GENERAL HARDY INEQUALITIES WITH OPTIMAL CONSTANTS AND REMAINDER TERMS

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One-dimensional Hardy inequalities with weights and remainder terms are studied. The corresponding optimal constants are discussed. Then by the process of symmetrization, Hardy inequalities with remainder terms in high-dimensional Sobolev spaces are obtained. This result gives a positive answer to the Brézis-Vázquez conjecture.

1. Introduction

In 1919, Hardy [7] proved the following inequality:

$$\int_0^\infty \frac{|u(t)|^p}{t^p}\mathrm{d}t \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |u'(t)|^p \mathrm{d}t, \quad u \in C_0^1(0,\infty), \tag{1.1}$$

where $1 < p < +\infty$. The readers can refer to [8] for the proof of this inequality. The best constant $(p/(p-1))^p$ in the above inequality was given by Landau [10].

It is pointed out in [9] that, in 1933, Leray [11] proved the following two inequalities:

$$\int_{\mathbb{R}^2\setminus B_1(0)} \frac{|u|^2}{|x|^2\ln^2|x|}\mathrm{d}x \le 4\int_{\mathbb{R}^2\setminus B_1(0)} |Du|^2\mathrm{d}x, \tag{1.2}$$

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\mathrm{d}x \le \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |Du|^2\mathrm{d}x, \tag{1.3}$$

where $u \in H_0^1$. Shen [13] obtained (1.2) for a bounded domain $\Omega \subset B_R(0)$ with $\ln^2|x|$ replaced by $\ln^2 R/|x|$. In 1995, Peral and Vázquez [12] showed that $(2/(n-2))^2$ is the best constant in (1.3).

In 1980, Shen [14] proved if ψ and ϕ satisfy $(\phi^{1/p}\psi^{1-1/p})' = (p-1)\psi$, then

$$\int_0^\infty \psi(t)|u(t)|^p\mathrm{d}t \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \phi(t)|u'(t)|^p\mathrm{d}t \tag{1.4}$$

for $u \in C_0^1(0,\infty)$. Moreover, if ψ and ϕ also satisfy $\phi(0)\psi^{p-1}(0) = 0$, then the above inequality is also true for $u \in C^1(0,\infty)$, see [16].

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It is proved in [15] that for $p > 1$,

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \mathrm{d}x \le \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |Du|^p \mathrm{d}x, \quad u \in W_0^{1,p}(\mathbb{R}^n). \tag{1.5}$$

García Azorero and Peral Alonso [5] proved (1.5) by using a different method. Similar to [12], it is showed that $(p/(n-p))^p$ is the best constant.

For Hardy inequalities with remainder terms, Brézis and Vázquez [4] proved recently that there exists a constant $C > 0$, depending only on n and Ω, such that

$$\int_\Omega |Du|^2 \mathrm{d}x \ge \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} \mathrm{d}x + C \int_\Omega |u|^2 \mathrm{d}x, \quad \forall u \in H_0^1(\Omega). \tag{1.6}$$

They asked whether the two terms on the right-hand side of (1.6) are just two terms of a series. Recently, Gazzola et al. [6] generalized (1.6) to the case of $n > p$. They proved that

$$\int_\Omega |Du|^p \mathrm{d}x \ge \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \mathrm{d}x + C \int_\Omega |u|^p \mathrm{d}x, \quad \forall u \in W_0^{1,p}(\Omega). \tag{1.7}$$

Another generalized form of (1.6) given by Adimurthi et al. [1] is

$$\int_\Omega |Du|^p \mathrm{d}x \ge \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \mathrm{d}x + C \sum_{j=1}^{k} \int_\Omega \frac{|u|^p}{|x|^p} \left(\prod_{i=1}^{j} \ln^{(i)} \frac{R}{|x|}\right)^2 \mathrm{d}x. \tag{1.8}$$

Our paper is organized as follows. In Section 1, we study one-dimensional Hardy inequalities with any weights and the corresponding optimal constants. We prove that the constant $(p/(p-1))^p$ $(p > 1)$ is the best constant in the inequality. Meanwhile, we give the relation between the weights in the Hardy inequalities, from which we can determine the other weight if one of the weights is given.

In Section 2, we deal with one-dimensional Hardy inequalities involving any weights and remainder terms $(p \ge 2)$. We also study the optimal constant in this inequality. We point out that the Hardy inequalities can be generalized in two different forms, see Theorem 3.3 (or Corollary 3.4) and Theorem 3.5 (or Corollary 3.6).

In Section 3, using the results established in Sections 1 and 2, we obtain Hardy inequalities with remainder terms in high-dimensional Sobolev spaces by the process of symmetrization. The remainder terms are allowed to be the combination of (1.6) and (1.8). This result gives a positive answer to the Brézis-Vázquez conjecture. Moreover, we obtain the expression of C. We also generalize the results to the case of $n = p$. Finally, for $n > p$ or $n = p$, we obtain the Hardy inequalities with another kind of remainder terms. This shows that the Brézis-Vázquez conjecture is also true for $n \ge p \ge 2$.

2. Hardy inequality with general weights

If $a \in (0, +\infty)$, we define

$$X = \{f \in C^1[0,a] \mid f(a) = 0\}, \qquad X_0 = \{f \mid f \in C_0^1[0,a]\}, \tag{2.1}$$

where $C_0^1[0,a]$ is the set of functions $f(x) \in C^1[0,a]$ with $f(0) = f(a) = 0$. If $a = +\infty$, we define

$$X = \{f \in C^1[0,+\infty) \mid \operatorname{supp} f \text{ is bounded}\}, \qquad X_0 = \{f \mid f \in C_0^1(0,\infty)\}, \tag{2.2}$$

where $C_0^1(0,\infty)$ is the set of functions $f \in C^1(0,\infty)$ with $\operatorname{supp} f$ being bounded. Let

$$\|f\|_{1,p,\phi} = \left(\int_0^a \phi(r)\,|f'(r)|^p \mathrm{d}r\right)^{1/p} \quad p > 1, \tag{2.3}$$

where $\phi \in C^1[0,a]$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$ and a is allowed to be $+\infty$. We denote the completion of X and X_0 with respect to the above norms by $W_\phi^{1,p}$ and $W_{0,\phi}^{1,p}$, respectively.

Theorem 2.1. *Assume f is a nonincreasing function. Then the following hold.*

(i) *For any $f \in W_\phi^{1,p}$,*

$$\int_0^a \psi(r)\,|f(r)|^p \,\mathrm{d}r \le \left(\frac{p}{p-1}\right)^p \int_0^a \phi(r)\,|f'(r)|^p \,\mathrm{d}r \tag{2.4}$$

if $\phi(r)$ and $\psi(r)$ satisfy

$$\left(\phi^{1/p}\psi^{1-1/p}\right)' = (p-1)\psi \tag{2.5}$$

and $\lim_{r\to 0}\phi(r)\psi^{p-1}(r) = 0$. *On the other hand, if $\phi(r)$ and $\psi(r)$ satisfy (2.5) but* $\lim_{r\to 0}\phi(r)\psi^{p-1}(r) \neq 0$, *then (2.5) is true for any $f \in W_{0,\phi}^{1,p}$.*

(ii) *Assume that $\phi \ge r^\alpha$ in some neighborhood of $r = 0$ for $\alpha > p-1$. If $a = \infty$ and $f \in W_\phi^{1,p}$, then the constant $(p/(p-1))^p$ in (2.4) is the best constant but is never achieved.*

Proof. (i) For the completeness, we repeat the proof as follows. Let $a = +\infty$ and $f \in X$ with $f(r) = 0$ if $r \ge R > 0$. Integrating by parts and applying (2.5), we have

$$-p\int_0^R |f|^{p-1}|f|'\phi^{1/p}\psi^{1-1/p}\mathrm{d}r = -\int_0^R \left(|f|^p\right)'\phi^{1/p}\psi^{1-1/p}\mathrm{d}r = (p-1)\int_0^R \psi|f|^p\mathrm{d}r. \tag{2.6}$$

Therefore, by the Hölder inequality, we get

$$(p-1)\int_0^R \psi|f|^p\mathrm{d}r \le p\left(\int_0^R \psi|f|^p\mathrm{d}r\right)^{(p-1)/p}\left(\int_0^R \phi|f'|^p\mathrm{d}r\right)^{1/p}. \tag{2.7}$$

This gives the result. Other cases can be proved similarly.

(ii) What we need to prove is

$$\inf_{f\in X}\frac{\int_0^\infty \phi|f'|^p\mathrm{d}r}{\int_0^\infty \psi|f|^p\mathrm{d}r} = \left(\frac{p-1}{p}\right)^p. \tag{2.8}$$

We insert in (2.4) the function

$$f_\epsilon(r)=\begin{cases}\left(\int_\epsilon^\infty \phi^{-1/(p-1)}\mathrm{d}r\right)^{1-1/p}, & 0\le r<\epsilon,\\ \left(\int_r^\infty \phi^{-1/(p-1)}\mathrm{d}r\right)^{1-1/p}, & \epsilon\le r<K,\\ a_0r+b_0, & K\le r<K+1,\\ 0, & r\ge K+1,\end{cases} \tag{2.9}$$

where K is a constant, a_0 and b_0 satisfy

$$a_0N+b_0=\left(\int_K^\infty \phi^{-1/(p-1)}\mathrm{d}r\right)^{1-1/p}=:C_K,\qquad a_0(N+1)+b_0=0. \tag{2.10}$$

Thus, $a_0=-C_K$ and $b_0=-C_K(K+1)$. Direct calculation shows that

$$\begin{aligned}\int_0^\infty \phi\,|\,f_\epsilon'\,|^p\mathrm{d}r=&\left(\frac{p-1}{p}\right)^p\left(\ln\int_K^\infty \phi^{-1/(p-1)}\mathrm{d}r-\ln\int_\epsilon^\infty \phi^{-1/(p-1)}\mathrm{d}r\right)\\&+\int_K^{K+1}\phi\,|\,a_0\,|^p\mathrm{d}r.\end{aligned} \tag{2.11}$$

Since $\psi(r)=\phi^{-1/(p-1)}(\int_r^a\phi^{-1/(p-1)}\mathrm{d}r)^{-p}$ (see Proposition 2.3), we have

$$\begin{aligned}\int_0^\infty \psi\,|\,f_\epsilon\,|^p=&\frac{\int_0^\epsilon\psi(r)\mathrm{d}r}{\left(\int_\epsilon^a\phi^{-1/(p-1)}\mathrm{d}r\right)^{1-p}}+\ln\int_K^\infty \phi^{-1/(p-1)}\mathrm{d}r-\ln\int_\epsilon^\infty \phi^{-1/(p-1)}\mathrm{d}r\\&+\int_K^{K+1}\psi\,|\,a_0r+b_0\,|^p\mathrm{d}r.\end{aligned} \tag{2.12}$$

By l'Hospital law,

$$\lim_{\epsilon\to0}\frac{\int_0^\epsilon\psi(r)\mathrm{d}r}{\left(\int_\epsilon^\infty\phi^{-1/(p-1)}\mathrm{d}r\right)^{1-p}}=\lim_{\epsilon\to0}\frac{\psi(\epsilon)}{(1-p)\left(\int_\epsilon^\infty\phi^{-1/(p-1)}\mathrm{d}r\right)^{-p}\phi^{-1/(p-1)}(\epsilon)}=\frac{1}{1-p}. \tag{2.13}$$

Therefore, we complete our proof since $\int_\epsilon^\infty\phi^{-1/(p-1)}\mathrm{d}r\to\infty$ as $\epsilon\to0$. □

Remark 2.2. If $\phi=r^{n-1}$, $n>p$, the function $f(r)=(\int_r^\infty\phi^{-1/(p-1)}\mathrm{d}r)^{1-1/p}$ does not belong to $W_\phi^{1,p}(0,\infty)$. But if $\int_0^\infty\phi^{-1/(p-1)}(\int_r^\infty\phi^{-1/(p-1)}\mathrm{d}r)^p\mathrm{d}r<\infty$, then $f(r)\in W_\phi^{1,p}(0,\infty)$ and $f(r)$ is an extremal function.

Before we close this section, we discuss the relation (2.5).

Proposition 2.3. *Assume that ϕ and ψ satisfy (2.5). If ϕ is given, then*

(i) $\psi(r) = \phi^{-1/(p-1)}(\int_r^a \phi^{-1/(p-1)}\mathrm{d}r)^{-p}$ *if $\phi(r) \geq r^\alpha$ in some neighborhood of $r = 0$ for some $\alpha > p-1$;*

(ii) $\psi(r) = \phi^{-1/(p-1)}(\int_0^r \phi^{-1/(p-1)}\mathrm{d}r)^{-p}$ *if $\phi(r) \leq r^\alpha$ in some neighborhood of $r = 0$ for some $\alpha < p-1$. In this case, (2.4) is true for $f \in W_{0,\phi}^{1,p}$;*

(iii) $\psi(r) = r^{-1}(\ln a'/r)^{-p}$ *for some $a' > a$, $a < \infty$ if $\phi(r) = r^\alpha$ in some neighborhood of $r = 0$ with $\alpha = p-1$.*

If ψ is given, then

(i) $\phi(r) = (p-1)\psi^{1-p}(\int_0^r \psi\,\mathrm{d}r)^p$ *if $\phi(r) \geq r^\alpha$ in some neighborhood of $r = 0$ for some $\alpha > -1$;*

(ii) $\phi(r) = (p-1)\psi^{1-p}(\int_r^a \psi\,\mathrm{d}r)^p$ *if $\phi(r) \geq r^\alpha$ in some neighborhood of $r = 0$ for some $\alpha < -1$. In this case, (2.4) is true for $f \in W_{0,\phi}^{1,p}$;*

(iii) $\phi(r) = (p-1)r^{p-1}(\ln a'/r)^{-p}$ *for some $a' > a$, $a < \infty$ if $\phi(r) = r^\alpha$ in some neighborhood of $r = 0$ with $\alpha = -1$.*

3. Hardy inequality with remainder terms

In this section, we are mainly concerned with the case $\phi(r) = r^\alpha$, $\alpha > p-1$ $(a = +\infty)$, and the case $\phi(r) = r^\alpha$, $\alpha = p-1$ $(a < +\infty)$, in some neighborhood of $r = 0$, which often occur in higher-dimensional Hardy inequalities. In these two cases, (2.4) is true for $f \in W_\phi^{1,p}$.

We introduce an identity.

Lemma 3.1. *Assume that $u_1 \in C^1[0,a]$, $u_1(a) = 0$, $0 < \phi_1 \in C[0,a]$, $0 < h_1 \in C^1(0,a]$, $(h_1^2)' = \phi_1^{-1}$, and $h_1^{-1}(0) = 0$. Let $u_1 = h_1 u_2$. Then,*

$$\int_0^a \phi_1 |u_1'|^2 \mathrm{d}r = \int_0^a \phi_1 |h_1'|^2 u_2^2 \mathrm{d}r + \int_0^a \phi_1 h_1^2 |u_2'|^2 \mathrm{d}r. \tag{3.1}$$

Proof. We have $u_1' = h_1' u_2 + h_1 u_2'$. Thus,

$$2\int_0^a \phi_1 u_2 u_2' h_1 h_1' \mathrm{d}r = \int_0^a \phi_1 u_2 u_2' (h_1^2)' \mathrm{d}r = \frac{u_2^2}{2}\bigg|_0^a = 0. \tag{3.2}$$

So the result follows.

Define

$$\lambda_1(\phi) = \inf_{u \in X} \frac{\int_0^a \phi |u'|^2 \mathrm{d}r}{\int_0^a \phi |u|^2 \mathrm{d}r}, \qquad X = \{u \in C^1[0,a] \mid u(a) = 0\}. \tag{3.3}$$

We have the following Poincaré inequality:

$$\lambda_1(\phi)\int_0^a \phi |u|^2 \mathrm{d}r \leq \int_0^a \phi |u'|^2 \mathrm{d}r. \tag{3.4}$$

□

Corollary 3.2. *Assume that u_i, h_i satisfy $u_i = h_i u_{i+1}$, where $h_1(0) = 0$, $0 < h_i \in C^1(0,a]$, $(h_i^2)' = \phi_i^{-1}$, and $\phi_{i+1} = \phi_1 \prod_{j=1}^{i} h_j^2$ for $i = 1,\dots,k$. Then,*

$$\int_0^a \phi_1 |u_1'|^2 dr \geq \sum_{i=1}^{k} \int_0^a \phi_1 \left|\frac{h_i'}{h_i}\right|^2 u_1^2 dr + \lambda_1(\phi_{k+1}) \int_0^a \phi_1 |u_1|^2 dr. \tag{3.5}$$

Proof. Applying the method of iterations in Lemma 3.1, and terminating the iteration process by (3.4), we can finish our proof. □

Theorem 3.3. *Assume $f \in W_\phi^{1,p}$ is nonincreasing. If ϕ and ψ satisfy (2.5), then for $p \geq 2$,*

$$\frac{4(p-1)}{p^2} \int_0^a \phi_1 \left|\left(f_1^{p/2}\right)'\right|^2 dr + \left(\frac{p-1}{p}\right)^p \int_0^a \psi |f|^p dr \leq \int_0^a \phi |f'|^p dr, \tag{3.6}$$

where $\phi_1 = \phi h^2(-h')^{p-2}$, h satisfies

$$-\frac{h'}{h} = \frac{p-1}{p}\left(\frac{\psi}{\phi}\right)^{1/p}, \quad h > 0, \tag{3.7}$$

and $f_1 = f/h$.

Proof. Let $f \in X$ and $f = f_1 h$. Then $f' = f_1' h + h' f_1 \leq 0$ since $f' \leq 0$. As a result,

$$\frac{f_1' h}{h' f_1} \geq -1. \tag{3.8}$$

Therefore, in view of the inequality

$$(1+x)^p \geq 1 + px + (p-1)x^2, \quad p \geq 2,\ x \geq -1, \tag{3.9}$$

and (3.7), we have

$$\begin{aligned} I :&= \int_0^a \left[\phi |f'|^p - \left(\frac{p-1}{p}\right)^p \psi |f|^p\right] dr \\ &= \int_0^a \phi |-h'|^p |f_1|^p \left(1 + \frac{f_1' h}{h' f_1}\right)^p - \left(\frac{p-1}{p}\right)^p \psi |h|^p |f_1|^p dr \\ &\geq \int_0^a \phi |-h'|^p |f_1|^p \left[p \frac{f_1' h}{h' f_1} + (p-1)\left(\frac{f_1' h}{h' f_1}\right)^2\right] := I_1 + I_2. \end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned} I_1 &= p \int_0^a \phi(-h')^p |f_1|^p \frac{f_1' h}{h' f_1} dr \\ &= p \int_0^a \phi |f_1|^{p-1} f_1' (-h')^{p-1} h\, dr = -\int_0^a f_1^p \left(\phi h (-h')^{p-1}\right)' dr = 0 \end{aligned} \tag{3.11}$$

because

$$\left(\phi h (-h')^{p-1}\right)' = C \psi^{(p-1)/p} h^p \left(p \frac{h'}{h} \phi^{1/p} + (p-1)\psi^{1/p}\right) = 0. \tag{3.12}$$

For I_2, we have

$$I_2 := (p-1)\int_0^a \phi h^2(-h')^{p-2} f_1^{p-2}\,|\,f_1'\,|^2 \mathrm{d}r = \frac{4(p-1)}{p^2}\int_0^a \phi h^2(-h')^{p-2}\left|\left(f_1^{p/2}\right)'\right|^2 \mathrm{d}r. \tag{3.13}$$

This completes our proof. □

Corollary 3.4. *Assume $f \in W_\phi^{1,p}$ is nonincreasing. If ϕ and ψ satisfy (2.5), then for $p \geq 2$,*

$$\frac{4(p-1)}{p^2}\left[\lambda_1(\phi_{i+1})\int_0^a \phi\left(-\frac{h'}{h}\right)^{p-2} f^p \mathrm{d}r + \sum_{i=1}^{k}\int_0^a \phi_1\left|\frac{h_i'}{h_i}\right|^2\left(\frac{f}{h}\right)^2 \mathrm{d}r\right] + \left(\frac{p-1}{p}\right)^p \int_0^a \psi|\,f\,|^p \mathrm{d}r \leq \int_0^a \phi|\,f'\,|^p \mathrm{d}r, \tag{3.14}$$

where h satisfies (3.7), $\phi_1 = \phi h^2(-h')^{p-2}$, $h_1(0) = 0$, $0 < h_i \in C^1(0,a]$, $(h_i^2)' = \phi_i^{-1}$, and $\phi_{i+1} = \phi_1 \prod_{j=1}^{i} h_i^2$ for $i = 1,\ldots,k$.

Proof. In fact, denote $f_1^{p/2} = u_1$ in (3.6), then by (3.5), we can complete our proof. □

Theorem 3.5. *Assume $f \in W_\phi^{1,p}$ is nonincreasing. If ϕ, ψ satisfy (2.5), then for $p \geq 2$,*

$$\int_0^a \phi h^p\,|\,f_1'\,|^p \mathrm{d}r + \left(\frac{p-1}{p}\right)^p \int_0^a \psi|\,f\,|^p \mathrm{d}r \leq \int_0^a \phi|\,f'\,|^p \mathrm{d}r, \tag{3.15}$$

where $f_1 = f/h$ and $-h'/h = ((p-1)/p)(\psi/\phi)^{1/p}$.

Proof. Similar to Theorem 3.3, applying the following inequality instead of (3.9),

$$(1+y)^p \geq 1 + py + |\,y\,|^p, \quad y > -1, \tag{3.16}$$

we can prove that

$$\int_0^a \phi|\,f'\,|^p \mathrm{d}r - \left(\frac{p-1}{p}\right)^p \int_0^a \psi|\,f\,|^p \mathrm{d}r \geq \int_0^a \phi h^p\,|\,f_1'\,|^p \mathrm{d}r. \tag{3.17}$$

□

Corollary 3.6. *Assume $f \in W_\phi^{1,p}$ is nonincreasing. Then for $p \geq 2$,*

$$\lambda_1(\phi_{k+1})\int_0^a \phi|\,f\,|^p \mathrm{d}r + \left(\frac{p-1}{p}\right)^p \int_0^a \left(\psi + \sum_{i=1}^{k}\frac{\psi_i}{\prod_{j=1}^{i} h_{j-1}^p}\right)|\,f\,|^p \mathrm{d}r \leq \int_0^a \phi|\,f'\,|^p \mathrm{d}r, \tag{3.18}$$

where $h_0 = h$, $\phi_{i+1} = \phi_i h_i^p$, and ϕ_i, ψ_i satisfy $-h_i'/h_i = ((p-1)/p)(\psi_i/\phi_i)$ for $i = 1,\ldots,k$.

Proof. Since $f' \le 0$, $h > 0$, and $h' < 0$, we have $f_1' = (hf' + h'f)/h^2 \le 0$. Therefore we can apply (3.15) again. Set $\phi_1 = \phi h^p$ and $f_2 = f_1/h_1$, then

$$\begin{aligned}\int_0^a \phi_1 |f_1'|^p \mathrm{d}r &\ge \left(\frac{p-1}{p}\right)^p \int_0^a \psi_1 |f_1|^p \mathrm{d}r + \int_0^a \phi_1 h_1^p |f_2'|^p \mathrm{d}r \\ &\ge \left(\frac{p-1}{p}\right)^p \int_0^a \psi_1 \frac{|f|^p}{h^p} \mathrm{d}r + \lambda_1(\phi_2) \int_0^a \psi_1 h_1^p |f_2|^p \mathrm{d}r \\ &= \left(\frac{p-1}{p}\right)^p \int_0^a \frac{\psi_1}{h^p} |f|^p \mathrm{d}r + \lambda_1(\phi_2) \int_0^a \phi_1 h_1^p \frac{|f|^p}{h_1^p h^p} \mathrm{d}r \\ &= \left(\frac{p-1}{p}\right)^p \int_0^a \frac{\psi_1}{h^p} |f|^p \mathrm{d}r + \lambda_1(\phi_2) \int_0^a \phi |f|^p \mathrm{d}r,\end{aligned} \tag{3.19}$$

which shows that the corollary is true when $k = 1$ due to Theorem 3.5. For any k, we can prove our result by the induction argument. □

4. Hardy inequalities in Sobolev spaces

We denote $\mathrm{e}^{(k)} = e^{\cdot^{\cdot^{\cdot^{\mathrm{e}\ (k\ \text{times})}}}}$, $\ln^{(1)} = \ln$, and $\ln^{(j)} = \ln \ln^{(j-1)}$ for $j \ge 2$. $W_0^{1,p}(\Omega)$ is the completion space of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| = |u|_p + |Du|_p$.

Theorem 4.1. *Let $0 \in \Omega \subset B_T(0) \subset \mathbb{R}^n$, and $n > p \ge 2$. Then for any $u \in W_0^{1,p}(\Omega)$,*

$$\begin{aligned}&\frac{(p-1)(n-p)^{p-2}}{p^p} \left(\sum_{i=1}^k \int_\Omega \frac{|u|^p}{|x|^p \left(\prod_{j=1}^i \ln^{(j)} R/|x|\right)^2} \mathrm{d}x + 4\lambda(\phi_{k+1}) \int_\Omega \frac{|u|^p}{|x|^{p-2}} \mathrm{d}x \right) \\ &\quad + \left(\frac{n-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} \mathrm{d}x \le \int_\Omega |Du|^p \mathrm{d}x,\end{aligned} \tag{4.1}$$

where $\phi_k = r \prod_{j=1}^k \ln^{(j)} R/r$ with $R = e^{(k-1)} T$.

Proof. For $x \notin \Omega$, define $u(x) = 0$. Let $|u|^*$ be the symmetric decreasing rearrangement of function $|u|$. Now observe that for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(B_T(0))$ with $|u|^* > 0$ and radially nonincreasing, and hence inequality (3.6) holds for $|u|^*$. We know that

$$\int_\Omega |Du|^p \mathrm{d}x = \int_{\omega_n} \int_0^T |Du|^p r^{n-1} r \,\mathrm{d}r \mathrm{d}\omega, \tag{4.2}$$

where ω_n denotes the area of the unit ball in $\mathbb{R}^n$. Taking $f = |u|^*$ in Theorem 3.3, we obtain

$$\frac{4(p-1)}{p^2} \int_0^a \phi_1 \left| \left(f_1^{p/2}\right)' \right|^2 \mathrm{d}r + \left(\frac{p-1}{p}\right)^p \int_0^a \psi |f|^p \mathrm{d}r \le \int_0^a \phi |f'|^p \mathrm{d}r, \tag{4.3}$$

where $\phi_1 = \phi h^2(-h')^{p-2}$. Choose $u_1 = f_1^{p/2}$ and $\phi = r^{n-1}$ in Corollary 3.2. Then $\psi = ((n-p)/p)^p r^{n-p-1}$ by Theorem 2.1, and $h = r^{1-n/p}$ by (3.7). By the definition in Corollary 3.2, we see that $\phi_1 = ((n-p)/p)^{p-2} r$, $h_1 = (\ln R/r)^{1/2}$, $\phi_2 = ((n-p)/p)^{p-2} r \ln R/r$, $h_2 = (\ln^{(2)} R/r)^{1/2}$, and so on. Noting that $\phi_1 |u_1|^2 = ((n-p)/p)^{p-2} |f|^p r^{n-1}/r^{p-2}$, we obtain

$$\begin{aligned} &\frac{(p-1)(n-p)^{p-2}}{p^p}\left(\sum_{i=1}^{k}\int_0^a \frac{|f|^p r^{n-1}}{r^p\left(\prod_{j=1}^{i}\ln^{(j)} R/r\right)^2}\mathrm{d}r + 4\lambda(\phi_{k+1})\int_0^a \frac{|f|^p r^{n-1}}{r^{p-2}}\mathrm{d}r\right) \\ &\quad + \left(\frac{n-p}{p}\right)^p \int_0^a |f|^p r^{n-1}/r^p \mathrm{d}r \le \int_0^a |f'|^p r^{n-1}\mathrm{d}r. \end{aligned} \tag{4.4}$$

Integrating both sides of the above inequality with respect to ω_n, we know that our theorem holds for $|u|^*$. It is well known that the symmetrization does not change the L^p-norm, it decreases gradient norm and increases the integrals $\int_\Omega (|u|^p/|x|^p)\,\mathrm{d}x$ and $\int_\Omega (|u|^p/|x|^p(\ln R/|x|)^2)\,\mathrm{d}x$, and so on. Therefore we complete our proof. □

When $p = 2$ in Theorem 4.1, we have the following theorem.

Theorem 4.2. *Let $0 \in \Omega \subset B_T(0) \subset \mathbb{R}^n$. Then for any $u \in W_0^{1,p}(\Omega)$,*

$$\begin{aligned} &\frac{1}{4}\sum_{i=1}^{k}\int_\Omega \frac{|u|^p}{|x|^p\left(\prod_{j=1}^{i}\ln^{(j)} R/|x|\right)^2}\mathrm{d}x + \lambda(\phi_{k+1})\int_\Omega |u|^2\mathrm{d}x \\ &\quad + \left(\frac{n-2}{2}\right)^2\int_\Omega \frac{|u|^2}{|x|^2}\mathrm{d}x \le \int_\Omega |Du|^2\mathrm{d}x, \end{aligned} \tag{4.5}$$

where $\phi_k = r\prod_{j=1}^{k}\ln^{(j)} R/r$ with $R = e^{(k-1)}T$.

Proof. This theorem can be proved by using Corollary 3.2 and the symmetrization process. □

Remark 4.3. Inequality (4.5) gives a positive answer to the Brézis-Vázquez conjecture, that is, $\lambda(\phi_{k+1})\int_\Omega |u|^2\,\mathrm{d}x$ and $((n-2)/2)^2\int_\Omega (|u|^2/|x|^2)\,\mathrm{d}x$ are two terms of a series indeed. Similarly, Theorems 4.1 and 4.2 show the correctness of the conjecture of Brézis and Vázquez.

Theorem 4.4. *Let $0 \in \Omega \subset B_T(0) \subset \mathbb{R}^n$, $n = p \ge 2$. Then for any $u \in W_0^{1,p}(\Omega)$,*

$$\begin{aligned} &\frac{4(p-1)^{p-1}}{p^p}\lambda_1(\phi_k)\int_\Omega \frac{|u|^p}{|x|^{p-2}\ln^{(p-2)} R/|x|}\mathrm{d}x \\ &\quad + \frac{(p-1)^{p-1}}{p^p}\sum_{j=2}^{k}\int_\Omega \frac{|u|^p}{|x|^p\ln^p R/|x|\prod_{i=2}^{j}\left(\ln^{(i)} R/|x|\right)^2}\mathrm{d}x \\ &\quad + \left(\frac{p-1}{p}\right)^p\int_\Omega \frac{|u|^p}{|x|^p\ln^p R/|x|}\mathrm{d}x \le \int_\Omega |Du|^p\mathrm{d}x, \end{aligned} \tag{4.6}$$

where $\phi_k = r\prod_{j=1}^{k}\ln(j)R/r$ and $R = \mathrm{e}^{(k-1)}T$.

Proof. Taking $\phi = r^{p-1}$, $\psi = 1/r(\ln R/r)^p$, $h = (\ln R/r)^{(p-1)/p}$, $\phi_1 = ((p-1)/p)^{p-2} r \ln R/r$, and $h_1 = (\ln^{(2)} R/r)^{1/2}$, we can complete our proof by Theorem 3.3 and Corollary 3.2 and using the symmetrization process similar to Theorem 4.1. □

Similar to Theorems 4.1 and 4.4, by using Theorem 3.5 instead of Theorem 3.3, we can obtain the following two theorems.

THEOREM 4.5. *Let $0 \in \Omega \subset B_T(0) \subset \mathbb{R}^n$, $n > p \geq 2$. Then for any $u \in W_0^{1,p}(\Omega)$,*

$$\begin{aligned} &\lambda_1(\phi_{k+1}) \int_\Omega |u|^p \,\mathrm{d}x \\ &\quad + \left(\frac{n-p}{p}\right)^p \int_\Omega \left(\frac{|u|^p}{|x|^p} + \sum_{j=1}^{k} \frac{|u|^p}{|x|^p \prod_{i=1}^{j} (\ln^{(i)} R/|x|)^p}\right) \mathrm{d}x \leq \int_\Omega |Du|^p \,\mathrm{d}x, \end{aligned} \tag{4.7}$$

where $\phi_k = ((n-p)/p)^2 r \prod_{j=1}^{k-1} (\ln^{(j)} R/r)$ and $R = \mathrm{e}^{(k-1)} T$.

THEOREM 4.6. *Let $0 \in \Omega \subset B_T(0) \subset \mathbb{R}^n$, $n = p \geq 2$. Then for any $u \in W_0^{1,p}(\Omega)$,*

$$\lambda_1(\phi_{k+1}) \int_\Omega |u|^p \,\mathrm{d}x + \left(\frac{p-1}{p}\right)^p \int_\Omega \sum_{j=1}^{k} \frac{|u|^p}{|x|^p \prod_{i=1}^{j} (\ln^{(i)} R/|x|)^p} \,\mathrm{d}x \leq \int_\Omega |Du|^p \,\mathrm{d}x, \tag{4.8}$$

where $\phi_k = r^{p-1} \prod_{i=1}^{k} (\ln^{(j)} R/r)^{p-1}$ and $R = \mathrm{e}^{(k-1)} T$.

Remark 4.7. The above theorems show that the Brézis-Vázquez conjecture is true for $n \geq p \geq 2$.

Now we consider the following weighted eigenvalue problem with a critical singular weight

$$\begin{aligned} -\operatorname{div}\left(|Du|^{p-2} Du\right) + \mu \frac{|u|^{p-2} u}{|x|^p (\ln R/|x|)^p} &= \lambda |u|^{p-2} u f(x), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{4.9}$$

where $n = p \geq 2$, $0 < \mu < (p/p-1)^p$, and $\lambda \in \mathbb{R}$. We look for a weak solution $u \in W_0^{1,p}(\Omega)$ of this problem and study asymptotic behavior of the first eigenvalues for different singular weights as μ increases to $(p/(p-1))^p$, after which the operator L_μ is no more bound from below. Let

$$\lambda_1(f, \mu) = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|Du|^p - \mu |u|^p / |x|^p) \,\mathrm{d}x}{\int_\Omega |u|^p f \,\mathrm{d}x} \tag{4.10}$$

and denote $\lambda_1(f) = \lambda_1(f, (p/(p-1))^p)$.

For $f(x) \in L_{\mathrm{loc}}^\infty(\Omega \setminus \{0\})$ with $f(x) > 0$, let $L^p(\Omega, f)$ be the set of all real-valued measurable functions $u(x)$ defined on Ω such that $f(x)|u(x)|^p$ is integrable over Ω. We define the norm in $L^p(\Omega, f)$ as $|u|_{p,f}^p = \int_\Omega f(x)|u|^p \,\mathrm{d}x$.

Theorem 4.8. *The above problem admits a positive weak solution $u \in W_0^{1,p}(\Omega)$, corresponding to the first eigenvalue $\lambda_1(f,\mu)$. Moreover, as μ increases to $(p/(p-1))^p$, $\lambda_1(f,\mu) \to \lambda_1(f) \geq 0$ for all $f \in \mathcal{R}_p$ and the limit $\lambda_1(f) > 0$, where*

$$\mathcal{R}_p = \left\{ f : \Omega \longrightarrow \mathbb{R}^+ \;\middle|\; f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}),\ \limsup_{|x|\to 0} f(x)|x|^p \left(\ln \frac{1}{|x|}\right)^p \left(\ln^{(2)} \frac{1}{|x|}\right)^2 < +\infty \right\}. \tag{4.11}$$

First we prove the following lemma.

Lemma 4.9. *If $f \in \mathcal{R}_p$, then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ is compact.*

Proof. If $f \in \mathcal{R}_p$, we have

$$\lim_{\epsilon \to 0} \sup_{x \in B_{\epsilon(0)}} f(x)|x|^p \left(\ln \frac{1}{|x|}\right)^p \left(\ln^{(2)} \frac{1}{|x|}\right)^2 < +\infty, \tag{4.12}$$

and hence for sufficiently small ϵ,

$$f(x) < \frac{C}{|x|^p (\ln 1/|x|)^p (\ln^{(2)} 1/|x|)^2} \quad \text{in } B_\epsilon = B_{\epsilon(0)}. \tag{4.13}$$

Let $u_m \subset W_0^{1,p}(\Omega)$ be bounded. Then there exists a subsequence, still denoted by u_m, $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_m \to u$ in L^p. By (4.13), we have

$$\begin{aligned} \int_\Omega |u_m - u|^p f(x)\,\mathrm{d}x \leq C \int_{B_\epsilon(0)} \frac{|u_m - u|^p}{|x|^p (\ln 1/|x|)^p (\ln^{(2)} 1/|x|)^2} \mathrm{d}x \\ + C \int_\Omega |u_m - u|^p \,\mathrm{d}x. \end{aligned} \tag{4.14}$$

By Theorem 4.4, we have

$$\begin{aligned} \int_{B_\epsilon(0)} \frac{|u_m - u|^p}{|x|^p (\ln 1/|x|)^p (\ln^{(2)} 1/|x|)^2} \mathrm{d}x &\leq \frac{1}{(\ln^{(2)} 1/\epsilon)^2} \int_{B_\epsilon(0)} \frac{|u_m - u|^p}{|x|^p |\ln 1/|x||^p} \mathrm{d}x \\ &\leq \frac{C}{(\ln^{(2)} 1/\epsilon)^2} |Du_m - Du|_p^p \longrightarrow 0 \end{aligned} \tag{4.15}$$

as $\epsilon \to 0$. Hence the proof follows. □

Proof of Theorem 4.8. We look for the critical points of the functional

$$I_\mu(u) = \frac{1}{p} \int_\Omega |Du|^p \,\mathrm{d}x - \frac{\mu}{p} \int_\Omega \frac{|u|^p}{|x|^p (\ln R/|x|)^p} \mathrm{d}x \tag{4.16}$$

which is Gateaux differentiable and coercive on $W_0^{1,p}(\Omega)$. We minimize this functional I_μ over the manifold $M = \{u \in W_0^{1,p}(\Omega) \mid \int_\Omega f(x)|u|^p \mathrm{d}x\,\mathrm{d}x = 1\}$ and let $\lambda_1(f,\mu)$ be the infimum. We can choose a special minimizing sequence $u_m \in M$ with $I_\mu(u_m) \to \lambda_1(f,\mu)$ and component of $DI_\mu(u_m)$ restricted to M tends to 0 strongly in $W_0^{-1,p'}(\Omega)$. By Theorem 4.4 and $\mu < (p/(p-1))$, we know that $\{u_m\}$ is a bounded sequence, hence we have for a subsequence, still denoted by $\{u_m\}$, $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_m \rightharpoonup u$ in $L^p(\Omega, |x|^{-p}(\ln R/|x|)^{-p})$, and $u_m \to u$ in $L^p(\Omega)$. By Lemma 4.9, we have $W_0^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega, f)$, hence $u \in M$. Further, u_m satisfies in $D'(\Omega)$

$$-\left(\operatorname{div}\left(|Du_m|^{p-2}Du_m\right) + \mu\frac{|u_m|^{p-2}u_m}{|x|^p(\ln R/|x|)^p}\right) = \lambda_m\left(|u_m|^{p-2}u_m f\right) + f_m, \tag{4.17}$$

where $f_m \to 0$ in $W_0^{-1,p'}(\Omega)$ and $\lambda_m \to \lambda$ as $m \to \infty$. By Theorem 2.1 in [2], we have $Du_m \to Du$ almost everywhere in Ω. By Brézis-Leib Lemma [3], we have

$$\begin{aligned}\lambda_{1,\mu}(f,\mu) &= |Du_m - Du|_p^p - \mu|u_m - u|_{p,|x|^{-p}\ln^{-p}R/|x|}^p \\ &\quad + |Du|_p^p - \mu|u|_{p,|x|^{-p}\ln^{-p}R/|x|}^p + o(1) \\ &\geq \left(\left(\frac{p-1}{p}\right)^p - \mu\right)|u|_{p,|x|^{-p}\ln^{-p}R/|x|}^p + \lambda_1(f,\mu) + o(1),\end{aligned} \tag{4.18}$$

hence we have $I_\mu(u) = \lambda_1(f,\mu)$. By Theorem 2.1 in [2], we conclude that u is a weak solution of (4.9) corresponding to $\lambda = \lambda_1(f,\mu)$. Similar to [1], we have $\lambda_1(f,\mu) \to \lambda_1(f)$. □

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