J. of Inequal. & Appl., 1997, Vol. 1, pp. 369–374
Reprints available directly from the publisher
Photocopying permitted by license only

Characterization of Trudinger's Inequality

T. OZAWA

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

(Received 21 January 1997)

A characterization of a sharp form of Trudinger's inequality is established in terms of the Gagliardo-Nirenberg inequality in the limiting case for Sobolev's imbeddings.

Keywords: Trudinger's inequality; Gagliardo-Nirenberg's inequality; Sobolev's inequality.

Mathematical Subject Classification: 31C, 26D

1 INTRODUCTION

Trudinger's inequality states that the Sobolev space $H^{n/p,p}(\mathbb{R}^n) = (1 - \Delta)^{-n/2p} L^p(\mathbb{R}^n)$ with $1 < p < \infty$ are continuously imbedded in the Orlicz space with defining convex function of expotential type [2, 15-17, 19, 23-25]. This replaces the usual Sobolev imbedding theorem in the limiting case where the L^∞ norm is beyond the control of the $H^{n/p,p}$ norm. Trudinger's inequality has a wide variety of applications to the partial differential equations in the case where the standard Sobolev estimates just fail, see for instance [15-18, 24, 25].

In [19] we have proved a sharp form of Trudinger's inequality which makes the dependence of functions on the dominant term more explicit: Let p and p' satisfy $1/p + 1/p' = 1$ and $1 < p < \infty$. Then there exist positive constants α and C such that for all $f \in H^{n/p,p}$ with $||(-\Delta)^{n/2p} f||_{L^p} \leq 1$

$$\int_{\mathbb{R}^n} (\exp(\alpha|f|^{p'}) - \sum_{\substack{0 \leq j < p-1 \\ j \in \mathbb{N}}} \frac{1}{j!}(\alpha|f|^{p'})^j) dx \leq C||f||_{L^p}^p. \qquad \text{(T1)}$$

Since the integrand in (T1) depends monotonically on α, the next important question consists in the estimation of the upper bound of α for (T1) with C possibly dependent on α. The restriction $\|(-\Delta)^{n/2p}f\|_{L^p} \le 1$ may be elminated by rescaling f by $f/\|(-\Delta)^{n/2p}f\|_{L^p}$ as follows:

$$\int_{\mathbb{R}^n}\left(\exp\left(\frac{\alpha|f|^{p'}}{\|(-\Delta)^{n/2p}f\|_{L^p}^{p'}}\right) - \sum_{\substack{0\le j<p-1\\ j\in\mathbb{N}}}\frac{1}{j!}\left(\frac{\alpha|f|^{p'}}{\|(-\Delta)^{n/2p}f\|_{L^p}^{p'}}\right)^j\right)dx$$
$$\le C\frac{\|f\|_{L^p}^{p'}}{\|(-\Delta)^{n/2p}f\|_{L^p}^{p'}}. \tag{T2}$$

The converse implication, however, seems nontrivial as was pointed out to me by Brezis. In [19] the proof of (T1) depends on the following inequality of Gagliardo-Nirenberg type: For any p with $1 < p < \infty$ there exists a constant M depending only on p and n such that for all $f \in H^{n/p,p}$ and all q with $p \le q < \infty$

$$\|f\|_{L^q} \le Mq^{1-1/p}\|(-\Delta)^{n/2p}f\|_{L^p}^{1-p/q}\|f\|_{L^p}^{p/q}. \tag{GN}$$

Given (GN), the power series expansion of the exponential function in the integrand in (T1) and estimation of the resulting $L^{p'j}$ norms on the basis of (GN) implies (T1) as is usual with an argument of the kind. A feature of (GN) lies in the explicit dependence of $\|(-\Delta)^{n/2p}f\|_{L^p}$ with sharp exponent on the dominant term as well as in the optimal growth rate $O(q^{1-1/p})$ in the constant factor. The proof of (GN) is reduced to the proof of the Hardy-Littlewood-Sobolev inequality with constant factor exhibiting the optimal growth with respect to the indices of target Lebesgue spaces.

In summary, the inequality (T1) in question originates from (GN) and leads to (T2). The purpose in this paper is to prove that those three inequalities are in fact equivalent, namely that (T2) implies (GN). Moreover, we characterize the upper bound of α in terms of the lower bound of M. To state the main result precisely, we introduce

$$\alpha_0 = \sup\{\alpha > 0;\ \text{There exists } C \text{ depending on } \alpha \text{ such that (T1) holds for all } f \in H^{n/p,p} \text{ with } \|(-\Delta)^{n/2p}f\|_{L^p} \le 1\},$$

$$M_0 = \inf\{M > 0;\ \text{There exists } r \ge p \text{ depending on } M \text{ such that (GN) holds for all } f \in H^{n/p,p} \text{ and all } q \text{ with } r \le q < \infty\},$$

$$\beta_0 = \limsup_{q\to\infty} \frac{\|f\|_{L^p}}{q^{1-1/p}\|(-\Delta)^{n/2p}f\|_{L^p}^{1-p/q}\|f\|_{L^p}^{p/q}}.$$

The main result in this paper is the following.

THEOREM *Inequalities (T1), (T2), and (GN) are equivalent. Moreover,* $1/\alpha_0 = p'eM_0^{p'} = p'e\beta_0^{p'}$.

We prove the theorem in the next section. The last assertion includes the fact that β_0 is independent of $f \in H^{n/p,p}$. Our result does not ensure the attainability of the best constants in (T1), (T2), and (GN). Although it deserves to be studied, it is outside the purpose of the paper. The equivalence described above is reminiscent of the characterization of the John-Nirenberg inequality by the growth property of the L^p mean of oscillation of function, see Garnett and Jones [10]. We shall return to the problem in the final section.

2 PROOF OF THE THEOREM

Since $\beta_0 \leq M_0$, it suffices to prove the (GN) $\Rightarrow$ (T1) with $1/\alpha_0 \leq p'e\beta_0^{p'}$ and that (T2) $\Rightarrow$ (GN) with $1/\alpha_0 \geq p'eM_0^{p'}$.

(I) (GN) $\Rightarrow$ (T1) with $1/\alpha_0 \leq p'e\beta_0^{p'}$: Suppose that (GN) holds. Let $f \in H^{n/p,p}$. For any $\varepsilon > 0$ there exists r such that for all q with $r \leq q < \infty$

$$||f||_{L^p} \leq (\beta_0 + \varepsilon)q^{1-1/p}||(-\Delta)^{n/2p} f||_{L^p}^{1-p/q}||f||_{L^p}^{p/q}. \tag{1}$$

Expanding the exponential function in the integrand in (T1), estimating the individual terms of the resulting expansion by means of (1), and using the monotone convergence theorem, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} (\exp(\alpha|f|^{p'}) - \sum_{\substack{p'j<r-1 \\ j\in\mathbb{N}}} \frac{1}{j!}(\alpha|f|^{p'})^j)dx \\ &\leq \sum_{\substack{p'j\geq r-1 \\ j\in\mathbb{N}}} \frac{1}{j!}(\alpha p'j(\beta_0+\varepsilon)^{p'})^j||f||_{L^p}^p, \end{aligned} \tag{2}$$

provided that the last series converges, namely $0 \leq \alpha < 1/(p'e(\beta_0 + \varepsilon)^{p'})$. By (GN) the finite sum of $(1/j!)(\alpha|f|^{p'})^j$ with j running over $p \leq p'j < r - 1$ is integrable and the resulting integral is bounded by a constant multiple of $||f||_{L^p}^p$, where we add a trivial remark that the sum over the empty set is understood to be zero. We have thus proved that (T1) holds whenever $0 \leq \alpha < 1/(p'e(\beta_0 + \varepsilon)^{p'})$. Hence $\alpha_0 \geq 1/(p'e(\beta_0 + \varepsilon)^{p'})$ and consequently $\alpha_0 \geq 1/(p'e\beta_0^{p'})$ since $\varepsilon > 0$ is arbitrary.

(II) (T2) $\Rightarrow$ (GN) with $1/\alpha_0 \geq p' e M_0^{p'}$: For any ε with $0 < \varepsilon < \alpha_0$ there exists C_ε such that (T2) holds for all $f \in H^{n/p,p}$ with α and C replaced respectively by $\alpha_0 - \varepsilon$ and C_ε. We regard the resulting integrand as the infinite series of $(1/j!)((\alpha_0 - \varepsilon)|f|^{p'}/||(-\Delta)^{n/2p} f||_{L^p}^{p'})^j$ with j running over $p - 1 \leq j < \infty$ and single one term out. This yields

$$||f||_{L^{p'j}} \leq (C_\varepsilon j!)^{1/p'j} (\alpha_0 - \varepsilon)^{-1/p'} ||(-\Delta)^{n/2p} f||_{L^p}^{1-(p-1)/j} ||f||_{L^p}^{(p-1)/j} \tag{3}$$

for all $j \in \mathbb{N}$ with $j \geq p - 1$. Let $q > p$ and let j satisfy $p'j \leq q < p'(j + 1)$. We use Hölder's inequality to interpolate (3) between $L^{p'j}$ and $L^{p'(j+1)}$ to obtain

$$||f||_{L^q} \leq (C_\varepsilon \Gamma(q/p'+2))^{1/p'j} (\alpha_0 - \varepsilon)^{-1/p'} ||(-\Delta)^{n/2p} f||_{L^p}^{1-p/q} ||f||_{L^p}^{p/q} \tag{4}$$

where Γ stands for the gamma function and we have used $(j + 1)! \leq \Gamma(q/p' + 2)$. By using Stirling's formula and $p'j \geq q - p'$, we see that for any $\delta > 0$ there exists r such that for all $f \in H^{n/p,p}$ and all q with $r \leq q < \infty$ (GN) holds with M replaced by $(p'e(\alpha_0 - \varepsilon))^{-/p'} + \delta$. This proves $M_0 \leq (p'e(\alpha_0 - \varepsilon))^{-/p'} + \delta$. Since ε and δ are arbitrary, $M_0 \leq (p'e\alpha_0)^{-1/p'}$. □

3 REMARKS

(1) There is another target space for imbeddings of the Sobolev spaces in the limiting case. That is BMO, the space of functions with bounded mean oscillation [12]. It is well known that for any p with $1 < p < \infty$, $H^{n/p,p} \hookrightarrow BMO$ with

$$||f||_{BMO} \leq C||(-\Delta)^{n/2p} f||_{L^p},$$

see [1, 3, 21, 22]. The equivalence proved in this paper is regarded as an analogue of the characterization of BMO due to Garnett and Jones [10], whereas it would seem unlikely that there is a simple relation between two classes of equivalent inequalities. An attempt in this direction proceeds as follows. Since L^q is realized as the complex interpolation between L^p and $BMO : L^q = [L^p, BMO]_{1-p/q}$ where $1 < p < q < \infty$ [4, 9, 11], we have

$$||f||_{L^q} \leq K||f||_{BMO}^{p/q} ||f||_{L^p}^{1-pq}.$$

Combining these inequalities above yields an inequality similar to (GN), while the resulting constant factor has a worse growth rate since K grows at least proportionally to q as is easily verified by the logarithmic function restricted near origin (a direct proof of the last inequality shows that K is a constant multiple of q when $p = 1$ [20]). Therefore a simple combination of these two inequalities results in an inequality weaker than (GN).

(2) There is another inequality arising from the limiting case of Sobolev's imbeddings. That is the Brezis-Gallouet-Wainger inequality [5-8, 19], which states that the L^∞ norm falls under the logarithmic control of the $H^{m,q}$ norm in the supercritical case $m > n/q$. The Brezis-Gallouet-Wainger inequality is derived from (GN) in [19] and therefore from equivalent Trudinger's inequality (T1). This reveals a simple relation between Trudinger's inequality and the Brezis-Gallouet-Wainger inequality though the proof of the implication proceeds in a roundabout way via (GN).

(3) One of the interesting applications of Trudinger's inequality to the partial differential equation is given by Vladimirov [26] for the uniqueness of weak solutions to the mixed problem for the nonlinear Schrödinger equation in two space dimensions where the energy norm does not control the L^∞ norm. Subsequently Vladimirov's argument is refined in [16, 17]. Actually the proof in [17] depends on a direct use of (GN) with $n = p = 2$ instead of (T1). Now that we have established the equivalence between (T1), (T2), and (GN), there is not much difference between the proofs in the papers refereed above. Nevertheless there are several applications of (GN) in the case where (T1) does not apply efficiently, [13, 14, 28] to name a few. In [27] the best constant in (GN) with $n = p = 2$ is described in terms of the associated variational problem.

References

[1] D.R. Adams, A note on Riesz potentials, *Duke Math. J.*, **42** (1975), 765–778.

[2] D.R. Adams, A sharp inequality of J. Moser for higher order derivatives, *Ann. of Math.*, **128** (1988), 385–398.

[3] D.R. Adams and R.J. Bagby, Translation-dilation invariant estimates for Riesz potentials, *Indiana Univ. Math. J.*, **23** (1974), 1051–1067.

[4] O. Blaso, Interpolation between $H^1_{B_0}$ and $L^p_{B_1}$, *Studia Math.*, **92** (1989), 205–210.

[5] H. Brezis, Laser beams and limiting cases of Sobolev inequalities, in "Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar Vol. II," *Research Notes in Math.*, **60** (1982), 86–97, Pitman.

[6] H. Brezis and T. Gallouet, Nonlinear Schrödinger evolution equations, *Nonlinear Anal. T.M.A.*, **4** (1980), 677–681.

[7] H. Brezis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. PDE*, **5** (1980), 773–789.

[8] H. Engler, An alternative proof of the Brezis-Wainger inequality, *Comm. PDE*, **14** (1989), 541–544.

[9] C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math.*, **129** (1972), 137–193.

[10] J.B. Garnett and P.W. Jones, The distance in BMO to L^∞, *Ann. of Math.*, **108** (1978), 373–393.

[11] S. Janson and P.W. Jones, Interpolation between H^p spaces: The complex Method, *J. Funct. Anal.*, **48** (1982), 58–80.

[12] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, **14** (1961), 415–426.

[13] H. Kozono and T. Ogawa, Two-dimensional Navier-Stokes flow in unbounded domains, *Math. Annalen*, **297** (1993), 1–31.

[14] H. Kozono, T. Ogawa and H. Sohr, Asymptotic behaviour in L^r for weak solutions of the Navier-Stokes equations in exterior domains, *Manuscripta Math.*, **74** (1992), 253–273.

[15] J. Moser, A sharp form of an inequality by N. Trudinger, *Indiana Univ. Math. J.*, **20** (1971) 1077–1092.

[16] T. Ogawa, A proof Trudinger's inequality and its application to nonlinear Schrödinger equation, *Nonlinear Anal. T.M.A.*, **14** (1990), 765–769.

[17] T. Ogawa and T. Ozawa, Trudinger type inequalities and uniqueness of weak solutions for the nonlinear Schrödinger mixed problem, *J. Math. Anal. Appl.*, **155** (1991), 531–140.

[18] T. Ogawa and T. Suzuki, *Nonlinear elliptic equations with critical growth related to the Trudinger inequality*, preprint.

[19] T. Ozawa, On critical cases of Sobolev's inequalities, *J. Funct. Anal.*, **127** (1995), 259–269.

[20] H.M. Reinmann and T. Rychener, Funktionen beschränkter mittlerer Oszillation, *Lecture Notes Math.* **487** (1975), Springer-Verlag.

[21] G. Stampacchia, The spaces $L^{p,\lambda}$, $N^{p,\lambda}$ and interpolation, *Ann. Sc. Norm. Sup. Pisa*, **19** (1965), 443–462.

[22] E.M. Stein and A. Zygmund, Boundedness of translation invariant operators on Hölder spaces and L^p-spaces, *Ann. of Math.*, **85** (1967), 337–349.

[23] R.S. Strichartz, A note on Trudinger's extension of Sobolev's inequalities, *Indiana Univ. Math. J.*, **21** (1972), 841–842.

[24] M. Struwe, Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces, *Ann. Inst. Henri Pointcaré, Analyse nonlinéaire*, **5** (1988), 425–464.

[25] N.S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.*, **17** (1967), 473–483.

[26] M.V. Vladimirov, On the solvability of a mixed problem for a nonlinear equation of Schrödinger type, *Soviet Math. Dokl.*, **29** (1984), 281–284.

[27] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.*, **87** (1983), 567–576.

[28] J.-Q. Yao, Comportement à l'infini des solutions d'une équation de Schrödinger non linéaire dans un domaine extérieur, *C.R. Acad. Sc. Paris*, **294** (1982), 163–166.