RIO-TYPE INEQUALITY FOR THE EXPECTATION OF PRODUCTS OF RANDOM VARIABLES

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Received 26 July 2004

We develop an inequality for the expectation of a product of n random variables generalizing the recent work of Dedecker and Doukhan (2003) and the earlier results of Rio (1993).

1. Introduction

Let $(\Omega,\mathcal{F},P)$ be a probability space and let (X,Y) be a bivariate random vector defined on it. Suppose that $E(X^2)<\infty$ and $E(Y^2)<\infty$. Hoeffding proved that

$$\operatorname{Cov}(X,Y)=\int_{\mathbb{R}^2}\left[P(X\le x,\ Y\le y)-P(X\le x)P(Y\le y)\right]dx\,dy. \tag{1.1}$$

In [5], Lehmann gave a simple proof of this identity and used it in his study of some concepts of dependence. This identity was generalized to functions $h(X)$ and $g(Y)$ with $E[h^2(X)]<\infty$ and $E[g^2(Y)]<\infty$ and with finite derivatives $h'(\cdot)$ and $g'(\cdot)$ by Newman [6]. Multidimensional versions of these results were proved by Block and Fang [1], Yu [13], and more recently by Prakasa Rao [7]. Related covariance identities for exponential and other distributions are given by Prakasa Rao in [9, 10].

Suppose that $\mathcal{M}$ is a sub-σ-algebra of $\mathcal{F}$ and Y is measurable with respect to $\mathcal{M}$. Let $\sigma(X)$ be the sub-σ-algebra generated by the random variable X. Define

$$\alpha(\mathcal{M},X)=\sup\left\{\left|P(A\cap B)-P(A)P(B)\right|,\ A\in\mathcal{M},\ B\in\sigma(X)\right\}. \tag{1.2}$$

Define

$$\begin{aligned} Q_X(u)&=\inf\left\{x:P\left(|X|>x\right)\le u\right\},\\ G_X(s)&=\inf\left\{z:\int_0^z Q_X(t)\,dt\ge s\right\},\\ H_{X,Y}(s)&=\inf\left\{t:E\left(|X|I_{[|Y|>t]}\right)\le s\right\}. \end{aligned} \tag{1.3}$$

Journal of Inequalities and Applications 2005:1 (2005) 7–14
DOI: 10.1155/JIA.2005.7

Rio [11] proved that

$$|\operatorname{Cov}(X,Y)| \le 2\int_0^{\alpha(\mathcal{M},X)/2} Q_Y(u)Q_X(u)\,du. \tag{1.4}$$

Related results are given in [12, page 9]. These results were generalized by Bradley [2] for a strong-mixing process and by Prakasa Rao [8] for rth-order joint cumulant under rth-order strong mixing. In a recent work, Dedecker and Doukhan [3] proved that

$$|E(XY)| \le \int_0^{\|E(X|\mathcal{M})\|_1} H_{X,Y}(t)\,dt \le \int_0^{\|E(X|\mathcal{M})\|_1} Q_Y o G_X(t)\,dt \tag{1.5}$$

and obtained an improved version of the above inequality. If X_i, $1 \le i \le n$, are positive-valued random variables, it is easy to see that

$$E(X_1X_2\cdots X_n) \le \int_0^1 Q_{X_1}(u)Q_{X_2}(u)\cdots Q_{X_n}(u)\,du. \tag{1.6}$$

For a proof, see [12, Lemma 2.1, page 35].

We now obtain an improved version of the above inequality following the techniques of Dedecker and Doukhan [3] and Block and Fang [1].

2. Main result

Let $\{X_i,\ 1 \le i \le n\}$ be a sequence of nonnegative random variables defined on a probability space $(\Omega,\mathcal{F},P)$. Then the random variable X_i can be represented in the form

$$X_i = \int_0^\infty I_{(x_i,\infty)}(X_i)\,dx_i, \tag{2.1}$$

where

$$I_{(x_i,\infty)}(X_i) = \begin{cases} 1 & \text{if } X_i > x_i, \\ 0 & \text{if } X_i \le x_i. \end{cases} \tag{2.2}$$

Hence

$$\begin{aligned} E(X_1X_2\cdots X_n) &= E\Big[X_1\Pi_{i=2}^n \int_0^\infty I_{(x_i,\infty)}(X_i)\,dx_i\Big] \\ &= \int_{\mathbb{R}_+^{n-1}} E\big[X_1\Pi_{i=2}^n I_{(x_i,\infty)}(X_i)\big]\,dx_2\cdots dx_n \\ &= \int_{\mathbb{R}_+^{n-1}} E\big[X_1 I_{[X_i>x_i,\ 2\le i\le n]}(X_2,\dots,X_n)\big]\,dx_2\cdots dx_n \end{aligned} \tag{2.3}$$

by the Fubini's theorem, where $\mathbb{R}_+^{n-1} = \{(x_2,\dots,x_n) : x_i \ge 0, 2 \le i \le n\}$. Observe that

$$E(X_1 I_{[X_i>x_i,\ 2\le i\le n]}(X_2,\dots,X_n)) \le \min\big(E[X_1], E(X_1 I_{[X_i>x_i,\ 2\le i\le n]}(X_2,\dots,X_n))\big) \tag{2.4}$$

and hence

$$E(X_1X_2\cdots X_n) \le \int_{\mathbb{R}_+^{n-1}} \left\{ \int_0^{EX_1} \chi_{(E[X_1 I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n)]>u)}(u)du \right\} dx_2\cdots dx_n. \tag{2.5}$$

Here $\chi_A(\cdot)$ denotes the indicator function of the set A. Let

$$g_{X_1}(x_2,\dots,x_n) = E[X_1 I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n)]. \tag{2.6}$$

Then

$$\begin{aligned} E(X_1X_2\cdots X_n) &\le \int_{\mathbb{R}_+^{n-1}} \left\{ \int_0^{EX_1} \chi_{[g_{X_1}(x_2,\dots,x_n)>u]}(u)du \right\} dx_2\cdots dx_n \\ &= \int_0^{E(X_1)} \left\{ \int_{[(x_2,\dots,x_n):g_{X_1}(x_2,\dots,x_n)>u]} 1\, dx_2\cdots dx_n \right\} du. \end{aligned} \tag{2.7}$$

Let

$$H_{X_1,X_2,\dots,X_n}(u) = \lambda[(x_2,\dots,x_n) : g_{X_1}(x_2,\dots,x_n) > u], \tag{2.8}$$

where λ is the Lebesgue measure on the space $\mathbb{R}_+^{n-1}$. Hence

$$E(X_1X_2\cdots X_n) \le \int_0^{E(X_1)} H_{X_1,X_2,\dots,X_n}(u)du. \tag{2.9}$$

Observe that

$$g_{X_1}(x_2,\dots,x_n) = E[X_1 I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n)] \le \int_0^{E[I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n)]} Q_{X_1}(u)du \tag{2.10}$$

from the Fréchet's inequality [4]. Here $Q_{X_1}(\cdot)$ is the generalized inverse of the function $T_{X_1}(x) = P(X_1 > x)$ as defined earlier. Let

$$M_{X_1}(y) = \int_0^y Q_{X_1}(t)dt. \tag{2.11}$$

Observe that $M_{X_1}(\cdot)$ is nondecreasing in y. Let $G_{X_1}(u) = \inf\{z : M_{X_1}(z) \ge u\}$ as defined earlier. Let

$$T_{X_2,\dots,X_n}(x_2,\dots,x_n) = P(X_i > x_i,\ 2 \le i \le n). \tag{2.12}$$

Note that

$$\begin{aligned} g_{X_1}(x_2,\dots,x_n) &\le M_{X_1}(E(I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n))), \\ g_{X_1}(x_2,\dots,x_n) > u &\Longrightarrow M_{X_1}(E(I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n))) > u \\ &\Longrightarrow E(I_{[X_i>x_i,\,2\le i\le n]}(X_2,\dots,X_n)) > G_{X_1}(u) \\ &\Longrightarrow P[X_i > x_i,\ 2\le i\le n] > G_{X_1}(u). \end{aligned} \tag{2.13}$$

Hence the set

$$[(x_2,\dots,x_n) \in \mathbb{R}_+^{n-1} : g_{X_1}(x_2,\dots,x_n) > u] \tag{2.14}$$

is contained in the set

$$[(x_2,\dots,x_n) \in \mathbb{R}_+^{n-1} : P(X_i > x_i,\ 2 \le i \le n) > G_{X_1}(u)]. \tag{2.15}$$

In particular, it follows that the Lebesgue measure of the former set is less than or equal to that of the latter. Let

$$Q^*_{X_2,\dots,X_n}(G_{X_1}(u)) \tag{2.16}$$

denote the Lebesgue measure of the set (2.15).

Then

$$H_{X_1,X_2,\dots,X_n}(u) \le Q^*_{X_2,\dots,X_n}(G_{X_1}(u)) \tag{2.17}$$

for all $0 \le u \le 1$. Hence

$$E(X_1X_2\cdots X_n) \le \int_0^{E(X_1)} Q^*_{X_2,\dots,X_n}(G_{X_1}(u))\,du. \tag{2.18}$$

We have proved the following inequality.

Theorem 2.1. *Let X_i, $1 \le i \le n$, be nonnegative random variables defined on a probability space $(\Omega,\mathcal{F},P)$. Then*

$$E(X_1X_2\cdots X_n) \le \int_0^{E(X_1)} H_{X_1,X_2,\dots,X_n}(u)\,du \le \int_0^{E(X_1)} Q^*_{X_2,\dots,X_n} o G_{X_1}(u)\,du, \tag{2.19}$$

where the functions H, Q^*, and G are as defined earlier.

3. Applications

We now suppose that the random variables $\{X_i,\ 1 \le i \le n\}$ are arbitrary but with

$$E|X_1X_2\cdots X_n| < \infty. \tag{3.1}$$

Define

$$\begin{gathered} g_{X_1}(x_2,\dots,x_n) = E(|X_1| I_{[|X_i|>x_i,\,2\le i\le n]}(X_2,\dots,X_n)), \\ H_{X_1,X_2,\dots,X_n}(u) = \lambda[(x_2,\dots,x_n) : g_{X_1}(x_2,\dots,x_n) \le u], \\ T_{X_2,\dots,X_n}(x_2,\dots,x_n) = P(|X_i| > x_i,\ 2 \le i \le n), \end{gathered} \tag{3.2}$$

and define $M_{X_1}(\cdot), Q_{X_1}(\cdot), Q^*_{X_2,\dots,X_n}$, and G_{X_1} accordingly. The following theorem follows by arguments analogous to those given in Section 2.

Theorem 3.1. *Let X_i, $1 \le i \le n$, be arbitrary random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Then*

$$E(|X_1X_2\cdots X_n|) \le \int_0^{E(|X_1|)} H_{X_1,X_2,\ldots,X_n}(u)du \le \int_0^{E(|X_1|)} Q^*_{X_2,\ldots,X_n} o G_{X_1}(u)du, \tag{3.3}$$

where the functions H, Q^*, and G are as defined above.

In particular, for $n = 2$, we have

$$E(|X_1X_2|) \le \int_0^{E(|X_1|)} H_{X_1,X_2}(u)du \le \int_0^{E(|X_1|)} Q_{X_2} o G_{X_1}(u)du \tag{3.4}$$

since $Q^*_X = Q_X$ for any univariate random variable X. Furthermore,

$$G_{X_1-E(X_1)}(u) \ge G_{X_1}\left(\frac{u}{2}\right), \quad 0 \le u \le 1 \tag{3.5}$$

(cf. [3]). Hence

$$E[|X_1X_2|] \le \int_0^{G^{-1}_{X_1}(E(|X_1|)/2)} Q_{X_2}(u)Q_{X_1}(u)du. \tag{3.6}$$

Therefore, for any two functions $f_i(\cdot)$, $i=1,2$, with $f_i(0)=0$ such that $E|f_1(X_1)f_2(X_2)| < \infty$, we obtain that

$$E[|f_1(X_1)f_2(X_2)|] \le \int_0^{G^{-1}_{f_1(X_1)}(E(|f_1(X_1)|)/2)} Q_{f_2(X_2)}(u)Q_{f_1(X_1)}(u)du. \tag{3.7}$$

Applying Theorem 3.1 for the random variables $X_1 - E(X_1), X_2, \ldots, X_n$, we get that

$$E[|(X_1 - E(X_1))X_2\cdots X_n|] \le \int_0^{E(|X_1-E(X_1)|)} Q^*_{X_2,\ldots,X_n} o G_{X_1-E(X_1)}(u)du. \tag{3.8}$$

But

$$G_{X_1-E(X_1)}(u) \ge G_{X_1}\left(\frac{u}{2}\right), \quad u \ge 0 \tag{3.9}$$

(cf. [3]). Hence

$$E[|(X_1 - E(X_1))X_2\cdots X_n|] \le \int_0^{E(|X_1-E(X_1)|)/2} Q^*_{X_2,\ldots,X_n} o G_{X_1}(u)du. \tag{3.10}$$

Observing that $G_{X_1}(\cdot)$ is the inverse of the function $M_{X_1}(y) = \int_0^y Q_{X_1}(t)dt$, it follows that

$$E[|(X_1 - E(X_1))X_2\cdots X_n|] \le \int_0^{G^{-1}_{X_1}(E(|X_1-E(X_1)|)/2)} Q^*_{X_2,\ldots,X_n}(u)Q_{X_1}(u)du. \tag{3.11}$$

Hence we have the following result.

Theorem 3.2. *Let X_i, $1 \le i \le n$, be arbitrary random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with $E|X_1| < \infty$ and $E|X_1X_2\cdots X_n| < \infty$. Then (3.11) holds.*

Observe that $Q_X^* = Q_X$ for any univariate random variable X. Let $n = 2$ in Theorem 3.2. Then $Q_{X_2}^* = Q_{X_2}$ and the above result reduces to

$$E[\,|\,(X_1 - E(X_1))X_2\,|\,] \le \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2}(u)Q_{X_1}(u)\,du. \tag{3.12}$$

As a further consequence, we get that

$$E[\,|\,(X_1 - E(X_1))(X_2 - E(X_2))\,|\,] \le \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2 - E(X_2)}(u)Q_{X_1}(u)\,du. \tag{3.13}$$

Since

$$Q_{X_2 - E(X_2)} \le Q_{X_2} + E\,|\,X_2\,|, \tag{3.14}$$

we obtain that

$$\begin{aligned} &E[\,|\,(X_1 - E(X_1))(X_2 - E(X_2))\,|\,] \\ &\quad\le \int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_2}(u)Q_{X_1}(u)\,du + E\,|\,X_2\,|\int_0^{G_{X_1}^{-1}(E(|X_1 - E(X_1)|)/2)} Q_{X_1}(u)\,du. \end{aligned} \tag{3.15}$$

Let

$$\alpha(X_1, X_2) = \max\left\{ G_{X_1}^{-1}\left(\frac{E(\,|\,X_1 - E(X_1)\,|\,)}{2}\right), G_{X_2}^{-1}\left(\frac{E(\,|\,X_2 - E(X_2)\,|\,)}{2}\right)\right\}. \tag{3.16}$$

Then it follows that

$$\begin{aligned} &E[\,|\,(X_1 - E(X_1))(X_2 - E(X_2))\,|\,] \\ &\le \int_0^{\alpha(X_1,X_2)} Q_{X_1}(u)Q_{X_2}(u)\,du + \frac{1}{2}\left(E\,|\,X_1\,|\int_0^{\alpha(X_1,X_2)} Q_{X_1}(u)\,du + E\,|\,X_2\,|\int_0^{\alpha(X_1,X_2)} Q_{X_2}(u)\,du\right). \end{aligned} \tag{3.17}$$

This inequality is different from the inequality in [12, page 9].

Let f_1 and f_2 be differentiable functions on $\mathbb{R}_+$ with $f_i(0) = 0$. Let X_i, $i = 1, 2$, be nonnegative random variables. Suppose that $E[f_i^2(X_i)] < \infty$, $i = 1, 2$. It is easy to see that

$$f_i(X_i) = \int_0^\infty f_i'(X_i)I_{(x_i,\infty)}(X_i)\,dx_i. \tag{3.18}$$

Then

$$\begin{aligned} E(f_1(X_1)f_2(X_2)) &= E\left[f_1(X_1)\int_0^\infty f_2'(X_2)I_{(x_2,\infty)}(X_2)\,dx_2\right] \\ &= \int_{\mathbb{R}_+} E[f_1(X_1)f_2'(X_2)I_{(x_2,\infty)}(X_2)]\,dx_2 \end{aligned} \tag{3.19}$$

by the Fubini's theorem. Observe that

$$\begin{aligned} & E(|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2]}(X_2)) \\ & \quad \le \min(E[|f_1(X_1)f_2'(X_2)|], E(|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2]}(X_2))) \end{aligned} \tag{3.20}$$

and hence

$$\begin{aligned} & |E(f_1(X_1)f_2(X_2))| \\ & \quad \le \int_{\mathbb{R}+}\left\{\int_0^{E[|f_1(X_1)f_2'(X_2)|]} \chi_{(E[|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2]}(X_2)]>u)}(u)du\right\}dx_2. \end{aligned} \tag{3.21}$$

Here $\chi_A(\cdot)$ denotes the indicator function of the set A. Let

$$g_{f_1(X_1),f_2'(X_2)}(x_2) = E[|f_1(X_1)f_2'(X_2)|I_{[X_2>x_2],}(X_2)]. \tag{3.22}$$

Then

$$\begin{aligned} |E(f_1(X_1)f_2(X_2))| &\le \int_{\mathbb{R}_+}\left\{\int_0^{E[|f_1(X_1)f_2'(X_2)|]} \chi_{([g_{f_1(X_1),f_2'(X_2)}(x_2)]>u)}(u)du\right\}dx_2 \\ &\le \int_0^{E[|f_1(X_1)f_2'(X_2)|]}\left\{\int_{[x_2:g_{f_1(X_1),f_2'(X_2)}(x_2)>u]} 1\,dx_2\right\}du. \end{aligned} \tag{3.23}$$

Let

$$H_{f_1(X_1),f_2'(X_2)}(u) = \inf\{x_2 : g_{f_1(X_1),f_2'(X_2)}(x_2) \le u\}. \tag{3.24}$$

Then it follows that

$$|E(f_1(X_1)f_2(X_2))| \le \int_0^{E[|f_1(X_1)f_2'(X_2)|]} H_{f_1(X_1),f_2'(X_2)}(u)du. \tag{3.25}$$

An analogous inequality holds by interchanging $f_1(X_1)$ and $f_2(X_2)$:

$$|E(f_1(X_1)f_2(X_2))| \le \int_0^{E[|f_1'(X_1)f_2(X_2)|]} H_{f_1'(X_1),f_2(X_2)}(u)du. \tag{3.26}$$

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