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Relative Boundedness and Compactness Theory for Second-order Differential Operators

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The problem considered is to give necessary and sufficient conditions for perturbations of a second-order ordinary differential operator to be either relatively bounded or relatively compact. Such conditions are found for three classes of operators. The conditions are expressed in terms of integral averages of the coefficients of the perturbing operator.

Keywords: Relatively bounded; relatively compact; maximal and minimal operators; perturbations.

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1 INTRODUCTION

We consider the second-order differential operator

$$L[y] = -(py')' + qy$$

and lower-order perturbations

$$B_j[y] = b_j y^{(j)} \qquad (j = 0, 1)$$

in the setting of the Hilbert space $L^2(a, \infty)$. We prove three theorems which give necessary and sufficient integral average conditions on b_j for relative boundedness or relative compactness of B_j with respect to L (with appropriate domain restrictions). We employ the following terminology (cf. Kato [9, pp. 190, 194] or Goldberg [7, p. 121]).

Suppose B, L are operators in a Hilbert space. We say B is *relatively bounded with respect to* L or simply L*-bounded* if $D(L) \subseteq D(B)$ and B is bounded on $D(L)$ with respect to the graph norm $\|\cdot\|_L$ of L defined by $\|y\|_L = \|y\| + \|Ly\|$, $y \in D(L)$, where $D(L)$ denotes the domain of L. In other words, B is L*-bounded* if $D(L) \subseteq D(B)$ and there exist nonnegative constants α and β such that

$$\|By\| \leq \alpha\|y\| + \beta\|Ly\|, \qquad y \in D(L).$$

A sequence $\{y_n\}$ is said to be L*-bounded* if there exists $K > 0$ such that $\|y_n\|_L < K, n \geq 1$.

B is called *relatively compact with respect to* L or simply L*-compact* if $D(L) \subseteq D(B)$ and B is compact on $D(L)$ with respect to the L-norm, i.e., B takes every L-bounded sequence into a sequence which has a convergent subsequence. For example, if L is the identity map, then L-boundedness (L-compactness) of B is equivalent to the usual operator norm boundedness (compactness) of B.

The space of complex-valued functions y with domain I such that $\|y\|_\infty := \operatorname{ess\,sup}_{t\in I}|y(t)| < \infty$ is denoted by $L^\infty(I)$. A local property is indicated by use of the subscript "loc," and AC is used to abbreviate absolutely continuous. The space of all complex-valued, n times continuously differentiable functions on I is denoted by $C^n(I)$; $C_c^n(I)$ denotes the restriction of $C^n(I)$ to functions with compact support contained in I; and $C_0^\infty(I)$ is the space of all complex-valued functions on I which are infinitely differentiable and have compact support contained in the interior of I. Throughout $\|\cdot\|$ denotes the norm in $L^2(a, \infty)$.

Let ℓ be a differential expression of the form $\ell[y] = -(py')' + qy$, where p and q are complex-valued, Lebesgue measurable functions on an interval I such that $1/p, q \in L_{\text{loc}}(I)$. Then the *maximal operator* L corresponding to ℓ has domain $D(L) = \{y \in L^2(I) : y, py' \in AC_{\text{loc}}(I), \ell[y] \in L^2(I)\}$ and action $L[y] = \ell[y] = -(py')' + qy$ $(y \in D(L))$. The *minimal operator* L_0 corresponding to ℓ is defined to be the minimal closed extension of L restricted to those $y \in D(L)$ which have compact support in the interior of I.

The maximal and minimal operators corresponding to $b_j y^{(j)}$ have similar definitions. See Naimark [10, §17] for properties of maximal and minimal operators.

Theorem 2.1 below is a special case of Theorem 1.3 in Anderson [1, pp. 31–32] and of Theorem 1.2 in Anderson [2]. This theorem covers the case in which the coefficients of L are eventually bounded above by the corresponding coefficients of an Euler operator, i.e., $0 < p(t) \leq Ct^2$, $|p'(t)| \leq Kt$, and $|q(t)| \leq M$ for some positive constants C, K, and M, and all t sufficiently large.

Before proving the other two second-order results, we give, in Theorem 2.2, conditions under which perturbation conditions for maximal operators are equivalent to those for minimal operators. The theorem is stated for operators of any order, and the key hypothesis is that the higher-order operator is limit-point at ∞. This result simplifies the proofs of the next two theorems since it suffices to consider only minimal operators.

The differential expression

$$\ell[y] = (-1)^n [p_0 y^{(n)}]^{(n)} + (-1)^{n-1} [p_1 y^{(n-1)}]^{(n-1)} + \cdots + p_n y,$$

$-\infty \leq a < t < b \leq \infty$, (where $p_0, p_1, \ldots, p_n$ are real-valued functions) is said to be *regular at* a if $a > -\infty$ and if the functions $1/p_0, p_1, \ldots, p_n$ are Lebesgue integrable in every interval $[a, \beta]$, $\beta < b$. Otherwise, ℓ *is singular at* a. Following usual terminology, ℓ is said to be *limit-point at* ∞ if its deficiency index equals n, i.e., the number of square-integrable solutions of $\ell[y] = \lambda y$, $\mathrm{Im}(\lambda) \neq 0$, is n.

In Theorem 3.1, unrestricted growth of p and q is allowed with q being the dominant term in the sense of (3.2) and (3.3). For example, the situation in which $p(t) = t^\alpha$ and $q(t) = Kt^\beta$ for some constants $K > 0$ and $\alpha \leq \beta + 2$ is included as a special case, as is the situation $p(t) = e^{\alpha t}$ and $q(t) = Ke^{\beta t}$ with $\alpha \leq \beta$. We give perturbation conditions on b_0 and b_1 which involve integral averages over intervals of length $\delta\sqrt{p(t)/q(t)}$ for some sufficiently small $\delta > 0$. The pointwise (sufficient) conditions in Everitt and Giertz [6, pp. 322–324] are recovered as a special case.

The last theorem, Theorem 4.1, deals with the case in which $p(t) = t^\alpha$ ($\alpha \geq 2$) dominates q in the sense of (4.3), e.g., $q(t) = Mt^\beta$ with $\beta \leq \alpha - 2$. As in Theorem 2.1, the perturbation conditions involve integral average conditions of b_0 and b_1 over intervals of length δt for some sufficiently small $\delta > 0$. Theorems 2.1, 3.1, and 4.1 overlap in the case that L is an Euler operator.

The proof of Theorems 3.1 requires a different approach from that used in Theorem 2.1. First, $L[y]^2$ is computed and a separation inequality of the form

$$C_1 \int_a^\infty |(py')'|^2 + C_2 \int_a^\infty pq|y'|^2 + C_3 \int_a^\infty q^2|y|^2 \le \int_a^\infty |L_0[y]|^2$$

($y \in D(L_0)$) is used. The separation inequality, used in proving Theorem 3.1, is derived in Everitt and Giertz [6, Theorem 1]. The proofs of Theorems 3.1 and 4.1 also rely on Theorem A, a special case of Theorem 2.1 in Brown and Hinton [4] on sufficient conditions for weighted interpolation inequalities.

In concluding §4 we show that Theorem 4.1 applies to the energy operator of the hydrogen atom, i.e.,

$$L[y] = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + V(x)\right] y,$$

on $0 < x \le 1$, and give (for $\ell > 1/2$) necessary and sufficient conditions that $V(x)$ is a relatively bounded (compact) perturbation of the operator $-y'' + \ell(\ell+1)x^{-2}y$. This application does not seem to have appeared in the literature.

Since self-adjoint operators for differential expressions are determined by restricting the domain of the maximal operator, a relatively bounded (compact) perturbation of the maximal operator is automatically a relatively bounded (compact) perturbation of such self-adjoint operators. For this reason, perturbation theorems for differential expressions are most useful when proved for maximal operators.

The authors express their appreciation to Mike Shaw for correcting an error in the proof of Theorem 2.2.

2 A p DOMINANT CASE AND AN EQUIVALENCE

Specializing the results of [2] to the second-order case yields the theorem below. This result is a case of p dominant and "small" in the sense of $|p'(t)| \le k\sqrt{p(t)}$.

THEOREM 2.1 *Let $I = [a, \infty)$. Suppose p and q are real-valued functions satisfying $p \in AC_{\text{loc}}(I)$, $q \in L^\infty(I)$, $p > 0$ on I, and*

$$|p'(t)| \le K\sqrt{p(t)}$$

a.e. on I for some positive constant K. Let L, B_j be the maximal operators associated with the differential expressions

$$\ell[y] = -(py')' + qy$$

and

$$v_j[y] = b_j y^{(j)} \qquad (j = 0, 1),$$

respectively, where each $b_j \in L_{\text{loc}}(I)$. For $j = 0, 1$ and $\delta > 0$, define

$$g_{j,\delta}(t) = \frac{1}{\sqrt{p(t)}} \int_t^{t+\delta\sqrt{p(t)}} \frac{|b_j(\tau)|^2}{p(\tau)^j} d\tau \qquad (t \in I)$$

Then the following hold for $j = 0, 1$.

(i) B_j is L-bounded if and only if $b_j \in L^2_{\text{loc}}(I)$ and

$$\sup_{a \le t < \infty} g_{j,\delta}(t) < \infty,$$

for some $\delta \in (0, 1/2)$.

(ii) B_j is L-compact if and only if $b_j \in L^2_{\text{loc}}(I)$ and

$$\lim_{t \to \infty} g_{j,\delta}(t) = 0,$$

for some $\delta \in (0, 1/2)$.

Proof This result is the special case of [1, Theorem 1.3] and [2, Theorem 1.2] in which $p = n = 2$, $s(t) = \sqrt{p(t)}$, $w \equiv 1$, $\alpha = 0$, $a_0 = q$, $a_1 = -p'/\sqrt{p}$, and $a_2 \equiv -1$. (In this case, $W \equiv 1$, $P_i = p^i$ for $i = 0, 1, 2$, and $N_0 = 1$.) □

Remark 2.1 Since a maximal operator is an extension of a minimal operator, Theorem 2.1 also holds if L and B_j are replaced by the minimal operators corresponding to ℓ and v_j, respectively.

THEOREM 2.2 *Consider the differential expressions $\ell = \sum_{i=0}^{n} a_i(t)D^i$ and $m = \sum_{j=0}^{n-1} b_j(t)D^j$, where each $a_i, b_j \in L^2_{\text{loc}}(a, \infty)$. Let L_0, M_0 and L_1, M_1 be the corresponding minimal and maximal operators, respectively. Suppose ℓ and m are symmetric. Let ℓ be regular at a and limit-point at ∞. Then M_1 is a relatively bounded (relatively compact) perturbation of L_1 if and only if M_0 is a relatively bounded (relatively compact) perturbation of L_0.*

Proof First suppose M_0 is L_0-bounded. Then $D(L_0) \subseteq D(M_0)$ and there exists $C_1 > 0$ such that

$$\|M_0 y\| \leq C_1 (\|y\| + \|L_0 y\|) \tag{2.1}$$

for all $y \in D(L_0)$. First we show that $D(L_1) \subseteq D(M_1)$. Let $y \in D(L_1)$. Since ℓ is regular at a and limit-point at ∞ (see Naimark [10, p. 31]),

$$D(L_0^*) = D(L_0) \oplus S$$

where L_0^* is the adjoint of L_0 and S is a finite-dimensional space of $C^n(a, \infty)$ functions with compact support. Since ℓ is symmetric, $L_0^* = L_1$. Therefore,

$$D(L_1) = D(L_0) \oplus S. \tag{2.2}$$

Hence there exists $y_0 \in D(L_0)$ and $y_c \in C_c^n(a, \infty)$ such that

$$y = y_0 + y_c. \tag{2.3}$$

Thus $\|my\| \leq \|my_0\| + \|my_c\|$. The first term on the right side is finite because $y_0 \in D(L_0) \subseteq D(M_0)$. The second term on the right is finite since $y_c^{(j)}$, $0 \leq j \leq n-1$, are continuous functions with compact support and $b_j \in L^2_{\text{loc}}(a, \infty)$. Hence $my \in L^2(a, \infty)$, and so $y \in D(M_1)$. Since $y \in D(L_1)$ was arbitrary, we have shown that $D(L_1) \subseteq D(M_1)$.

Next we show that there exists a constant C such that

$$\|M_1 y\| \leq C (\|y\| + \|L_1 y\|)$$

for all $y \in D(L_1)$. Let $y \in D(L_1)$. Write y as in (2.3). Before proceeding further, we state and prove a lemma.

LEMMA 2.1 *Let X and S be subspaces of a Banach space B, where X is closed, S is finite-dimensional, and $X \cap S = \{0\}$. Then there exists a constant $K > 0$ such that $\|x + s\| \geq K\|s\|$ for all $x \in X$ and $s \in S$.*

Proof The proof is by contradiction. Suppose no such K exists. Then there exist sequences $\{x_n\} \subset X$ and $\{s_n\} \subset S$ with $\|s_n\| = 1$ for all n such that $\|x_n + s_n\| \to 0$ as $n \to \infty$. Let $C = \{s \in S : \|s\| = 1\}$. Then C is closed and bounded. Since S is finite-dimensional, C is compact. Therefore, $\{s_n\}$ has a convergent subsequence which we relabel as $\{s_\ell\}$. So there exists s^* such that $\|s_\ell - s^*\| \to 0$ as $\ell \to \infty$. Note that $\|s^*\| = 1$ and

$$\begin{aligned}\|x_\ell - x_m\| &= \|x_\ell + s_\ell - s_\ell + s_m - s_m - x_m\| \\ &\leq \|x_\ell + s_\ell\| + \|s_\ell - s_m\| + \|s_m + x_m\|.\end{aligned}$$

By hypothesis, the first and last terms on the right side approach 0 as $\ell \to \infty$ and $m \to \infty$, respectively. Since $\{s_\ell\}$ is convergent, $\|s_\ell - s_m\| \to 0$ as $\ell, m \to \infty$. Therefore, $\{x_\ell\}$ is Cauchy. Since X is closed, there exists $x^* \in X$ such that $\|x_\ell - x^*\| \to 0$ as $\ell \to \infty$. We have

$$\|(x_\ell + s_\ell) - (x^* + s^*)\| \le \|x_\ell - x^*\| + \|s_\ell - s^*\| \to 0$$

as $\ell \to \infty$. Hence $x_\ell + s_\ell \to x^* + s^*$. By hypothesis, $x_\ell + s_\ell \to 0$. Since limits in B are unique, $x^* + s^* = 0$. Finally, since $X \cap S = \{0\}$, $x^* = s^* = 0$, which contradicts $\|s^*\| = 1$. This completes the proof of Lemma 2.1. □

Returning to the proof of the theorem, an application of Lemma 2.1 (with $X =$ graph of L_1 with graph norm $\|y\|_L = \|y\| + \|L_1 y\|$, and S as in (2.2)) yields the existence of a constant k for all y as in (2.3),

$$\|y\| + \|L_1 y\| = \|y\|_L \ge k\|y_c\|_L \ge k\|y_c\|. \tag{2.4}$$

Since a linear operator acting on a finite-dimensional space is bounded, there exists $C_2 > 0$ such that

$$\|M_1 y_c\| \le C_2\|y_c\| \tag{2.5}$$

for all $y_c \in S$. By hypothesis, $D(L_0) \subseteq D(M_0)$. Thus since $M_0 \subset M_1$, $M_1 y_0 = M_0 y_0$. So $M_1 y = M_1 y_0 + M_1 y_c = M_0 y_0 + M_1 y_c$. Now, use of (2.1), (2.5), and (2.4) produces

$$\begin{aligned}\|M_1 y\| &\le \|M_0 y_0\| + \|M_1 y_c\| \le C_1(\|y_0\| + \|L_0 y_0\|) + C_2\|y_c\| \\ &\le C_1(\|y_0\| + \|L_0 y_0\|) + \frac{C_2}{k}\left[\|y\| + \|L_1 y\|\right].\end{aligned} \tag{2.6}$$

Note that

$$\|y_0\| = \|y - y_c\| \le \|y\| + \|y_c\| \le \left(1 + \frac{1}{k}\right)\|y\| + \frac{1}{k}\|L_1 y\|, \tag{2.7}$$

where the last bound follows from (2.4). Since $y_0 \in D(L_0)$ and $L_0 \subset L_1$, $L_0 y_0 = L_1 y_0$. Thus $\|L_0 y_0\| = \|L_1 y_0\| = \|L_1(y_0 + y_c) - L_1 y_c\| = \|L_1 y - L_1 y_c\| \le \|L_1 y\| + \|L_1 y_c\|$. Since L_1 is a bounded operator when acting on the finite-dimensional space S, there exists $C_3 > 0$ independent of y_c such that $\|L_1 y_c\| \le C_3\|y_c\|$. Another application of (2.4) and substitution in the previous estimate gives $\|L_0 y_0\| \le \|L_1 y\| + \frac{C_3}{k}\left[\|y\| + \|L_1 y\|\right]$. Use of this bound and (2.7) in (2.6) yields $\|M_1 y\| \le C\left(\|y\| + \|L_1 y\|\right)$ for some constant C independent of $y \in D(L_1)$. Since $y \in D(L_1)$ was arbitrary, M_1 is L_1-bounded.

Now suppose M_0 is L_0-compact. Then $D(L_0) \subseteq D(M_0)$ and $\{f_n\} \subset D(L_0)$ with $\|f_n\| + \|L_0 f_n\| \leq C$ for all n implies that $\{M_0 f_n\}$ contains a convergent subsequence. To show that M_1 is L_1-compact, suppose $\{y_n\} \subset D(L_1)$ with $\|y_n\| \leq C_1$ and $\|L_1 y_n\| \leq C_2$ for all n. Since L_0-compactness of M_0 implies L_0-boundedness of M_0, M_1 is L_1-bounded by the first part of the proof. Therefore, $D(L_1) \subseteq D(M_1)$. By (2.3), $y_n = y_{n0} + y_{nc}$ where $y_{n0} \in D(L_0)$ and $y_{nc} \in S$. So $M_1 y_n = M_1 y_{n0} + M_1 y_{nc}$. Since $y_{n0} \in D(L_0) \subseteq D(M_0)$ and $M_0 \subset M_1$, $M_1 y_{n0} = M_0 y_{n0}$. Therefore, $M_1 y_n = M_0 y_{n0} + M_1 y_{nc}$. Since $L_0 \subset L_1$, $L_0 y_{n0} = L_1 y_{n0}$. By hypothesis, $\{L_1 y_n\}$ is bounded in $L^2(a, \infty)$. Thus

$$\begin{aligned}\|L_1 y_{n0}\| &= \|L_1 y_n - L_1 y_{nc}\| \leq \|L_1 y_n\| + \|L_1 y_{nc}\| \\ &\leq C_2 + C\|y_{nc}\| \leq C_2 + \frac{C}{k}\|y_n\| \leq C_2 + \frac{C}{k}C_1,\end{aligned}$$

for some positive constants C and k, where we have used the fact that L_1 is bounded when acting on the finite-dimensional space S, Lemma 2.1 (as in (2.4)), and the hypothesis that $\{y_n\}$ is bounded. Therefore $\{L_1 y_{n0}\}$, and hence $\{L_0 y_{n0}\}$ is bounded in $L^2(a, \infty)$. Since M_0 is L_0-compact, $\{M_0 y_{n0}\}$ contains a convergent subsequence $\{M_0 y_{n_j 0}\}$. Since M_1 is a bounded operator on S, $\|M_1 y_{n_j c}\| \leq C\|y_{n_j c}\| \leq (C/k)\|y_{n_j}\| \leq \widetilde{C}$, where the last two inequalities follow from Lemma 2.1 and boundedness of $\{y_n\}$. Therefore, $\{M_1 y_{n_j c}\}$ is bounded in a finite-dimensional subspace of $L^2(a, \infty)$ and hence contains a convergent subsequence. So $\{M_1 y_n\}$ contains a convergent subsequence. Therefore, M_1 is L_1-compact.

Next suppose that M_1 is L_1-bounded. Then $D(L_1) \subseteq D(M_1)$ and

$$\|M_1 y\| \leq C(\|y\| + \|L_1 y\|) \tag{2.8}$$

for all $y \in D(L_1)$. First we show that $D(L_0) \subseteq D(M_0)$. Let $y \in D(L_0)$. Then, by a theorem in Naimark [11, p. 68], there exists $\{y_k\} \subset C_c^n(a, \infty)$ such that $y_k \to y$ and $L_0 y_k \to L_0 y$ as $k \to \infty$. Since $L_0 \subset L_1$, (2.8) holds for $y_k - y$ for each k:

$$\begin{aligned}\|M_1 y_k - M_1 y\| &\leq C\left(\|y_k - y\| + \|L_1(y_k - y)\|\right) \\ &= C\left(\|y_k - y\| + \|L_0 y_k - L_0 y\|\right)\end{aligned}$$

It follows that $M_1 y_k \to M_1 y$. Since order$(m) <$ order(l), the functions $y_k \in C_c^n(a, \infty)$ are smooth enough (specifically, $C_c^{n-1}(a, \infty)$) that $y_k \in D(M_0)$. Since $M_0 \subset M_1$, $M_0 y_k = M_1 y_k$; and so $M_0 y_k \to M_1 y$. Therefore, $\{y_k\}$ and $\{M_0 y_k\}$ are convergent. Since M_0 is a closed operator, $y \in D(M_0)$ and $M_0 y_k \to M_0 y$. Therefore, $D(L_0) \subseteq D(M_0)$.

Since $L_0 \subset L_1$, (2.8) holds for all $y \in D(L_0)$, i.e., $\|M_1 y\| \le C(\|y\| + \|L_0 y\|)$ for all $y \in D(L_0)$. Since $D(L_0) \subseteq D(M_0)$ and $M_0 \subset M_1$, $M_1 y = M_0 y$ for $y \in D(L_0)$. Hence $\|M_0 y\| \le C(\|y\| + \|L_0 y\|)$ for all $y \in D(L_0)$. Therefore, M_0 is L_0-bounded.

Finally, suppose M_1 is L_1-compact. Then $D(L_1) \subseteq D(M_1)$ and if $\{f_n\} \subset D(L_1)$ with

$$\|f_n\| + \|L_1 f_n\| \le C \tag{2.9}$$

for all n, then $\{M_1 f_n\}$ contains a convergent subsequence. Since L_1-compactness of M_1 implies L_1-boundedness of M_1, M_0 is L_0-bounded by the proof of the previous part. Therefore, $D(L_0) \subseteq D(M_0)$. Suppose $\{y_n\} \subseteq D(L_0)$ with $\|y_n\| + \|L_0 y_n\| \le C$ for all n. Since $L_0 \subseteq L_1$, $\{y_n\} \subseteq D(L_1)$ and $L_0 y_n = L_1 y_n$. Hence (2.9) holds with f_n replaced by y_n. Therefore, $\{M_1 y_n\}$ contains a convergent sequence. Since $\{y_n\} \subseteq D(L_0) \subseteq D(M_0)$ and $M_0 \subset M_1$, it follows that $\{M_0 y_n\}$ contains a convergent subsequence. Therefore, M_0 is L_0-compact. □

3 A q DOMINANT RESULT

The following theorem establishes relative boundedness and compactness of perturbations when q dominates p in the sense of (3.2) and (3.3).

THEOREM 3.1 *Let $I = [a, \infty)$. Let p and q be $AC_{\text{loc}}(I)$ real-valued functions such that $p > 0$ on I,*

$$q(t) \ge K, \tag{3.1}$$

$$|p'(t)| \le A_1 \sqrt{p(t)q(t)} \tag{3.2}$$

and

$$p(t)^{1/2} |q'(t)| \le A_2 q(t)^{3/2} \tag{3.3}$$

for $t \in I$ and some positive constants K, A_1, and A_2 with $A_2 < 1$.

Let L, B_j be the maximal operators associated with the differential expressions

$$\ell[y] = -(py')' + qy$$

and

$$v_j[y] = b_j y^{(j)} \qquad (j = 0, 1)$$

respectively, where each $b_j \in L_{\text{loc}}(I)$. *For* $j = 0, 1$ *and* $\delta > 0$, *define*

$$g_{j,\delta}(t) = \sqrt{\frac{q(t)}{p(t)}} \int_t^{t+\delta\sqrt{\frac{p(t)}{q(t)}}} \frac{|b_j(\tau)|^2}{p(\tau)^j q(\tau)^{2-j}}\, d\tau. \tag{3.4}$$

Then the following hold for $j = 0, 1$.

(i) B_j *is L-bounded if and only if* $b_j \in L^2_{\text{loc}}(I)$ *and*

$$\sup_{a \le t < \infty} g_{j,\delta}(t) < \infty \tag{3.5}$$

for some $\delta \in (0, 1/(A_1 + A_2))$.

(ii) B_j *is L-compact if and only if* $b_j \in L^2_{\text{loc}}(I)$ *and*

$$\lim_{t \to \infty} g_{j,\delta}(t) = 0 \tag{3.6}$$

for some $\delta \in (0, 1/(A_1 + A_2))$.

The following theorem is a special case of Theorem 2.1 in Brown and Hinton [4]. It gives sufficient conditions for weighted interpolation inequalities.

THEOREM A *Let* $I = [a, \infty)$ *and* $0 \le j \le 1$. *Let* N, W, *and* P *be positive measurable functions such that* $N \in L_{\text{loc}}(I)$ *and* W^{-1}, $P^{-1} \in L_{\text{loc}}(I)$. *Suppose there exists* $\varepsilon_0 > 0$ *and a positive continuous function* $f = f(t)$ *on* I *such that*

$$S_1(\varepsilon) := \sup_{t \in I} \left\{ f^{2(2-j)} T_{t,\varepsilon}(P) \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} N \right] \right\} < \infty$$

and

$$S_2(\varepsilon) := \sup_{t \in I} \left\{ f^{-2j} T_{t,\varepsilon}(W) \left[\frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} N \right] \right\} < \infty$$

for all $\varepsilon \in (0, \varepsilon_0)$, *where* $T_{t,\varepsilon}(P) = \frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} P^{-1}$ *with a similar definition for* $T_{t,\varepsilon}(W)$. *Then there exists* $K > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$ *and* $y \in D$,

$$\int_I N|y^{(j)}|^2 \le K \left\{ \varepsilon^{-2j} S_2(\varepsilon) \int_I W|y|^2 + \varepsilon^{2(2-j)} S_1(\varepsilon) \int_I P|y''|^2 \right\},$$

where $D = \{y : y' \in AC_{\text{loc}}(I),\ \int_I W|y|^2 < \infty,\ \text{and}\ \int_I P|y''|^2 < \infty\}$.

Proof of Theorem 3.1 From Dunford and Schwartz [5; XIII 6.14], L is regular at a and limit-point at ∞. In view of Theorem 2.2, it suffices to prove the result for minimal rather than maximal operators. Let L_0 and $B_{j,0}$ denote the minimal operators associated with ℓ and v_j, respectively.

In Everitt and Giertz [6, Theorem 1], the separation inequality

$$\int_a^\infty |(py')'|^2 + C_1 \int_a^\infty pq\,|y'|^2 + C_2 \int_a^\infty q^2|y|^2 \le \int_a^\infty |L_0[y]|^2, \quad (3.7)$$

valid for all $y \in D(L_0)$ and some positive constants C_1 and C_2, is established. Their proof uses (3.3) and shows that $C_1 = 1 + w$ and $C_2 = w$ where $w = 1 - A_2$. Thus we have for all $y \in D(L_0)$ that

$$\int_a^\infty [(py')']^2 + (1+\omega) \int_a^\infty pq\,(y')^2 + \omega \int_a^\infty q^2y^2 \le \int_a^\infty L_0[y]^2. \quad (3.8)$$

To make use of (3.8) in subsequent calculations, we will estimate $\int_a^\infty p^2|y''|^2$ in terms of $\int_a^\infty q^2|y|^2$ and $\int_a^\infty |(py')'|^2$, where $y \in D(L_0)$. Note that $(py')' = py'' + p'y'$, and so $\int_a^\infty p^2|y''|^2 = \int_a^\infty |(py')' - p'y'|^2 \le 2\left[\int_a^\infty |(py')'|^2 + \int_a^\infty |p'y'|^2\right]$ by the inequality $|\alpha - \beta|^2 \le 2(|\alpha|^2 + |\beta|^2)$. Use of (3.2) gives

$$\int_a^\infty p^2|y''|^2 \le 2\int_a^\infty |(py')'|^2 + 2A_1^2 \int_a^\infty pq\,|y'|^2. \quad (3.9)$$

Next we will estimate the last integral in terms of $\int_a^\infty q^2|y|^2$ and the integral on the left. We apply Theorem A with $N = pq$, $W = q^2$, $P = p^2$, $j = 1$, and ε_0 and $f = f(t)$ to be chosen below. By the definitions of S_1 and S_2 in Theorem A,

$$S_1(\varepsilon) = \sup_{t\in I}\left\{f^2 \frac{1}{\varepsilon f}\left(\int_t^{t+\varepsilon f} \frac{1}{p^2}\right) \frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} pq\right\}$$

and

$$S_2(\varepsilon) = \sup_{t\in I}\left\{\frac{1}{f^2} \frac{1}{\varepsilon f}\left(\int_t^{t+\varepsilon f} \frac{1}{q^2}\right) \frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} pq\right\}.$$

Basic estimates are obtained from the following lemma in [4, pp. 575–576].

LEMMA 3.1 *Let s and w be positive, $AC_{\mathrm{loc}}(I)$ functions such that $|s'(t)| \leq N_0$ and $|s(t)w'(t)| \leq M_0 w(t)$ a.e. on I for some constants N_0 and M_0. Then for fixed $t \in I$, $0 < \varepsilon < \frac{1}{N_0}$, and $t \leq \tau \leq t + \varepsilon s(t)$, we have that*

$$(1 - \varepsilon N_0)s(t) \leq s(\tau) \leq (1 + \varepsilon N_0)s(t)$$

and

$$\exp\left(-\frac{M_0}{N_0}\right) w(t) \leq w(\tau) \leq \exp\left(\frac{M_0}{N_0}\right) w(t).$$

Note that if p and q are constant functions, then the choice $f = \sqrt{p/q}$ implies that $S_1 = S_2 \equiv 1$. Of course, p and q need not be constant functions. However, this choice of f and Lemma 3.1 (with $s = f$ and $w = p$ or q) imply that $p(\tau)$ and $q(\tau)$ are nearly constant for $t \leq \tau \leq t + \varepsilon f$ and ε sufficiently small. Thus for $f = \sqrt{p/q}$, we have

$$|f'| = \left|\frac{1}{2}p^{-1/2}p'q^{-1/2} - \frac{1}{2}p^{1/2}q^{-3/2}q'\right| \leq N_0$$

by (3.2) and (3.3), where $N_0 = (A_1 + A_2)/2$. Also,

$$\left|\frac{fp'}{p}\right| = \frac{|p'|}{\sqrt{pq}} \leq A_1$$

and

$$\left|\frac{fq'}{q}\right| = \frac{\sqrt{p}|q'|}{q^{3/2}} \leq A_2.$$

Let $0 < \varepsilon < \varepsilon_0$, where $\varepsilon = 1/N_0 = 2/(A_1 + A_2)$. Then by Lemma 3.1, we have for fixed $t \in I$ and $t \leq \tau \leq t + \varepsilon f(t)$,

$$(1 - \varepsilon N_0)f(t) \leq f(\tau) \leq (1 + \varepsilon N_0)f(t), \tag{3.10}$$

$$\exp\left(\frac{-A_1}{N_0}\right) p(t) \leq p(\tau) \leq \exp\left(\frac{A_1}{N_0}\right) p(t), \tag{3.11}$$

and

$$\exp\left(\frac{-A_2}{N_0}\right) q(t) \leq q(\tau) \leq \exp\left(\frac{A_2}{N_0}\right) q(t). \tag{3.12}$$

It follows that $S_1(\varepsilon) \leq C_1$ and $S_2(\varepsilon) \leq C_2$ for some positive constants C_1 and C_2. By Theorem A, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a constant $C > 0$ such that

$$\int_a^\infty pq\,|y'|^2 \leq C\int_a^\infty q^2|y|^2 + \varepsilon\int_a^\infty p^2|y''|^2$$

for all $y \in D(L)$ and hence for all $y \in D(L_0)$. Use of this estimate in (3.9) gives $(1 - 2A_1^2\varepsilon)\int_a^\infty p^2|y''|^2 \le 2C A_1^2 \int_a^\infty q^2|y|^2 + 2\int_a^\infty |(py')'|^2$ for all $y \in D(L_0)$. Choose $\varepsilon > 0$ such that $\varepsilon < 1/2A_1^2$. Then there exist positive constants K_1 and K_2 such that

$$\int_a^\infty p^2|y''|^2 \le K_1 \int_a^\infty q^2|y|^2 + K_2 \int_a^\infty |(py')'|^2 \tag{3.13}$$

for all $y \in D(L_0)$.

(i) *Sufficiency*. Suppose (3.5) holds for some $\delta = (0, 1/(A_1 + A_2))$. Fix $j \in \{0, 1\}$. To prove that $B_{j,0}$ is L_0-bounded, we will employ Theorem A again. To be specific, choose $N = |b_j|^2$, $W = q^2$, $P = p^2$, $\varepsilon_0 = \delta$, and $f = \sqrt{p/q}$. By (3.11) and (3.12), we have for $0 < \varepsilon < \delta$, using $f = f(t)$, $p = p(t)$, etc, that

$$\begin{aligned} S_1(\varepsilon) &\le \widetilde{C} \sup_{t\in I} \left\{ f^{2(2-j)} \frac{1}{p^2} \frac{1}{\varepsilon f} \int_t^{t+\varepsilon f} |b_j|^2 \right\} \\ &\le C \sup_{t\in I} \left\{ \frac{f^{2(2-j)}}{p^2} \frac{p^j q^{2-j}}{\varepsilon} \frac{1}{f} \int_t^{t+\varepsilon f} \frac{|b_j(\tau)|^2}{p(\tau)^j q(\tau)^{2-j}}\, d\tau \right\} \end{aligned}$$

for some constants $\widetilde{C}$ and C. By the definition (3.4) of $g_{j,\delta}$ and the choice of f, for all $\varepsilon \in (0, \delta)$,

$$S_1(\varepsilon) \le \frac{C}{\varepsilon} \sup_{t\in I} g_{j,\delta}(t) \le \frac{C_1}{\varepsilon} \tag{3.14}$$

where the last inequality holds by (3.5) for some constant $C_1 > 0$. Similarly,

$$S_2(\varepsilon) \le C \sup_{t\in I} \left\{ \frac{1}{f^{2j}} \frac{1}{q^2} \frac{p^j q^{2-j}}{\varepsilon} g_{j,\delta}(t) \right\} = \frac{C}{\varepsilon} \sup_{t\in I} g_{j,\delta}(t) \le \frac{C_2}{\varepsilon} \tag{3.15}$$

for some constant $C_2 > 0$ and all $\varepsilon \in (0, \delta)$. By Theorem A, there exists a constant $K > 0$ such that for all $\varepsilon \in (0, \delta)$ and $y \in D(L_0)$,

$$\int_a^\infty |b_j y^{(j)}|^2 \le K \left\{ C_2 \varepsilon^{-2j-1} \int_a^\infty q^2|y|^2 + C_1 \varepsilon^{3-2j} \int_a^\infty p^2|y''|^2 \right\} \tag{3.16}$$

where we have used (3.14) and (3.15). Since $j = 0$ or 1, $3 - 2j > 0$ and so the coefficient of the last integral in (3.16) can be made arbitrarily small by choosing $\varepsilon \in (0, \delta)$ sufficiently small. This observation and (3.13) imply that for any $\varepsilon_1 \in (0, \delta)$, there exists a constant $M > 0$ such that

$$\int_a^\infty |b_j y^{(j)}|^2 \le M \int_a^\infty q^2|y|^2 + \varepsilon_1 \int_a^\infty |(py')'|^2$$

for all $y \in D(L_0)$. It follows from this estimate and the separation inequality (3.8) that $B_{j,0}$ is L_0-bounded.

Necessity. For the proof of necessity of (3.5), we work with maximal operators. Suppose B_j is L-bounded. Fix $j \in \{0, 1\}$ and $\delta \in (0, 1/(A_1 + A_2))$. Let ϕ be a function in $C_0^\infty(\mathcal{R})$ such that $\phi \equiv 1$ on $[0, 1]$ and support $(\phi) = [-2, 2]$. Define for $t \geq a$

$$h_0(t) = \phi(t), \quad h_1(t) = t\phi(t). \tag{3.17}$$

Then $h_j \in C_0^\infty(\mathcal{R})$ and $h_j^{(j)}(t) = 1$ on $[0, 1]$ for $j = 0, 1$. For each $r \geq a$, define

$$h_{j,r}(t) = \delta^j f(r)^j h_j(u), \qquad t \geq a, \tag{3.18}$$

where

$$f = \sqrt{p/q}, \qquad u = \frac{t-r}{\delta f(r)}.$$

Then

$$h_{j,r}^{(j)}(t) = 1, \qquad r \leq t \leq r + \delta f(r), \tag{3.19}$$

and

$$\text{support}(h_{j,r}) = [r - 2\delta f(r), r + 2\delta f(r)]. \tag{3.20}$$

By the definition of B_j, $(B_j h_{j,r})(t) = b_j(t) h_{j,r}^{(j)}(t)$, $t \geq a$, and so

$$b_j = B_j h_{j,r} \quad \text{on } [r, r + \delta f(r)].$$

Thus for $r \geq a$, we have (by the definition (3.4))

$$\begin{aligned} g_{j,\delta} &= \sqrt{\frac{q(r)}{p(r)}} \int_r^{r+\delta\sqrt{\frac{p(r)}{q(r)}}} \frac{|b_j|^2}{p^j q^{2-j}} = \frac{1}{f(r)} \int_r^{r+\delta f(r)} \frac{|B_j h_{j,r}|^2}{p^j q^{2-j}} \\ &\leq \frac{1}{f(r)} \frac{C}{p(r)^j q(r)^{2-j}} \int_r^{r+\delta f(r)} |B_j h_{j,r}|^2 \end{aligned}$$

for some constant $C > 0$, where we have used (3.11) and (3.12). Now, by the definition of $f = \sqrt{p/q}$,

$$g_{j,\delta}(r) \leq \frac{C}{f(r)^{2j+1} q(r)^2} \int_a^\infty |B_j h_{j,r}|^2 = \frac{C}{f(r)^{2j+1} q(r)^2} \|B_j h_{j,r}\|^2 \tag{3.21}$$

for $r \geq a$. By the hypothesis that B_j is L-bounded, we obtain

$$\sqrt{g_{j,\delta}(r)} \leq \frac{C}{f(r)^{j+1/2} q(r)} \left(\|h_{j,r}\| + \|L h_{j,r}\|\right) \tag{3.22}$$

for $r \geq a$ and a different constant C. Estimates for the terms in the graph norm of $h_{j,r}$ are given in the following lemma.

LEMMA 3.2 *Let p and q be positive, $AC_{\mathrm{loc}}[a,\infty)$ functions such that (3.2) and (3.3) hold. Let $f=\sqrt{p/q}$. For each $r\geq a$, define $h_{j,r}$ as in (3.18). Then there exist positive constants C_1, C_2, C_3, and C_4 such that for $r\geq a$ and $j=0$ or 1,*

$$\|h_{j,r}\|\leq C_1 f(r)^{j+1/2}, \tag{3.23}$$

$$\|qh_{j,r}\|\leq C_2 f(r)^{j+1/2}q(r), \tag{3.24}$$

$$\|(ph'_{j,r})'\|\leq C_3 f(r)^{j+1/2}q(r), \tag{3.25}$$

and

$$\|Lh_{j,r}\|\leq C_4 f(r)^{j+1/2}q(r). \tag{3.26}$$

Proof of Lemma 3.2 Fix $r\geq a$, and $j\in\{0,1\}$. By (3.20) and the change of variable $u=(t-r)/\delta f(r)$,

$$\begin{aligned}\|h_{j,r}\|^2&=\int_a^\infty |h_{j,r}(t)|^2dt=\int_{r-2\delta f(r)}^{r+2\delta f(r)}\delta^{2j}f(r)^{2j}|h_j(u)|^2dt\\&=\delta^{2j}f(r)^{2j}\int_{-2}^{2}|h_j(u)|^2\delta f(r)du=Cf(r)^{2j+1}\end{aligned}$$

for some positive constant C which is independent of r. (Note that $\int_{-2}^2|h_j(u)|^2du$ is finite since h_j is continuous on $\mathcal{R}$.) This establishes (3.23).

Next we use (3.11) and (3.12) to estimate

$$\begin{aligned}\|qh_{j,r}\|^2&=\int_a^\infty q(t)^2|h_{j,r}(t)|^2dt=\int_{r-2\delta f(r)}^{r+2\delta f(r)}q(t)^2\delta^{2j}f(r)^{2j}|h_j(u)|^2dt\\&\leq Cf(r)^{2j}q(r)^2\int_{r-2\delta f(r)}^{r+2\delta f(r)}|h_j(u)|^2dt\\&=Cf(r)^{2j}q(r)^2\int_{-2}^{2}|h_j(u)|^2\delta f(r)du=\widetilde{C}f(r)^{2j+1}q(r)^2\end{aligned}$$

for some positive constants C and $\widetilde{C}$ independent of r. So (3.24) holds.

Since $(\alpha+\beta)^2\leq 2(\alpha^2+\beta^2)$,

$$\begin{aligned}\|(ph'_{j,r})'\|^2&=\|ph''_{j,r}+p'h'_{j,r}\|^2\leq 2\|ph''_{j,r}\|^2+2\|p'h'_{j,r}\|^2\\&\leq 2\int_{r-2\delta f(r)}^{r+2\delta f(r)}p(t)^2|h''_{j,r}(t)|^2dt\\&\quad+2\int_{r-2\delta f(r)}^{r+2\delta f(r)}A_1^2p(t)q(t)|h'_{j,r}(t)|^2dt\end{aligned}$$

by (3.2). Note that $h'_{j,r} = \delta^j f(r)^j h'_j(u)\frac{du}{dt} = \delta^{j-1} f(r)^{j-1} h'_j(u)$ and $h''_{j,r}(t) = \delta^{j-2} f(r)^{j-2} h''_j(u)$. By (3.10), (3.11), and (3.12), we obtain (for some constants K_1 and K_2 independent of r)

$$\begin{aligned}
\|(ph'_{j,r})'\|^2 &\le K_1 p(r)^2 f(r)^{2j-4} \int_{r-2\delta f(r)}^{r+2\delta f(r)} |h''_j(u)|^2 dt \\
&\quad + K_2 p(r) q(r) f(r)^{2j-2} \int_{r-2\delta f(r)}^{r+2\delta f(r)} |h'_j(u)|^2 dt \\
&= K_2 \delta p(r)^2 f(r)^{2j-3} \int_{-2}^{2} |h''_j(u)|^2 du \\
&\quad + K_2 \delta p(r) q(r) f(r)^{2j-1} \int_{-2}^{2} |h'_j(u)|^2 du \\
&\le C[p(r)^2 f(r)^{2j-3} + p(r) q(r) f(r)^{2j-1}]
\end{aligned}$$

where C is a constant independent of r. Since $f = \sqrt{p/q}$, we find that $\|(ph'_{j,r})'\|^2 \le 2Cf(r)^{2j+1} q(r)^2$. Hence (3.25) holds.

By the action of L and an inequality used earlier,

$$\|Lh_{j,r}\|^2 = \| -(ph'_{j,r})' + qh_{j,r}\|^2 \le 2\|(ph'_{j,r})'\|^2 + 2\|qh_{j,r}\|^2.$$

By (3.24) and (3.25), this implies that $\|Lh_{j,r}\|^2 \le 2(C_3^2 + C_2^2) f(r)^{2j+1} q(r)^2$. This establishes (3.26) and completes the proof of Lemma 3.2. □

Returning to the proof of Theorem 3.1, use of (3.23) and (3.26) in (3.22) yields

$$\sqrt{g_{j,\delta}(r)} \le \frac{CC_1}{q(r)} + CC_4 \tag{3.27}$$

for all $r \ge a$. By hypothesis, q is bounded away from 0. Hence $g_{j,\delta}$ is bounded above on $[a, \infty)$. Therefore, (3.5) holds for any $\delta \in (0, 1/(A_1 + A_2))$. This establishes the necessity of (3.5) for L-boundedness of B_j.

(ii) *Sufficiency*. Suppose (3.6) holds for some $\delta \in (0, 1/(A_1 + A_2))$. It suffices to show that $B_{j,0}$ ($j = 0, 1$) is L_0-compact (by Theorem 2.2). Fix $j \in \{0, 1\}$. For $y \in D(L_0)$ and each positive integer $N > a$, define

$$R_{j,N}[y] = \begin{cases} B_{j,0}[y] = b_j y^{(j)} & \text{on } [a, N], \\ 0 & \text{on } [N, \infty]. \end{cases}$$

Set

$$\psi_j(N) = \sup_{N \le t < \infty} g_{j,\delta}(t).$$

By hypothesis,

$$\lim_{N\to\infty} \psi_j(N) = 0$$

and (3.5) holds. Hence $B_{j,0}$ is L_0-bounded by (i). So $D(L_0) \subset D(B_{j,0})$. For $y \in D(L_0)$,

$$\|B_{j,0}[y] - R_{j,N}[y]\| = \left\{ \int_N^\infty |b_j y^{(j)}|^2 \right\}^{1/2} \tag{3.28}$$

by the definition of $R_{j,N}$.

Next we apply Theorem A on the interval $[N, \infty]$ in the same manner used to derive (3.16). As in the proofs of (3.14) and (3.15), we find that

$$S_1(\varepsilon) \le C_1 \sup_{N\le t<\infty} g_{j,\delta}(t) = C_1 \psi_j(N)$$

and

$$S_2(\varepsilon) \le C_2 \psi_j(N)$$

for some positive constants C_1 and C_2 and all $\varepsilon \in (0, \delta)$. It follows that there exists a constant $K > 0$ such that for all $\varepsilon \in (0, \delta)$ and $y \in D(L_0)$,

$$\begin{aligned}\int_N^\infty |b_j y^{(j)}|^2 &\le K \left\{ \varepsilon^{-2j} C_2 \psi_j(N) \int_a^\infty q^2 |y|^2 \right. \\ &\qquad \left. + \varepsilon^{2(2-j)} C_1 \psi_j(N) \int_a^\infty p^2 |y''|^2 \right\} \\ &\le C\psi_j(N) \left\{ \int_a^\infty q^2|y|^2 + \int_a^\infty |(py')'|^2 \right\}\end{aligned}$$

for some constant $C > 0$, where the last inequality holds by (3.13). Now by the separation inequality (3.8) and the definition of the graph norm $\|\cdot\|_L$, we have

$$\int_N^\infty |b_j y^{(j)}|^2 \le C\psi_j(N) \int_a^\infty |L_0[y]|^2 \le C\psi_j(N) \|y\|_L^2$$

for some constant $C > 0$ (independent of N) and all $y \in D(L_0)$. Combining this inequality with (3.28) yields

$$\sup_{\substack{y\in D(L_0)\\ y\neq 0}} \frac{\|B_{j,0}[y] - R_{j,N}[y]\|}{\|y\|_L} \le C\sqrt{\psi_j(N)} \to 0$$

as $N \to \infty$. Therefore, $R_{j,N} \to B_{j,0}$ as $N \to \infty$ with respect to $\|\cdot\|_L$.

Next we show that each $R_{j,N}$ is L_0-compact. Because the argument is standard, we give only a brief sketch here. Let $\{y_k\}$ be an L_0-bounded sequence. Clearly, $\{y_k\}$ is bounded in $L^2(a, N)$. An application of Theorem A yields that $\{y_k\}$ and $\{y_k'\}$ are equicontinuous on $[a, N]$. By the Arzela-Ascoli Theorem, $\{y_k\}$ has a subsequence $\{y_{k_m}\}$ such that $\{y_{k_m}\}$ and $\{y'_{k_m}\}$ converge uniformly on $[a, N]$. This fact, combined with the hypothesis that $b_j \in L^2_{\text{loc}}(a, \infty)$, implies that $\{R_{j,N} y_{k_m}\}$ is a Cauchy sequence in $L^2(a, N)$. Since $\{y_k\}$ was an arbitrary L_0-bounded sequence, each $R_{j,N}$ is L_0-compact. So $B_{j,0}$ is the uniform limit of L_0-compact operators. Therefore, $B_{j,0}$ is L_0-compact.

Necessity. Fix $j \in \{0, 1\}$. Suppose $B_{j,0}$ is L_0-compact and that (3.6) does not hold for any $\delta \in (0, 1/(A_1 + A_2))$. We show that this leads to a contradiction. By hypothesis, for any $\delta \in (0, 1/(A_1 + A_2))$, there exists $\varepsilon > 0$ and a sequence $\{r_\ell\} \subset \mathcal{R}$ such that $r_\ell \to \infty$ and

$$g_{j,\delta}(r_\ell) \geq \varepsilon$$

for all $\ell \geq 1$. Fix $\delta \in (0, 1/(A_1 + A_2))$. Define $h_{j,r}$ as in (3.18). Then by (3.21), we have that for some constant $C > 0$,

$$g_{j,\delta}(r_\ell) \leq \frac{C}{f(r_\ell)^{2j+1} q(r_\ell)^2} \|B_{j,0} h_{j,r_\ell}\|^2$$

for all $\ell \geq 1$. For $r, t \in [a, \infty)$, define

$$\sigma_{j,r}(t) = \frac{1}{f(r)^{j+1/2} g(r)} h_{j,r}(t).$$

Then for all $\ell \geq 1$,

$$\varepsilon \leq C \|B_{j,0} \sigma_{j,r_\ell}\|^2 \tag{3.29}$$

and

$$\|\sigma_{j,r_\ell}\|_L^2 = \frac{1}{f(r_\ell)^{2j+1} q(r_\ell)^2} \|h_{j,r_\ell}\|_L^2 \leq M$$

for some constant $M > 0$ by Lemma 3.2 and (3.1). (See (3.22) and (3.27).) Thus the sequence $\{\sigma_{j,r_\ell}\}$ is L_0-bounded. Since $B_{j,0}$ is L_0-compact, $\{B_{j,0}\sigma_{j,r_\ell}\}$ has a subsequence converging to some $y_0 \in L^2(a, \infty)$. Since the supports of the $\sigma_{j,r}$ tend to infinity as $r \to \infty$, this implies $y_0 = 0$ thereby contradicting (3.29). □

Everitt and Giertz consider the operators R and T_0 defined by

$$R[y] = ir(ry)' \qquad (y \in D(R))$$

and

$$T_0[y] = -(py')' + qy \qquad (y \in D(T_0))$$

with $D(R) = \{y \in L^2(a, \infty) : y = AC_{\mathrm{loc}}[a, \infty), R[y] \in L^2(a, \infty), y(a) = 0, ry \in L^2(a, \infty)\}$ and $D(T_0) = \{y \in L^2(a, \infty) : y' \in AC_{\mathrm{loc}}[a, \infty), T_0[y] \in L^2(a, \infty), y(a) = 0\}$. They prove that the pointwise conditions

$$|r(t)|^4 \leq C_1 p(t)q(t), \quad |r(t)|^2 \leq C_2 q(t), \quad |r'(t)|^2 \leq C_3 q(t)$$

(for some positive constants C_1, C_2, and C_3 and $t \in [a, \infty)$) are sufficient for R to be T_0-bounded. By identifying $b_0 = i\,r\,r'$ and $b_1 = i\,r^2$, we find that their result is a special case of Theorem 3.1. Furthermore, Theorem 3.1 generalizes the sufficient pointwise conditions of Everitt and Giertz to integral average conditions which are necessary and sufficient for relative boundedness of B_j with respect to L.

Example 3.1 Let $p(t) = t^\alpha$, $q(t) = Kt^\beta$, and $a = 1$, where α, β, and K are constants with $K > 0$. Then (3.2) and (3.3) are each equivalent to $\alpha \leq \beta + 2$. Assume that this relationship between α and β is satisfied. Also, assume that $K > \beta^2$. Then $A_2 < 1$, where A_2 is the constant in (3.3). From (3.4), we have (up to multiplicative constants)

$$g_{0,\delta}(t) = t^{(\beta-\alpha)/2} \int_t^{t+\delta t^{(\alpha-\beta)/2}} \frac{|b_0(\tau)|^2}{\tau^{2\beta}}\, d\tau$$

and

$$g_{1,\delta}(t) = t^{(\beta-\alpha)/2} \int_t^{t+\delta t^{(\alpha-\beta)/2}} \frac{|b_1(\tau)|^2}{\tau^{\alpha+\beta}}\, d\tau.$$

For example, by Theorem 3.1, B_0, B_1 are L-bounded if $|b_0(t)| \leq C_0 t^\beta$ and $|b_1(t)| \leq C_1 t^{(\alpha+\beta)/2}$ for some positive constants C_0 and C_1 and $t \in [1, \infty)$. $\square$

Example 3.2 Let $p(t) = e^{\alpha t}$ and $q(t) = Ke^{\beta t}$ with $\alpha \leq \beta$ and $K > \beta^2 e^{(\alpha-\beta)a}$. Then (3.2) and (3.3) hold with $A_2 < 1$. By definition,

$$g_{0,\delta}(t) = e^{(\beta-\alpha)t/2} \int_t^{t+\delta e^{(\alpha-\beta)t/2}} \frac{|b_0(\tau)|^2}{e^{2\beta\tau}}\, d\tau$$

and

$$g_{1,\delta}(t) = e^{(\beta-\alpha)t/2} \int_t^{t+\delta e^{(\alpha-\beta)t/2}} \frac{|b_1(\tau)|^2}{e^{(\alpha+\beta)\tau}}\, d\tau$$

(up to multiplicative constants). For example, by Theorem 3.1, B_0, B_1 are L-compact if $|b_0(t)| \leq C_0 e^{(\beta-\varepsilon)t}$ and $|b_1(t)| \leq C_1 e^{(\alpha+\beta-\varepsilon)t}$ for some positive constants C_0, C_1, and ε and all $t \in [a, \infty)$. □

4 A p DOMINANT CASE WITH p LARGE

In the next theorem, we consider the situation in which $p(t) = t^\alpha$ and $|q(t)| \leq Mt^{\alpha-2}$ for some constants $\alpha \geq 2$ and $M > 0$, $t \in [a, \infty)$.

THEOREM 4.1 *Let $I = [a, \infty)$. Let L, B_j be the maximal operators associated with the differential expressions*

$$\ell[y] = -(t^\alpha y')' + qy \tag{4.1}$$

and

$$v_j[y] = b_j y^{(j)} \qquad (j = 0, 1), \tag{4.2}$$

respectively, where $a > 0$, $\alpha \geq 2$, q is a real-valued $L_{\text{loc}}(I)$ function such that

$$|q(t)| \leq Mt^{\alpha-2} \qquad (a \leq t < \infty) \tag{4.3}$$

for some sufficiently small positive constant M, and each $b_j \in L_{\text{loc}}(I)$. For $j = 0, 1$ and $\delta > 0$, define

$$g_{j,\delta}(t) = \frac{1}{t} \int_t^{t+\delta t} \frac{|b_j(\tau)|^2}{\tau^{2(\alpha+j-2)}}\, d\tau.$$

Then the following hold for $j = 0, 1$.

(i) B_j is L-bounded if and only if $b_j \in L^2_{\text{loc}}(I)$ and

$$\sup_{a \leq t < \infty} g_{j,\delta}(t) < \infty \tag{4.4}$$

for some $\delta \in (0, 1/2)$.

(ii) B_j is L-compact if and only if $b_j \in L^2_{\text{loc}}(I)$ and

$$\lim_{t \to \infty} g_{j,\delta}(t) = 0 \tag{4.5}$$

for some $\delta \in (0, 1/2)$.

Proof We first consider the case $q = 0$. Then ℓ is a one-term operator which is limit point ($\ell[y] = 0$ has the non $L^2(a, \infty)$ soltuion $y \equiv 1$). Thus by Theorem 2.2 it suffices to prove the result for the minimal operators L_0 and $B_{j,0}$ associated, respectively with ℓ and v_j.

Let $y \in C_0^\infty(a, \infty)$ be real valued. Since $(t^\alpha y')' = t^\alpha y'' + \alpha t^{\alpha-1} y'$, we have after an integration by parts that

$$\begin{aligned}\int_a^\infty [(t^\alpha y')']^2 &= \int_a^\infty [t^{2\alpha}(y'')^2 + 2\alpha t^{2\alpha-1} y'y'' + \alpha^2 t^{2\alpha-2}(y')^2] \\ &= \int_a^\infty t^{2\alpha}(y'')^2 + (\alpha - \alpha^2)\int_a^\infty t^{2\alpha-2}(y')^2\end{aligned} \tag{4.6}$$

from which we conclude that

$$\int_a^\infty [(t^\alpha y')']^2 \le \int_a^\infty t^{2\alpha}(y'')^2 \tag{4.7}$$

since $\alpha > 1$. On the other hand, the Hardy inequality ([8, pp. 245–246]) gives that for $\gamma \neq -1$,

$$\int_a^b t^\gamma w(t)^2 dt \le \frac{4}{(\gamma+1)^2}\int_a^b t^{\gamma+2} w'(t)^2 dt \tag{4.8}$$

for all $w \in AC_{\text{loc}}[a, b]$ such that $w(a) = w(b) = 0$. Applying (4.8) to $w = y'$ and $\gamma = 2\alpha - 2$ gives

$$\int_a^\infty t^{2\alpha-2}(y')^2 \le \frac{4}{(2\alpha-1)^2}\int_a^\infty t^{2\alpha}(y'')^2 \tag{4.9}$$

Substitution of (4.9) into (4.6) and simplifying yields that

$$\int_a^\infty t^{2\alpha}(y'')^2 \le (2\alpha-1)^2 \int_a^\infty [(t^\alpha y')']^2. \tag{4.10}$$

Again employing the lemma of Everitt and Giertz [6, p. 313], we have that (4.10) holds for all $y \in D(L_0)$.

(i) *Sufficiency*. Suppose (4.4) holds for some $\delta \in (0, 1/2)$. Fix $j \in \{0, 1\}$. We will apply Theorem A with $N = |b_j|^2$, $W = t^{2\alpha-4}$, $P = t^{2\alpha}$, $\varepsilon_0 = \delta$, and $f(t) = t$. For $0 < \varepsilon < \delta$, we have (by Lemma 3.1) for some constant C_1,

$$S_1(\varepsilon) \le C_1 \sup_{a \le t < \infty}\left\{t^{2(2-j)} t^{-2\alpha} t^{2(\alpha+j-2)} \frac{1}{\varepsilon} g_{j,\delta}(t)\right\} = \frac{C_2}{\varepsilon}\sup_{a \le t < \infty}\{g_{j,\delta}(t)\} < \infty$$

by (4.4). Similarly, for $0 < \varepsilon < \delta$ and some constant C_2,

$$S_2(\varepsilon) \le C_2 \sup_{a \le t < \infty} \left\{ t^{-2j} \frac{1}{t^{2\alpha-4}} t^{2(\alpha+j-2)} \frac{1}{\varepsilon} g_{j,\delta}(t) \right\} = \frac{C_2}{\varepsilon} \sup_{a \le t < \infty} \{g_{j,\delta}(t)\} < \infty.$$

Hence by Theorem A, there exists $K > 0$ such that for all $\varepsilon \in (0, \delta)$ and $y \in C_0^\infty(a, \infty)$, $\int_a^\infty |b_j y^{(j)}|^2 \le K\{\varepsilon^{-2j-1} \int_a^\infty t^{2\alpha-4}|y|^2 + \varepsilon^{3-2j} \int_a^\infty t^{2\alpha}|y''|^2\}$ for $j = 0, 1$. Applying the Hardy-type inequality (4.8) twice to the middle integral produces $\int_a^\infty t^{2\alpha-4}|y|^2 \le \frac{16}{(2\alpha-3)^2(2\alpha-1)^2} \int_a^\infty t^{2\alpha}|y''|^2$ for $y \in C_0^\infty(a, \infty)$. Thus for a different constant $K > 0$,

$$\int_a^\infty |b_j y^{(j)}|^2 \le K \int_a^\infty t^{2\alpha}|y''|^2$$

for all $y \in C_0^\infty(a, \infty)$. By (4.10), we have for all $y \in C_0^\infty(a, \infty)$ that,

$$\|B_{j,0}y\|^2 = \int_a^\infty |b_j y^{(j)}|^2 \le K(2\alpha-1)^2 \int_a^\infty |(t^\alpha y')'|^2 = K(2\alpha-1)\|L_0 y\|^2. \tag{4.11}$$

To extend (4.11) to functions in $D(L_0)$, let $u \in D(L_0)$. Then there exists a sequence $\{u_n\}$ of $C_0^\infty(a, \infty)$ functions such that $u_n \to u$ and $L_0 u_n \to L_0 u$. Replacing y in (4.11) by $u_n - u_m$, we see that $\{B_{j,0}u_n\}$ is a Cauchy sequence in $L^2(a, \infty)$. Since $B_{j,0}$ is a closed operator, this implies that $B_{j,0}u_n \to B_{j,0}u$. Hence, by replacing y in (4.11) by u_n and letting $n \to \infty$, we have established that (4.11) holds for functions in $D(L_0)$. Therefore, $B_{j,0}$ is L_0-bounded.

Necessity. As in the proof of necessity in Theorem 3.1 (i), we will work directly with maximal operators. Fix $j \in \{0, 1\}$ and $\delta \in (0, 1/2)$. Suppose B_j is L-bounded. For each $r \ge a$, define the function $h_{j,r}$ as in (3.18) with the exception that f is replaced by $f(r) = r$. Then

$$h_{j,r}(t) = \delta^j r^j h_j(u), \qquad t \ge a, \tag{4.12}$$

where $u = (t - r)/\delta r$ and h_j is defined by (3.17). Also, (3.19), (3.20) hold for the choice $f(r) = r$.

By the definition of $g_{j,\delta}$,

$$\begin{aligned} g_{j,\delta}(r) &= \frac{1}{r}\int_r^{r+\delta r} \frac{|b_j(\tau)|^2}{\tau^{2(\alpha+j-2)}}\,d\tau = \frac{1}{r}\int_r^{r+\delta r} \frac{|B_j h_{j,r}(\tau)|^2}{\tau^{2(\alpha+j-2)}}\,d\tau \\ &\le \frac{1}{r^{2\alpha+2j-3}} \int_r^{r+\delta r} |B_j h_{j,r}(\tau)|^2 d\tau. \end{aligned}$$

Now, after replacing the interval of integration by the larger interval $[a, \infty)$, we obtain $g_{j,\delta}(r) \le \frac{1}{r^{2\alpha+2j-3}} \|B_j h_{j,r}\|^2$. Since B_j is L-bounded, there exists a positive constant C such that for all $r \ge a$,

$$\sqrt{g_{j,\delta}(r)} \le \frac{C}{r^{\alpha+j-3/2}} \left(\|h_{j,r}\| + \|Lh_{j,r}\|\right). \tag{4.13}$$

Next we state and prove the analog of Lemma 3.2 for the present setting.

LEMMA 4.1 *For $j = 0, 1$ define $h_{j,r}$ by (4.12). Let $q = 0$ in (4.1). Then there exist positive constants C_1 and C_2 such that for $r \ge a$ and $j = 0$ or 1,*

$$\|h_{j,r}\| \le C_1 r^{j+1/2}, \tag{4.14}$$

and

$$\|Lh_{j,r}\| = \|(t^\alpha h'_{j,r})'\| \le C_2 r^{\alpha+j-3/2}. \tag{4.15}$$

Proof of Lemma 4.1 Fix $r \ge a$ and $j \in \{0, 1\}$. Using the compact support of $h_{j,r}$, we have

$$\|h_{j,r}\|^2 = \int_{r-2\delta r}^{r+2\delta r} \delta^{2j} r^{2j} |h_j(u)|^2 dt = \delta^{2j} r^{2j} \int_{-2}^{2} |h_j(u)|^2 \delta r\, du = Cr^{2j+1}$$

for some positive constant C independent of r. (Recall that h_j, defined by (3.17), belongs to $C_0^\infty(a, \infty)$ and does not depend on r.) So (4.14) holds.

Next we estimate, using (4.7), that

$$\begin{aligned}
\|(t^\alpha h'_{j,r})'\|^2 &\le \int_{r-2\delta r}^{r+2\delta r} t^{2\alpha} |h''_{j,r}(t)|^2 dt \\
&\le \widetilde{C} r^{2\alpha} \int_{r-2\delta r}^{r+2\delta r} \left| \delta^j r^j h''_j(u) \frac{1}{\delta^2 r^2} \right|^2 dt \\
&= K r^{2\alpha+2j-4} \int_{-2}^{2} |h''_j(u)|^2 \delta r\, du = \widetilde{K} r^{2\alpha+2j-3}
\end{aligned}$$

for some positive constants C, $\widetilde{C}$, K, and $\widetilde{K}$ independent of r. This establishes (4.15) and Lemma 4.1. □

We now continue with the proof of Theorem 4.1. By combining (4.14) and (4.15) with (4.13), we find that $\sqrt{g_{j,\delta}(r)} \leq \frac{K_1}{r^{\alpha-2}} + K_2$ for some positive constants K_1 and K_2 and all $r \geq a$. Since $\alpha \geq 2$, the right side can be bounded above independently of r. Therefore, (4.4) holds for any $\delta \in (0, 1/2)$. Thus (4.4) is necessary for the L-boundedness of B_j.

(ii) The proof that (4.5) is necessary and sufficient for the L-compactness of B_j is essentially the same as the proof of Theorem 3.1 (ii) and is therefore omitted. This completes the proof of Theorem 4.1. □

To allow for a q term we note that if q satisfies (4.3), then it is a b_0 pertubation term satisfying (4.4) and is thus an L-bounded perturbation of $(t^\alpha y')'$. Further the above proof shows from (4.11) that the relative bound is proportional to M. Therefore the relative bound can be made less than one by taking M sufficiently small. This fact, together with the following general result, completes the proof of Theorem 4.1: If B is a relatively bounded perturbation of A with relative bound $b < 1$, i.e., $D(A) \subset D(B)$ and $\|Bx\| \leq a\|x\| + b\|Ax\|$, and C is a relatively bounded (relatively compact) perturbation of A, then C is a relatively bounded (relatively compact) perturbation of $A + B$. Note that if $|q(t)| \leq Mt^{\Delta}$, $\Delta < \alpha - 2$, then it is a b_0 perturbation of $(t^\alpha y')'$ which is relatively compact. In this case there is no restriction on M.

Finally, we apply Theorem 4.1 to the energy operator of the hydrogen atom

$$L[y] = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + V(x)\right] y, \qquad 0 < x \leq 1 \tag{4.16}$$

where $\ell > 1/2$. First we define a unitary transformation U from $L^2(0, 1)$ onto $L^2(t_0, \infty)$, $t_0 = 1/(2\ell - 1)$, by

$$(Uy)(t) = z(t) = x^\ell y(x), \qquad t = x^{1-2\ell}/(2\ell - 1). \tag{4.17}$$

From the formulas in [3] it follows by straightforward calculations that if

$$K[z] = -(P(t)\dot{z})^{\cdot} + Q(t)z, \qquad \cdot = d/dt, \qquad t_0 \leq t < \infty, \tag{4.18}$$

where $P(t) = x^{-4\ell}$ and $Q(t) = V(x)$, t as in (4.17), then the minimal operator L_0 and maximal operator L_1 determined by (4.16) are unitarily equivalent to the minimal operator K_0 and maximal operator K_1 determined by (4.18), i.e., $K_0 = UL_0U^{-1}$, $K_1 = UL_1U^{-1}$. This means that relative boundedness (compactness) criteria for one operator translates into relative boundedness (compactness) criteria for the other. Let B_0 be the maximal operator associated with multiplication by $Q(t)$ in $L^2(t_0, \infty)$, and let C_0 be the maximal operator associated with multiplication by $V(x)$ in $L^2(0, 1)$. Define, for $0 < \varepsilon < 1, 0 < x \le 1$,

$$g_\varepsilon(x) = \frac{1}{x}\int_{x(1-\varepsilon)}^{x} u^4|V(u)|^2 du,$$

and set $\varepsilon_0 = 1 - (3/2)^{1-2\ell}$. Let $\widetilde{L}_1$ be the maximal operator associated with (4.16) in the case $V(x) \equiv 0$. We now show C_0 is a relatively bounded (compact) perturbation of $\widetilde{L}_1$ if and only if $V \in L^2_{\text{loc}}(0, 1)$ and

$$\sup_{0<x\le 1} g_\varepsilon(x) < \infty \qquad \left(\lim_{x\to 0} g_\varepsilon(x) = 0\right)$$

for some $\varepsilon \in (0, \varepsilon_0)$.

For the proof we apply Theorem 4.1 to $\widetilde{K}_1 = U\widetilde{L}_1U^{-1}$. Clearly $Q \in L^2_{\text{loc}}(t_0, \infty)$ is equivalent to $V \in L^2_{\text{loc}}(0, 1)$. Also with $P(t) = x^{-4\ell} = t^{4\ell/(2\ell-1)}$ we have $\alpha = 4\ell/(2\ell - 1)$ in Theorem 4.1 and the change of variable $u = [(2\ell - 1)\tau]^{1/(1-2\ell)}$ shows that

$$\begin{aligned} g_{0,\delta}(t) &:= \frac{1}{t}\int_t^{t+\delta t} \frac{|Q(\tau)|^2}{\tau^{2(\alpha-2)}}\, d\tau \\ &= x^{2\ell-1}\int_{x(1-\varepsilon)}^{x} u^4|V(u)|^2(2\ell-1)^c u^{-2\ell} du \end{aligned}$$

where $\varepsilon := 1-(1+\delta)^{1/(1-2\ell)}$, $c := (2\ell+3)/(2\ell-1)$. On $x(1-\varepsilon) \le u \le x$ we have

$$1 \le \left(\frac{x}{u}\right)^{2\ell} \le \frac{1}{(1-\varepsilon)^{2\ell}};$$

hence the boundedness of $g_{0,\delta}(t)$ on $1 \le t < \infty$ is equivalent to the boundedness of $g_\varepsilon(x)$ on $0 < x \le 1$. Similarly $g_{0,\delta}(t) \to 0$ as $t \to \infty$ if and only if $g_\varepsilon(x) \to 0$ as $x \to 0^+$. This completes the proof.

In particular, a Coulomb type potential $V(x) = c/x$ is a relatively compact perturbation of $\widetilde{L}_1$.

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