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# Goluzin's Extension of the Schwarz-Pick Inequality

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For a function $f$ holomorphic and bounded, $|f| < 1$, with the expansion

$$f(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$$

in the disk $D = \{|z| < 1\}$, $n \geq 1$, we set

$$\Gamma(z, f) = (1 - |z|^2)|f'(z)|/(1 - |f(z)|^2),$$

$$A = |a_n|/(1 - |a_0|^2), \qquad \text{and} \qquad \Upsilon(z) = z^n(z + A)/(1 + Az).$$

Goluzin's extension of the Schwarz-Pick inequality is that

$$\Gamma(z, f) \leq \Gamma(|z|, \Upsilon), \qquad z \in D.$$

We shall further improve Goluzin's inequality with a complete description on the equality condition. For a holomorphic map from a hyperbolic plane domain into another, one can prove a similar result in terms of the Poincaré metric.

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## 1 INTRODUCTION

Let $D = \{|z| < 1\}$, let $\mathcal{B}$ be the family of all the holomorphic functions $f : D \to D$, and let $\mathcal{F}$ be the family of $f \in \mathcal{B}$ univalent and $f(D) = D$. For $f \in \mathcal{B}$ and $z \in D$ we set

$$\Gamma(z, f) = \frac{(1 - |z|^2)|f'(z)|}{1 - |f(z)|^2}.$$

The Pick version of the Schwarz inequality, or simply, the Schwarz-Pick inequality then reads that

$$\Gamma(z, f) \leq 1$$

everywhere in $D$. Furthermore,

$$\boxed{\exists z_o \in D,\ \Gamma(z_o, f) = 1} \Rightarrow \boxed{f \in \mathcal{F}} \Rightarrow \boxed{\Gamma(z, f) = 1\ \forall z \in D}.$$

For $f \in \mathcal{B}$ we set

$$\Phi_1(z) = \frac{z(z + \Gamma(0, f))}{1 + \Gamma(0, f)z}, \quad z \in D.$$

The case $n = 1$ of G. M. Goluzin's theorem [1, Theorem 3], [2, p. 335, Theorem 6], then reads that

$$\Gamma(z, f) \leq \Gamma(|z|, \Phi_1) \tag{1.1}$$

at each $z \in D$. Since for $\Xi$ in [3] we have

$$\Xi(z, f) \equiv \frac{\Gamma(0, f)(1 + |z|^2) + 2|z|}{1 + |z|^2 + 2\Gamma(0, f)|z|} = \Gamma(|z|, \Phi_1), \tag{1.2}$$

our former result [3, Theorem 1] is actually a rediscovery of (1.1). The present author regrets overlooking (1.1) of Goluzin. However, we dare to note the following two items.

**(I)** The equality condition described in [3, Theorem 1] is more detailed than Goluzin's.

**(II)** The proof of [3, Theorem 1] is quite different from Goluzin's; it depends on a further analysis of $\Gamma(z, f)$ in [3, Theorem 2].

Goluzin, *loc. cit.*, actually obtained a result under the condition that

$$f(z) = f(0) + \sum_{k=n}^{\infty} \frac{f^{(k)}(0)}{k!} z^k, \quad z \in D, \tag{1.3}$$

for $f \in \mathcal{B}$, where $n \geq 1$ and, possibly, $f^{(n)}(0) = 0$.

The purpose of the present paper is to extend the cited result for $f$ of (1.3) with a complete description for the equality condition.

## 2 EXTENSION

For $f \in \mathcal{B}$ with the expansion (1.3) we set

$$A = \frac{|f^{(n)}(0)|}{n!(1 - |f(0)|^2)}.$$

As will be seen, $0 \leq A \leq 1$. We furthermore set

$$B = \begin{cases} 1, & \text{for } A = 1; \\ \dfrac{|f^{(n+1)}(0)|}{(n+1)!(1 - |f(0)|^2)(1 - A^2)}, & \text{for } A < 1. \end{cases}$$

We shall observe that $0 \leq B \leq 1$. Set

$$\Phi_k(z) = \frac{z^k(z + A)}{1 + Az}, \qquad (k = 0, 1, 2, \cdots),$$

$$\Psi(z) = \frac{z(z + B)}{1 + Bz}, \qquad \text{and}$$

$$R_n(z) = \frac{|z|^n\big(1 - \Phi_0(|z|)^2\big)\big(1 - \Gamma(|z|, \Psi)\big)}{1 - \Phi_n(|z|)^2} \qquad (\text{for } n \geq 1 \text{ of } (1.3)),$$

for $z \in D$. For $f \in \mathcal{B}$ with (1.3) one can prove that $\Gamma(0, f) = \Gamma(0, \Phi_n)$. Furthermore, $R_n(z) \geq 0$ and $R_n(0) = 0$.

Set $G_\lambda(z) = z^\lambda$, $z \in D$, and

$$\mathcal{F}_\lambda = \{T \circ G_\lambda;\ T \in \mathcal{F}\}, \qquad \lambda = 1, 2, \cdots,$$

so that $\mathcal{F} = \mathcal{F}_1$. Note that for $f \in \mathcal{B}$ with (1.3), the $n$-th derivative of $\dfrac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}$ at $z = 0$ is $\dfrac{f^{(n)}(0)}{1 - |f(0)|^2}$. Hence, for $f \in \mathcal{B}$ with (1.3) to be in $\mathcal{F}_n$, it is necessary and sufficient that $A = 1$. We further note that

$$\Gamma(z, f) = \Gamma(|z|, G_\lambda)$$

for $f \in \mathcal{F}_\lambda$ and $z \in D$.

For $a \in D$ we set

$$E(a) = \begin{cases} D, & \text{if } a = 0; \\ \left\{-\dfrac{a}{|a|}r;\ 0 \leq r < 1\right\}, & \text{if } a \neq 0. \end{cases}$$

THEOREM 1 *For $f \in \mathcal{B}$ with the Taylor expansion (1.3) we have the inequality*

$$\Gamma(z, f) \leq \Gamma(|z|, \Phi_n) - R_n(z) \tag{2.1}$$

*at each $z \in D$. The equality in (2.1) holds at a point $z \neq 0$ if and only if either $f$ is in $\mathcal{F}_n$ or $f$ is of the specified form*

$$f(w) = T(w^n S(w)) \tag{2.2}$$

*in $D$, where $T,\ S \in \mathcal{F}$ and $S(a) = 0,\ a \in D$. For $f \in \mathcal{F}_n$ the equality holds in (2.1) everywhere in $D$, whereas for $f$ of (2.2), the equality in (2.1) holds at each point $z \in E(a)$.*

Each $f(w) = T(w^{n+1}) \in \mathcal{F}_{n+1}$ is of the form (2.2) with $S(w) = w$. However, one can observe that $f \notin \mathcal{F}_{n+1}$ for $f$ of (2.2) if and only if $a \neq 0$.

Goluzin's cited extension for $f \in \mathcal{B}$ of (1.3) is that

$$\Gamma(z, f) \leq \Gamma(|z|, \Phi_n), \tag{2.3}$$

an inequality weaker than (2.1). The inequality (2.1) implies (2.3). Furthermore, as will be observed, $R_n(z) \equiv 0$ if and only if $f \in \mathcal{F}_n$ or $f$ is of the form (2.2). Again the equality condition for (2.3) in Goluzin's is not complete enough.

Let $\mathcal{F}^{n+1}$ be the family of all functions $T_1 \cdots T_{n+1}$, products of $T_k \in \mathcal{F},\ k = 1, \cdots, n+1,\ n \geq 1$. Then $f$ of (2.2) is in $\mathcal{F}^{n+1}$. For the proof we let $S(w) = \dfrac{\varepsilon(w-a)}{1-\bar{a}w}$, $|\varepsilon| = 1$, and $T(b) = 0$. The equation $w^n S(w) = b$, or,

$$\varepsilon w^n (w-a) - b(1-\bar{a}w) = 0 \tag{2.4}$$

has exactly $n+1$ roots, $c_1, c_2, \cdots c_{n+1}$, say, in $D$. Actually, on the circle $\{|w| = 1\}$ we have

$$|\varepsilon w^n (w-a)| = |w-a| = |1-\bar{a}w| > |b(1-\bar{a}w)|.$$

The Rouché theorem on the equation yields that the equation (2.4) has the same number of roots as that of $w^n(w-a) = 0$ in $D$. It is now easy to have the expression

$$f(w) = \delta \prod_{k=1}^{n+1} \frac{w-c_k}{1-\overline{c_k}w},$$

for a constant $\delta$, $|\delta| = 1$.

The converse is true in case $n = 1$; see [3] where $\mathcal{G} = \mathcal{F}^2$. However, for $n > 1$, we have $f \in \mathcal{F}^{n+1}$ which is not of the form (2.2). For example,

$$f(w) = \prod_{k=2}^{n+2} \frac{kw - 1}{k - w}$$

is in $\mathcal{F}^{n+1}$. Suppose that $f$ is of the form (2.2). Then $f'(0) = 0$. On the other hand,

$$f'(0) = \frac{(-1)^n}{(n+2)!} \sum_{k=2}^{n+2} \left(k - \frac{1}{k}\right) \neq 0.$$

This is a contradiction.

## 3 PROOF OF THEOREM 1

LEMMA *For each $f \in \mathcal{B}$ and at each $z \in D$, one has*

$$|f(z)| \leq \frac{|z| + |f(0)|}{1 + |f(0)||z|}. \tag{3.1}$$

*The equality in (3.1) at a point $z \neq 0$ holds if and only if $f \in \mathcal{F}$. For $f \in \mathcal{F}$ with $f(a) = 0$, the equality in (3.1) holds at all points $z \in E(a)$.*

*Proof* It follows from the Schwarz lemma that

$$\left|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\right| \leq |z|, \qquad z \in D. \tag{3.2}$$

On the other hand,

$$\frac{|f(z)| - |f(0)|}{1 - |f(0)||f(z)|} \leq \left|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\right|, \qquad z \in D. \tag{3.3}$$

Combining (3.2) and (3.3) one has (3.1). The equality in (3.1) at $z \neq 0$ holds if and only if $f \in \mathcal{F}$ from (3.2) and $\operatorname{Re}\left(\overline{f(0)}f(z)\right) = |f(0)f(z)|$ with $|f(z)| \geq |f(0)|$ from (3.3). Thus, in case $f \in \mathcal{F}$ with $f(0) = 0$, the equality in (3.1) holds in the whole $D$, whereas in case $f \in \mathcal{F}$ with $f(a) = 0$, $a \neq 0$, the equality in (3.1) holds for $z$ with

$$f(z) \in E(-f(0)) \cap \{w;\ |w| \geq |f(0)|\}.$$

Hence the equality in (3.1) holds at all points $z \in E(a)$.

*Proof of Theorem 1* To prove (2.1) we may suppose that $z \neq 0$ because the equality holds at $z = 0$.

Set

$$g(w) = \frac{1}{w^n} \cdot \frac{f(w) - f(0)}{1 - \overline{f(0)} f(w)}$$

and

$$h(w) = w^n g(w), \qquad w \in D,$$

so that $|g(0)| = A$ and, in case $g \in \mathcal{B}$, one has $\Gamma(0, g) = B$.

In case $A = 1$ or $|g(w)| \equiv 1$, we conclude that $f \in \mathcal{F}_n$ for which the equality in (2.1) holds at each point of $D$.

In case $A < 1$, we can apply (1.1) to $g \in \mathcal{B}$ to have

$$\Gamma(z, g) \leq \Gamma(|z|, \Psi) \equiv Q(|z|), \tag{3.4}$$

whence

$$\left| \frac{h'(z)}{z^n} - \frac{nh(z)}{z^{n+1}} \right| = |g'(z)| \leq \frac{Q(|z|)\left(|z|^{2n} - |h(z)|^2\right)}{|z|^{2n}(1 - |z|^2)},$$

so that

$$|h'(z)| \leq \frac{np}{r} + \frac{Q(r)(r^{2n} - p^2)}{r^n(1 - r^2)}, \tag{3.5}$$

where $|z| = r$ and $|h(z)| = p$, $0 < r < 1$, $0 \leq p < 1$.

It now follows from (3.5) that

$$\frac{\Gamma(z, f)}{1 - r^2} = \frac{\Gamma(z, h)}{1 - r^2} \leq \frac{\frac{np}{r} + \frac{Q(r)(r^{2n} - p^2)}{r^n(1 - r^2)}}{1 - p^2} \equiv F(p).$$

Note that $Q(r) > 0$ for $r > 0$. For each $r$, $0 < r < 1$, the function $F(p)$ is strictly increasing for $p$, $0 \leq p < r^n$. To prove this, we consider the numerator of the derivative $F'(p)$, that is,

$$\varphi(p) = \frac{n}{r} p^2 - \frac{2Q(r)(1 - r^{2n})}{r^n(1 - r^2)} p + \frac{n}{r}.$$

Since the product of the roots of the equation $\varphi(p) = 0$ is 1, at most one root is in the interval $0 < p < r^n$.

Goluzin, *loc. cit.*, proved that

$$\Theta(r) \equiv n\left(r^n + \frac{1}{r^n}\right) - \frac{2(1-r^{2n})}{r^{n-1}(1-r^2)} > 0$$

for $0 < r < 1$. Hence

$$\varphi(r^n) = r^{n-1}\left[n\left(r^n + \frac{1}{r^n}\right) - \frac{2Q(r)(1-r^{2n})}{r^{n-1}(1-r^2)}\right] \geq r^{n-1}\Theta(r) > 0.$$

Since $\varphi(0) > 0$, and $\varphi(r^n) > 0$ we thus conclude that the equation $\varphi(p) = 0$ has no root in the interval $0 < p < r^n$, so that $\varphi(p) > 0$ for all $p$, $0 \leq p \leq r^n$. Therefore $F'(p) > 0$ for $0 \leq p \leq r^n$.

We now apply our Lemma to $g$ to have

$$p = |h(z)| = r^n|g(z)| \leq r^n \cdot \frac{r+A}{1+Ar} < r^n. \tag{3.6}$$

Hence

$$\Gamma(z, f) \leq (1-r^2)F\left(\frac{r^n(r+A)}{1+Ar}\right). \tag{3.7}$$

This is just (2.1).

The equality in (3.7) holds if and only if those in (3.4) and in (3.5) for $p = \dfrac{r^n(r+A)}{1+Ar}$, and furthermore the equality

$$|g(z)| = \frac{r+A}{1+Ar}, \tag{3.8}$$

all hold at the same time. The equality (3.8) is valid if and only if

$$g \in \mathcal{F}, \qquad g(a) = 0, \qquad \text{and} \qquad z \in E(a). \tag{3.9}$$

The equality in (3.4) holds in the whole $D$ for $g \in \mathcal{F}$; in this case $Q(|z|) = 1$. To prove that the equality in (3.5) holds under (3.9) for $p = \dfrac{r^n(r+A)}{1+Ar}$ and for $z \in E(a)$, we set

$$g(w) = \frac{\varepsilon(w-a)}{1-\overline{a}w}, \qquad |\varepsilon| = 1.$$

In case $a = 0$, we have $A = |g(0)| = 0$ and $h(w) \equiv \varepsilon w^{n+1}$. Hence the equality in (3.5) holds for $p = \frac{r^n(r+A)}{1+Ar} = r^n$. In case $a \neq 0$, we have for $z = -\frac{a}{|a|}r$ $(0 < r < 1)$ of $E(a)$ that

$$g(z) = -\frac{a}{|a|}\varepsilon|g(z)| \quad \text{and} \quad g'(z) = \varepsilon|g'(z)|,$$

so that

$$\frac{h'(z)}{z^n} = \varepsilon\left[\frac{n}{r}|g(z)| + |g'(z)|\right] \quad \text{and} \quad \frac{nh(z)}{z^{n+1}} = \varepsilon \cdot \frac{n}{r}|g(z)|.$$

Hence

$$\left|\frac{h'(z)}{z^n} - \frac{nh(z)}{z^{n+1}}\right| = \left|\frac{h'(z)}{z^n}\right| - \left|\frac{nh(z)}{z^{n+1}}\right|.$$

We thus have the equality in (3.5) for $p = \frac{r^n(r+A)}{1+Ar}$ because $\Gamma(z, g) = 1$.

*Remark* We can further improve (2.1) for $f \notin \mathcal{F}_n$. For this purpose we apply Theorem 1 to $g \in \mathcal{B}$ in the proof of Theorem 1 to have

$$\Gamma(z, g) \leq Q_1(|z|), \tag{3.10}$$

where $Q_1(|z|)$ is the right-hand side of (2.1) applied to the present $g$. We then follow the lines in the proof of Theorem 1 replacing (3.4) with (3.10). The resulting inequality in terms of $f$ is rather complicated.

## 4 POINCARÉ METRIC

A domain $\Omega$ in the plane $C = \{|z| < +\infty\}$ is called hyperbolic if its boundary in $C$ contains at least two points. Each hyperbolic domain $\Omega$ has the Poincaré metric $P_\Omega(z)|dz|$. Namely,

$$1/P_\Omega(z) = (1 - |w|^2)|\phi'(w)|, \qquad z = \phi(w),$$

for a holomorphic, universal covering projection $\phi$ from $D$ onto $\Omega$; the choice of $\phi$ and $w$ is immaterial as far as $z = \phi(w)$ is satisfied. The Poincaré distance $d_\Omega(z_1, z_2)$ of $z_1$ and $z_2$ in $\Omega$ is the minimum of all the integrals $\int_\gamma P_\Omega(z)|dz|$ along the rectifiable curves $\gamma$ connecting $z_1$ and $z_2$ within $\Omega$. Given $z_1$ and $z_2$ in $D$, for each $w_1 \in D$ with $z_1 = \phi(w_1)$ we have $w_2 \in D$ with $z_2 = \phi(w_2)$ such that

$$d_\Omega(z_1, z_2) = d_D(w_1, w_2).$$

Let $\Omega$ and $\Sigma$ be hyperbolic domains in $C$, and let $f : \Omega \to \Sigma$ be holomorphic. For $c \in \Omega$ and $n \geq 1$ we suppose that

$$f(z) = f(c) + \sum_{k=n}^{\infty} \frac{f^{(k)}(c)}{k!}(z-c)^k \tag{4.1}$$

in a disk of center $c$ contained in $\Omega$. Again, $f^{(n)}(c) = 0$ is admissible. Set

$$\Lambda_I(z) = \frac{\partial}{\partial z} \log P_\Omega(z) \quad \text{and} \quad \Lambda_{II}(z) = \frac{\partial}{\partial z} \log P_\Sigma(f(z))$$

for $z \in \Omega$. Set

$$A(c) = \frac{P_\Sigma(f(c))}{P_\Omega(c)^n} \frac{|f^{(n)}(c)|}{n!},$$

$$B(c) = \begin{cases} 1, & \text{for } A(c) = 1; \\ \dfrac{\Theta(c)}{1 - A(c)^2}, & \text{for } A(c) < 1, \end{cases}$$

where, in case $n = 1$,

$$\Theta(c) = \frac{P_\Sigma(f(c))}{P_\Omega(c)^2} \left| \frac{f''(c)}{2} + f'(c)\left\{\Lambda_{II}(c) - \Lambda_I(c)\right\} \right|,$$

and, in case $n > 1$,

$$\Theta(c) = \frac{P_\Sigma(f(c))}{P_\Omega(c)^{n+1}} \left| \frac{f^{(n+1)}(c)}{(n+1)!} - \frac{f^{(n)}(c)}{(n-1)!} \Lambda_I(c) \right|;$$

as will be seen, $0 \leq A(c) \leq 1$ and $0 \leq B(c) \leq 1$ (for $n \geq 1$ of (4.1)). Furthermore, set

$$\Phi_{k,c}(z) = \frac{z^k(z + A(c))}{1 + A(c)z}, \quad (k = 0, 1, 2, \cdots),$$

$$\Psi_c(z) = \frac{z(z + B(c))}{1 + B(c)z}, \quad \text{and}$$

$$R_{n,c}(z) = \frac{|z|^n \left(1 - \Phi_{0,c}(|z|)^2\right)(1 - \Gamma(|z|, \Psi_c))}{1 - \Phi_{n,c}(|z|)^2} \quad (\text{for } n \geq 1 \text{ of } (4.1)),$$

for $z \in D$.

In particular, if $\Omega = \Sigma = D$ and $c = 0$, then we have

$$A = A(0), \quad B = B(0), \quad \Phi_k = \Phi_{k,0}, \quad \Psi = \Psi_0, \quad \text{and} \quad R_n = R_{n,0}.$$

THEOREM 2 *For a holomorphic function $f : \Omega \to \Sigma$ with the Taylor expansion (4.1) we have the inequality*

$$\frac{P_\Sigma(f(z))}{P_\Omega(z)} |f'(z)| \le \Gamma\big(\tanh d_\Omega(z,c), \Phi_{n,c}\big) - R_{n,c}\big(\tanh d_\Omega(z,c)\big) \quad (4.2)$$

*at each $z \in \Omega$.*

For the equality in (4.2), see just after the proof.

*Proof of Theorem 2* Let $\phi$ and $\psi$ be universal covering projections from $D$ onto $\Omega$ and $\Sigma$, respectively, such that $c = \phi(0)$ and $f(c) = \psi(0)$. Let $F$ be the single-valued branch of $\psi^{-1} \circ f \circ \phi$ in $D$ such that $F(0) = 0$. Since $\psi(F(w)) = f(\phi(w))$, $w \in D$, we have

$$\psi'(F(w))F'(w) = f'(\phi(w))\phi'(w),$$

$$\psi''(F(w))F'(w)^2 + \psi'(F(w))F''(w) = f''(\phi(w))\phi'(w)^2 + f'(\phi(w))\phi''(w),$$

and for $n > 2$, we have, by induction,

$$(\star) + n\psi''(F(w))F'(w)F^{(n-1)}(w) + \psi'(F(w))F^{(n)}(w) =$$

$$f^{(n)}(\phi(w))\phi'(w)^n + \frac{n(n-1)}{2} f^{(n-1)}(\phi(w))\phi'(w)^{n-2}\phi''(w) + (\#),$$

where the terms containing

$$F'(w), \cdots, F^{(n-2)}(w)$$

appear in $(\star)$, and those containing

$$f'(\phi(w)), \cdots, f^{(n-2)}(\phi(w))$$

appear in (#). We thus have

$$\frac{F^{(n)}(0)}{n!} = \frac{f^{(n)}(c)}{n!} \frac{\phi'(0)^n}{\psi'(0)};$$

actually, in case $n > 1$, we have $F^{(k)}(0) = f^{(k)}(c) = 0$ for $1 \le k \le n-1$. Consequently,

$$\frac{|F^{(n)}(0)|}{n!} = \frac{P_\Sigma(f(c))}{P_\Omega(c)^n} \frac{|f^{(n)}(c)|}{n!}.$$

We now calculate $\dfrac{|F^{(n+1)}(0)|}{(n+1)!}$. In case $n = 1$, it follows that

$$\frac{|F''(0)|}{2} = \left|\frac{\phi'(0)^2}{\psi'(0)}\right| \left|\frac{f''(c)}{2} + \frac{1}{2}\frac{\phi''(0)}{\phi'(0)^2} \cdot f'(c) - \frac{1}{2}\frac{\psi''(0)}{\psi'(0)^2} \cdot f'(c)^2\right|,$$

which, together with

$$\Lambda_I(c)\phi'(0) = -\frac{1}{2}\frac{\phi''(0)}{\phi'(0)} \quad \text{and} \quad \Lambda_{II}(c)\psi'(0) = -\frac{1}{2}\frac{\psi''(0)}{\psi'(0)} \cdot f'(c),$$

shows that $\dfrac{|F''(0)|}{2} = \Theta(c)$. In case $n > 1$, it follows from $0 = f^{(k)}(c) = F^{(k)}(0)$ $(1 \le k \le n-1)$ that

$$\frac{|F^{(n+1)}(0)|}{(n+1)!} = \left|\frac{\phi'(0)^{n+1}}{\psi'(0)}\right| \left|\frac{f^{(n+1)}(c)}{(n+1)!} + \frac{1}{2}\frac{\phi''(0)}{\phi'(0)^2} \cdot \frac{f^{(n)}(c)}{(n-1)!}\right| = \Theta(c).$$

Given $z \in \Omega$, we choose $w \in D$ with $z = \phi(w)$ and $d_\Omega(z, c) = d_D(w, 0) = \operatorname{arctanh}|w|$. We apply now Theorem 1 to $F$ at $w \in D$. Since

$$\Gamma(w, F) = \frac{P_\Sigma(f(z))}{P_\Omega(z)} \cdot |f'(z)|,$$

the requested (4.2) follows from (2.1) with $|w| = \tanh d_\Omega(z, c)$.

One can give the equality conditions in terms of $F$ to (4.2); they are left as exercises. The formulation appears not to have a good geometric interpretation. It is easy to see that the equality in (4.2) holds at $z = c$.

In case $\Omega = \Sigma = D$, the inequality (4.2) becomes

$$\Gamma(z, f) \le \Gamma\left(\left|\frac{z-c}{1-\bar{c}z}\right|, \Phi_{n,c}\right) - R_{n,c}\left(\left|\frac{z-c}{1-\bar{c}z}\right|\right)$$

at each $z \in D$. In this case, for $n \ge 1$,

$$A(c) = \frac{|f^{(n)}(c)|}{n!}\,\frac{(1-|c|^2)^n}{1-|f(c)|^2},$$

and in case $n = 1$,

$$\Theta(c) = \frac{(1-|c|^2)^2}{1-|f(c)|^2}\left|\frac{f''(c)}{2} - \frac{\bar{c}f'(c)}{1-|c|^2} + \frac{\overline{f(c)}f'(c)^2}{1-|f(c)|^2}\right|$$

and in case $n > 1$,

$$\Theta(c) = \frac{(1-|c|^2)^{n+1}}{1-|f(c)|^2}\left|\frac{f^{(n+1)}(c)}{(n+1)!} - \frac{\bar{c}}{1-|c|^2}\cdot\frac{f^{(n)}(c)}{(n-1)!}\right|.$$

## References


[1] G.M. Goluzin, Some estimations of derivatives of bounded functions (in Russian), *Mat. Sbornik*, **16**(58) No. 3 (1945), 295–306.

[2] G.M. Goluzin, Geometric theory of functions of a complex variable. Translations of Mathematical Monographs, vol. 26, *Amer. Math. Soc.*, Providence, 1969.

[3] S. Yamashita, The Pick version of the Schwarz lemma and comparison of the Poincaré densities. *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **19** (1994), 291–322.