrix, f_λ and g_λ are real-valued functions. Since f_1 and g_1 are non-negative functions and take 0 only at the origin, these have positive minimums on the unit circle. Both f_λ and g_λ uniformly converge to f_1 and g_1, respectively. Therefore if we take λ sufficiently close to 1, we are able to make

$$A^{-1} < X_\lambda < A.$$

We take

$$\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then

$$\langle (A + A^{-1})\phi | \phi \rangle = 2 + \frac{1}{2},$$

however

$$\langle (X_\lambda + X_\lambda^{-1})\phi | \phi \rangle = 2\lambda + \frac{1}{2\lambda},$$

and

$$(2\lambda + \frac{1}{2\lambda}) - (2 + \frac{1}{2}) = (\lambda - 1)(2 + \frac{1}{2\lambda}) > 0.$$

Thus we have

$$X_\lambda + X_\lambda^{-1} \not\leq A + A^{-1}.$$

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