SOME INEQUALITIES FOR SUMS OF NONNEGATIVE DEFINITE MATRICES IN QUATERNIONS

YONGGE TIAN AND GEORGE P. H. STYAN

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Some matrix versions of the Cauchy-Schwarz and Frucht-Kantorovich inequalities are established over the quaternionic algebra. As applications, a group of inequalities for sums of Hermitian nonnegative definite matrices over the quaternionic algebra are derived.

Let $a = a_0 + a_1 i + a_2 j + a_3 k$ be a quaternion, where $a_0, \ldots, a_3$ are numbers from the real field $\mathbb{R}$ and the three imaginary units i, j, and k satisfy

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j. \tag{1}$$

The collection of all quaternions is denoted by $\mathbb{H}$ and is called the real quaternionic algebra. This algebra was first introduced by Hamilton in 1843 (see [5, 6]), and is often called the Hamilton quaternionic algebra.

It is well known that $\mathbb{H}$ is an associative division algebra over $\mathbb{R}$. For any $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$, the conjugate of $a = a_0 + a_1 i + a_2 j + a_3 k$ is defined to be $\overline{a} = a_0 - a_1 i - a_2 j - a_3 k$, which satisfies

$$\overline{\overline{a}} = a, \qquad \overline{a+b} = \overline{a} + \overline{b}, \qquad \overline{ab} = \overline{b}\,\overline{a} \tag{2}$$

for all $a, b \in \mathbb{H}$. The norm of a is defined to be $|a| = \sqrt{a\overline{a}} = \sqrt{\overline{a}a} = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. Let $A = (a_{st})$ be an $m \times n$ matrix over $\mathbb{H}$, where $a_{st} \in \mathbb{H}$. The conjugate transpose of A is defined to be $A^* = (\overline{a_{ts}})$. A square matrix A over $\mathbb{H}$ is called Hermitian if $A^* = A$. General properties of matrices over $\mathbb{H}$ can be found in [13, 18].

Because $\mathbb{H}$ is noncommutative, one cannot directly extend various results on complex numbers to quaternions. On the other hand, $\mathbb{H}$ is known to be algebraically isomorphic to the two matrix algebras consisting of

$$\psi(a) \overset{\text{def}}{=} \begin{bmatrix} a_0 + a_1 i & -(a_2 + a_3 i) \\ a_2 - a_3 i & a_0 - a_1 i \end{bmatrix} \in \mathbb{C}^{2\times 2}, \qquad \phi(a) \overset{\text{def}}{=} \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \tag{3}$$

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respectively. Moreover, it is shown in [13] that the diagonal matrix $\operatorname{diag}(a,a)$ satisfies the following universal similarity factorization equality (USFE):

$$P\operatorname{diag}(a,a)P^{*} = \psi(a), \tag{4}$$

where

$$P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -j & k \end{bmatrix} \tag{5}$$

is a unitary matrix over $\mathbb{H}$, that is, $PP^{*} = P^{*}P = I_2$; the diagonal matrix $\operatorname{diag}(a,a,a,a)$ satisfies the following USFE:

$$Q\operatorname{diag}(a,a,a,a)Q^{*} = \phi(a), \tag{6}$$

where the matrix Q has the following independent expression:

$$Q = Q^{*} = \frac{1}{2}\begin{bmatrix} 1 & i & j & k \\ -i & 1 & k & -j \\ -j & -k & 1 & i \\ -k & j & -i & 1 \end{bmatrix}, \tag{7}$$

which is a unitary matrix over $\mathbb{H}$.

The two equalities in (4) and (6) reveal two fundamental facts that the quaternion a is an eigenvalue of multiplicity two for the complex matrix $\psi(a)$ and an eigenvalue of multiplicity four for the real matrix $\phi(a)$.

In general, for any $m \times n$ matrix $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}^{m\times n}$, where $A_0,\dots,A_3 \in \mathbb{R}^{m\times n}$, the block-diagonal matrix $\operatorname{diag}(A,A)$ satisfies the following universal factorization equality:

$$P_{2m}\operatorname{diag}(A,A)P_{2n}^{*} = \begin{bmatrix} A_0 + A_1 i & -(A_2 + A_3 i) \\ A_2 - A_3 i & A_0 - A_1 i \end{bmatrix} \stackrel{\text{def}}{=} \Psi(A) \in \mathbb{C}^{2m\times 2n}, \tag{8}$$

where P_{2m} and P_{2n}^{*} are the following two unitary matrices over $\mathbb{H}$:

$$P_{2m} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_m & -iI_m \\ -jI_m & kI_m \end{bmatrix}, \qquad P_{2n}^{*} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & jI_n \\ iI_n & -kI_n \end{bmatrix}. \tag{9}$$

In particular, if $m = n$, then (8) becomes a USFE over $\mathbb{H}$. Let $A = A_0 + A_1 i + A_2 j + A_3 k \in \mathbb{H}^{m\times n}$, where $A_0,\dots,A_3 \in \mathbb{R}^{m\times n}$. Then the block-diagonal matrix $\operatorname{diag}(A,A,A,A)$ satisfies the following universal factorization equality:

$$Q_{4m}\operatorname{diag}(A,A,A,A)Q_{4n}^{*} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix} \stackrel{\text{def}}{=} \Phi(A) \in \mathbb{R}^{4m\times 4n}, \tag{10}$$

where Q_{4t} is the following unitary matrix over $\mathbb{H}$:

$$Q_{4t} = Q_{4t}^* = \frac{1}{2}\begin{bmatrix} I_t & iI_t & jI_t & kI_t \\ -iI_t & I_t & kI_t & -jI_t \\ -jI_t & -kI_t & I_t & iI_t \\ -kI_t & jI_t & -iI_t & I_t \end{bmatrix}, \quad t = m, n. \tag{11}$$

In particular, if $m = n$, then (10) becomes a USFE over $\mathbb{H}$. Result (10) was also shown in Tian [13] in the investigation of various universal block-matrix factorizations. The two universal block-matrix factorizations in (8) and (10) can be used to extend various results in complex and real matrix theory to quaternionic matrices.

For a general $m \times n$ matrix A over $\mathbb{C}$, the Moore-Penrose inverse $A^\dagger$ of A is defined to be the unique $n \times m$ matrix X satisfying the four Penrose equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. General properties of the Moore-Penrose inverse can be found in [2, 3].

The Moore-Penrose inverse $A^\dagger$ of a matrix A over $\mathbb{H}$ is defined to be the matrix X over $\mathbb{H}$ satisfying the four Penrose equations $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$. The existence and uniqueness of $A^\dagger$ of A over $\mathbb{H}$ can be shown through the following Lemma 1(g).

Some consequences derived from (8) and (10) are given below, which will be used in the sequel.

Lemma 1. *Let* $A, B \in \mathbb{H}^{m\times n}$, $C \in \mathbb{H}^{n\times p}$, *and* $\lambda \in \mathbb{R}$. *Then*

(a) $A = B \Leftrightarrow \Psi(A) = \Psi(B)$;
(b) $\Psi(A+B) = \Psi(A) + \Psi(B)$;
(c) $\Psi(AC) = \Psi(A)\Psi(C)$;
(d) $\Psi(\lambda A) = \Psi(A\lambda) = \lambda\Psi(A)$;
(e) $\Psi(A^*) = \Psi^*(A)$;
(f) *if A is nonsingular, then* $\Psi(A^{-1}) = \Psi^{-1}(A)$ *and* $A^{-1} = (1/2)E_{2m}\Psi^{-1}(A)E_{2m}^*$, *where* $E_{2m} = [I_m, jI_m]$;
(g) $A^\dagger$ *satisfies* $\Psi(A^\dagger) = \Psi^\dagger(A)$ *and* $A^\dagger = (1/2)E_{2n}\Psi^\dagger(A)E_{2m}^*$.

The two factorizations in (8) and (10) enable us to extend various results on real and complex matrices into quaternionic matrices. In the past several years, various inequalities for quaternions and matrices in quaternions were considered; see, for example, [11, 12, 15, 16, 17, 19]. In this paper, we will consider some basic matrix inequalities in Löwner partial ordering over $\mathbb{H}$. As applications, we give a group of matrix inequalities for sums of Hermitian nonnegative definite matrices over $\mathbb{H}$.

In complex matrix analysis, two Hermitian matrices A and B of the same order are said to satisfy the Löwner partial ordering $A \leqslant B$ if $B - A$ is nonnegative definite. It was shown in Marshall and Olkin [9] that if the complex matrix A of order n is Hermitian positive definite with its eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0$, while an $n \times p$ complex matrix X satisfies $X^*X = I_p$, then

$$(X^*AX)^{-1} \leqslant X^*A^{-1}X \leqslant \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}(X^*AX)^{-1}. \tag{12}$$

Various extensions of (12) for complex matrices are also investigated in the literature (see, e.g., [1, 4, 7, 8, 9, 10]).

LEMMA 2. *Let $A \in \mathbb{C}^{n\times n}$ be a nonnull Hermitian nonnegative definite matrix with rank $r \leqslant n$ and the r positive eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and let X be an $n\times p$ complex matrix. Then*

$$X^*P_AX(X^*AX)^{\dagger}X^*P_AX \leqslant X^*A^{\dagger}X \leqslant \frac{(\lambda_1+\lambda_r)^2}{4\lambda_1\lambda_r}X^*P_AX(X^*AX)^{\dagger}X^*P_AX, \tag{13}$$

where $P_A = AA^{\dagger}$ is the orthogonal projector onto the range (column space) of A.

The inequality on the left-hand side of (13) was first given by Baksalary and Puntanen [1], the inequality on the right-hand side of (13) was established by Drury et al. [4]. The left-hand side of (13) was extended to a more general situation by Pečarić et al. [10] as follows.

LEMMA 3. *Let $A \in \mathbb{C}^{n\times n}$ be a nonnegative definite matrix and let $P \in \mathbb{C}^{n\times p}$ and $Q \in \mathbb{C}^{n\times q}$. Then*

$$\begin{gathered} Q^*AQ \geqslant Q^*AP(P^*AP)^{\dagger}P^*AQ, \\ \operatorname{rank}\left[Q^*AQ - Q^*AP(P^*AP)^{\dagger}P^*AQ\right] = \operatorname{rank}[AP,AQ] - \operatorname{rank}(AP). \end{gathered} \tag{14}$$

Moreover, the following statements are equivalent:

(a) *the equality in (14) holds;*

(b) *Range $(AQ) \subseteq$ Range (AP), that is, there is a Z such that $APZ = AQ$;*

(c) *$AQ = AP(P^*AP)^{\dagger}P^*AQ$.*

The following general result was shown in [14].

LEMMA 4. *Let $A_1,\ldots,A_k \in \mathbb{C}^{n\times n}$ be Hermitian nonnegative definite matrices, and let $N_1,\ldots,$ $N_k \in \mathbb{C}^{n\times p}$. Then*

$$\sum_{i=1}^{k}N_i^*A_iN_i \geqslant \left(\sum_{i=1}^{k}A_iN_i\right)^*\left(\sum_{i=1}^{k}A_i\right)^{\dagger}\left(\sum_{i=1}^{k}A_iN_i\right), \tag{15}$$

with equality if and only if there is a Z such that $A_iZ = A_iN_i$, $i = 1,\ldots,k$. Furthermore, let $X_1,\ldots,X_k \in \mathbb{C}^{n\times q}$. Then

$$\sum_{i=1}^{k}N_i^*A_iN_i \geqslant \left(\sum_{i=1}^{k}X_i^*A_iN_i\right)^*\left(\sum_{i=1}^{k}X_i^*A_iX_i\right)^{\dagger}\left(\sum_{i=1}^{k}X_i^*A_iN_i\right), \tag{16}$$

with equality if and only if there is a Z such that $(A_iX_i)Z = A_iN_i$, $i = 1,\ldots,k$.

In this paper, we consider the extensions of the above inequalities to quaternionic matrices. It is well known that any Hermitian matrix $A \in \mathbb{H}^{n\times n}$ can be decomposed as $A = PJP^*$, where $P \in \mathbb{H}^{n\times n}$ satisfies $PP^* = P^*P = I_n$ and J is a real diagonal matrix, the entries in J are called the eigenvalues of A; see, for example, Zhang [18]. If the diagonal entries in J are nonnegative, A is said to be nonnegative definite. If the diagonal entries of J are all positive, A is said to be positive definite.

From Lemma 1(a) and (e), one derive the following simple result.

Lemma 5. *Let $A \in \mathbb{H}^{n\times n}$. Then A is Hermitian if and only if $\Psi(A)$ is Hermitian; A is Hermitian nonnegative definite (positive definite) if and only if $\Psi(A)$ is Hermitian nonnegative definite (positive definite).*

Two Hermitian nonnegative definite matrices $A, B \in \mathbb{H}^{n\times n}$ are said to satisfy the matrix inequality $A \leqslant B$ in Löwner partial ordering if $B - A$ is nonnegative definite.

Our main results on matrix inequalities in Löwner partial ordering are presented below.

Theorem 6. *Let $A \in \mathbb{H}^{n\times n}$ be a nonnull Hermitian nonnegative definite matrix with rank$(A) = r \leqslant n$, the r positive eigenvalues of A be $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, and let $X \in \mathbb{H}^{n\times p}$. Then*

$$X^*P_AX(X^*AX)^{\dagger}X^*P_AX \leqslant X^*A^{\dagger}X \leqslant \frac{(\lambda_1+\lambda_r)^2}{4\lambda_1\lambda_r}X^*P_AX(X^*AX)^{\dagger}X^*P_AX, \tag{17}$$

where $P_A = AA^{\dagger}$ is the orthogonal projector onto the range of A.

Proof. Since the r positive eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$, A can be decomposed as $A = PJP^*$, where $PP^* = P^*P = I_n$, $J = \mathrm{diag}(\lambda_1,\dots,\lambda_r,0,\dots,0)$. Thus, $\Psi(A) = \Psi(P)\Psi(J)\Psi^*(P)$ and $\Psi(P)\Psi^*(P) = \Psi^*(P)\Psi(P) = I_{2n}$. This implies that $\Psi(A)$ is a Hermitian nonnegative definite matrix over $\mathbb{C}$. Note that the diagonal elements of $\Psi(J)$ are eigenvalues of $\Psi(A)$ and that the maximum and minimum positive eigenvalues of $\Psi(A)$ are λ_1 and λ_r, respectively. Thus

$$\begin{aligned}&\Psi^*(X)P_{\Psi(A)}\Psi(X)\big[\Psi^*(X)\Psi(A)\Psi(X)\big]^{\dagger}\Psi^*(X)P_{\Psi(A)}\Psi(X)\\&\quad\leqslant \Psi^*(X)\Psi^{\dagger}(A)\Psi(X)\\&\quad\leqslant \frac{(\lambda_1+\lambda_r)^2}{4\lambda_1\lambda_r}\Psi^*(X)P_{\Psi(A)}\Psi(X)\big[\Psi^*(X)\Psi(A)\Psi(X)\big]^{\dagger}\Psi^*(X)P_{\Psi(A)}\Psi(X).\end{aligned} \tag{18}$$

Applying Lemma 1(c), (d), (e), and (g) to (18) gives

$$\begin{aligned}\Psi\big[X^*P_AX(X^*AX)^{\dagger}X^*P_AX\big] &\leqslant \Psi(X^*A^{\dagger}X)\\&\leqslant \frac{(\lambda_1+\lambda_r)^2}{4\lambda_1\lambda_r}\Psi\big[X^*P_AX(X^*AX)^{\dagger}X^*P_AX\big].\end{aligned} \tag{19}$$

Applying Lemma 5 to (19) gives (17). □

Similarly, one can derive from Lemmas 3, 4, and 5 the following two theorems.

Theorem 7. *Let $A \in \mathbb{H}^{n\times n}$ be a nonnegative definite matrix and let $P \in \mathbb{H}^{n\times p}$ and $Q \in \mathbb{H}^{n\times q}$. Then*

$$Q^*AQ \geqslant Q^*AP(P^*AP)^{\dagger}P^*AQ, \tag{20}$$

and with equality in (20) if and only if $AQ = AP(P^*AP)^{\dagger}P^*AQ$.

Theorem 8. *Let $A_1,\dots,A_k \in \mathbb{H}^{n\times n}$ be Hermitian nonnegative definite matrices and let $N_1,\dots,N_k \in \mathbb{H}^{n\times p}$. Then*

$$\sum_{i=1}^{k} N_i^*A_iN_i \geqslant \left(\sum_{i=1}^{k} A_iN_i\right)^* \left(\sum_{i=1}^{k} A_i\right)^{\dagger} \left(\sum_{i=1}^{k} A_iN_i\right), \tag{21}$$

with equality if and only if there is a Z such that $A_iZ = A_iN_i$, $i = 1,\dots,k$. Furthermore, let $X_1,\dots,X_k \in \mathbb{H}^{n\times q}$. Then

$$\sum_{i=1}^{k} N_i^*A_iN_i \geqslant \left(\sum_{i=1}^{k} X_i^*A_iN_i\right)^* \left(\sum_{i=1}^{k} X_i^*A_iX_i\right)^{\dagger} \left(\sum_{i=1}^{k} X_i^*A_iN_i\right), \tag{22}$$

with equality if and only if there is a Z such that $(A_iX_i)Z = A_iN_i$, $i = 1,\dots,k$.

Various special cases can be derived from (17), (20), (21), and (22). For example, letting $N_i = A_i$, $i = 1,\dots,k$ in (21) gives

$$\sum_{i=1}^{k} A_i^3 \geqslant \left(\sum_{i=1}^{k} A_i^2\right) \left(\sum_{i=1}^{k} A_i\right)^{\dagger} \left(\sum_{i=1}^{k} A_i^2\right), \tag{23}$$

with equality if and only if there is a Z such that $A_iZ = A_i^2$, $i = 1,\dots,k$; letting $N_i = I_n$ and $X_i = A_i$, $i = 1,\dots,k$ in (22) gives

$$\sum_{i=1}^{k} A_i \geqslant \left(\sum_{i=1}^{k} A_i^2\right) \left(\sum_{i=1}^{k} A_i^3\right)^{\dagger} \left(\sum_{i=1}^{k} A_i^2\right), \tag{24}$$

with equality if and only if there is a Z such that $A_i^2Z = A_i$, $i = 1,\dots,k$. Letting $N_i = A_i^t$, $i = 1,\dots,k$ in (21), where t is a positive integer, yields

$$\sum_{i=1}^{k} A_i^{2t+1} \geqslant \left(\sum_{i=1}^{k} A_i^{t+1}\right) \left(\sum_{i=1}^{k} A_i\right)^{\dagger} \left(\sum_{i=1}^{k} A_i^{t+1}\right), \tag{25}$$

with equality if and only if there is a Z such that $A_iZ = A_i^{t+1}$, $i = 1,\dots,k$. Its dual inequality by (22) is

$$\sum_{i=1}^{k} A_i \geqslant \left(\sum_{i=1}^{k} A_i^{t+1}\right) \left(\sum_{i=1}^{k} A_i^{2t+1}\right)^{\dagger} \left(\sum_{i=1}^{k} A_i^{t+1}\right), \tag{26}$$

with equality if and only if there is a Z such that $A_i^{t+1}Z = A_i$, $i = 1,\dots,k$.

If A_i is Hermitian positive definite and $N_i = A_i^{-1}B_i$, $i = 1,\dots,k$, then (21) becomes

$$\sum_{i=1}^{k} B_i^* A_i^{-1} B_i \geqslant \left(\sum_{i=1}^{k} B_i\right)^* \left(\sum_{i=1}^{k} A_i\right)^{-1} \left(\sum_{i=1}^{k} B_i\right), \tag{27}$$

with equality if and only if $A_1^{-1}B_1 = \cdots = A_k^{-1}B_k$. Its dual inequality by (22) is

$$\sum_{i=1}^{k} A_i \geqslant \left(\sum_{i=1}^{k} B_i\right) \left(\sum_{i=1}^{k} B_i^* A_i^{-1} B_i\right)^{\dagger} \left(\sum_{i=1}^{k} B_i\right)^*, \tag{28}$$

with equality if and only if there is a Z such that $B_iZ = A_i$, $i = 1,\dots,k$.

Letting $N_i = A_i^{\dagger}$, $i = 1,\dots,k$ in (21) yields

$$\sum_{i=1}^{k} A_i^{\dagger} \geqslant \left(\sum_{i=1}^{k} P_{A_i}\right) \left(\sum_{i=1}^{k} A_i\right)^{\dagger} \left(\sum_{i=1}^{k} P_{A_i}\right), \tag{29}$$

with equality if and only if there is a Z such that $A_iZ = P_{A_i}$, $i = 1,\dots,k$.

Letting $N_i = A_i^{\dagger}X_i$, $i = 1,\dots,k$ in (22) gives

$$\sum_{i=1}^{k} X_i^* A_i^{\dagger} X_i \geqslant \left(\sum_{i=1}^{k} X_i^* P_{A_i} X_i\right) \left(\sum_{i=1}^{k} X_i^* A_i X_i\right)^{\dagger} \left(\sum_{i=1}^{k} X_i^* P_{A_i} X_i\right), \tag{30}$$

with equality if and only if there is a Z such that $(A_iX_i)Z = A_iA_i^{\dagger}X_i$, $i = 1,\dots,k$. In particular, if all A_i are Hermitian positive definite, then

$$\sum_{i=1}^{k} X_i^* A_i^{-1} X_i \geqslant \left(\sum_{i=1}^{k} X_i^* X_i\right) \left(\sum_{i=1}^{k} X_i^* A_i X_i\right)^{\dagger} \left(\sum_{i=1}^{k} X_i^* X_i\right), \tag{31}$$

with equality if and only if there is a Z such that $(A_iX_i)Z = X_i$, $i = 1,\dots,k$. The above inequality can be written equivalently as

$$\sum_{i=1}^{k} X_i^* A_i X_i \geqslant \left(\sum_{i=1}^{k} X_i^* X_i\right) \left(\sum_{i=1}^{k} X_i^* A_i^{-1} X_i\right)^{\dagger} \left(\sum_{i=1}^{k} X_i^* X_i\right), \tag{32}$$

with equality if and only if there is a Z such that $X_iZ = A_iX_i$, $i = 1,\dots,k$.

Letting $X_i = \sqrt{w_i}I_n$, $i = 1,\dots,k$ with $\sum_{i=1}^{k} w_i = 1$ in the above inequality gives

$$\sum_{i=1}^{k} w_i A_i^{\dagger} \geqslant \left(\sum_{i=1}^{k} w_i P_{A_i}\right) \left(\sum_{i=1}^{k} w_i A_i\right)^{\dagger} \left(\sum_{i=1}^{k} w_i P_{A_i}\right), \tag{33}$$

with equality if and only if there is a Z such that $A_iZ = A_iA_i^{\dagger}$, $i = 1,\dots,k$. In particular,

$$w_1A_1^{-1} + \cdots + w_kA_k^{-1} \geqslant (w_1A_1 + \cdots + w_kA_k)^{-1}, \tag{34}$$

with equality if and only if $A_1 = \cdots = A_k$.

Theorem 9. *Let $A_1,\ldots,A_k \in \mathbb{H}^{n\times n}$ be nonnull Hermitian nonnegative definite matrices. Then*

$$\sum_{i=1}^{k} A_i^{\dagger} \leqslant \frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{k} P_{A_i}\right)\left(\sum_{i=1}^{k} A_i\right)^{\dagger}\left(\sum_{i=1}^{k} P_{A_i}\right), \tag{35}$$

where M and m are, respectively, the maximum and minimum positive eigenvalues of A_1, $\ldots,A_k$.

In fact, let $A = \operatorname{diag}(A_1,\ldots,A_k)$ and $X = [I_n,\ldots,I_n]$. Then $X^*P_AX = P_{A_1} + P_{A_2} + \cdots + P_{A_k}$, $X^*AX = A_1 + \cdots + A_k$, and $X^*A^{\dagger}X = A_1^{\dagger} + \cdots + A_k^{\dagger}$. In this case, the right-hand side of (17) becomes (35).

Combining (29) and (35) yields a two-side inequality for the sum $\sum_{i=1}^{k} A_i^{\dagger}$

$$S\left(\sum_{i=1}^{k} A_i\right)^{\dagger} S \leqslant \sum_{i=1}^{k} A_i^{\dagger} \leqslant \frac{(m+M)^2}{4mM} S\left(\sum_{i=1}^{k} A_i\right)^{\dagger} S, \tag{36}$$

where $S = \sum_{i=1}^{k} AA_i^{\dagger}$, where M and m are, respectively, the maximum and minimum positive eigenvalues of $A_1,\ldots,A_k$.

If $A_1,\ldots,A_k$ are nonnull Hermitian nonnegative definite, so are $A_1^{\dagger},\ldots,A_k^{\dagger}$ and M^{-1} and m^{-1} are, respectively, the minimum and maximum positive eigenvalues of $A_1^{\dagger},\ldots,A_k^{\dagger}$. Replacing A_i with $A_i^{\dagger}$, $i = 1,\ldots,k$ and replacing M and m with M^{-1} and m^{-1}, respectively, in (36), we obtain the following two-side inequality for the sum $\sum_{i=1}^{k} A_i$:

$$S\left(\sum_{i=1}^{k} A_i^{\dagger}\right)^{\dagger} S \leqslant \sum_{i=1}^{k} A_i \leqslant \frac{(m+M)^2}{4mM} S\left(\sum_{i=1}^{k} A_i^{\dagger}\right)^{\dagger} S, \tag{37}$$

where $S = \sum_{i=1}^{k} AA_i^{\dagger}$, M and m are, respectively, the maximum and minimum positive eigenvalues of $A_1,\ldots,A_k$.

It is well known in complex matrix theory that if a complex matrix A is Hermitian, then $AA^{\dagger} = A^{\dagger}A$. If a quaternionic matrix A is Hermitian, then $\Psi(A)$ is Hermitian by Lemma 5. Hence, $\Psi(A)\Psi^{\dagger}(A) = \Psi^{\dagger}(A)\Psi(A)$. From this equality and Lemma 1(a), (c), and (g), one can obtain that if a quaternionic matrix A is Hermitian, then $AA^{\dagger} = A^{\dagger}A$. Notice that $S = \sum_{i=1}^{k} P_{A_i}$ is Hermitian. It follows that $SS^{\dagger} = S^{\dagger}S$. On the other hand, it is easy to verify that for any nonnegative definite matrices $A_1,\ldots,A_k$ over $\mathbb{C}$

$$\operatorname{Range}\left(\sum_{i=1}^{k} P_{A_i}\right) = \operatorname{Range}\left(\sum_{i=1}^{k} A_i\right) = \operatorname{Range}\left(\sum_{i=1}^{k} A_i^{\dagger}\right). \tag{38}$$

Thus

$$SS^{\dagger}\left(\sum_{i=1}^{k} A_i\right) = \left(\sum_{i=1}^{k} A_i\right)S^{\dagger}S = \sum_{i=1}^{k} A_i, \qquad SS^{\dagger}\left(\sum_{i=1}^{k} A_i^{\dagger}\right) = \left(\sum_{i=1}^{k} A_i^{\dagger}\right)S^{\dagger}S = \sum_{i=1}^{k} A_i^{\dagger}. \tag{39}$$

These matrix equalities can be extended to any nonnegative definite matrices $A_1,\ldots,A_k$ over $\mathbb{H}$ through Lemmas 1 and 5. In such cases, Pre- and post-multiplying (36) and (37) by $S^\dagger$ yields the following two inequalities:

$$\begin{aligned} \frac{4mM}{(m+M)^2}\sum_{i=1}^k S^\dagger A_i^\dagger S^\dagger \leqslant \left(\sum_{i=1}^k A_i\right)^\dagger \leqslant \sum_{i=1}^k S^\dagger A_i^\dagger S^\dagger, \\ \frac{4mM}{(m+M)^2}\sum_{i=1}^k S^\dagger A_i S^\dagger \leqslant \left(\sum_{i=1}^k A_i^\dagger\right)^\dagger \leqslant \sum_{i=1}^k S^\dagger A_i S^\dagger \end{aligned} \tag{40}$$

for nonnull Hermitian nonnegative definite matrices $A_1,\dots,A_k$ over $\mathbb{H}$, where M and m are, respectively, the maximum and minimum positive eigenvalues of $A_1,\dots,A_k$.

If $A_1,\dots,A_k$ are Hermitian positive definite over $\mathbb{H}$, then (36) reduces to

$$k^2\left(\sum_{i=1}^k A_i\right)^{-1} \leqslant \sum_{i=1}^k A_i^{-1} \leqslant k^2\frac{(m+M)^2}{4mM}\left(\sum_{i=1}^k A_i\right)^{-1}, \tag{41}$$

where M and m are, respectively, the maximum and minimum positive eigenvalues of $A_1,\dots,A_k$. In particular, when $k=2$, (41) becomes

$$4(A+B)^{-1} \leqslant A^{-1}+B^{-1} \leqslant \frac{(m+M)^2}{mM}(A+B)^{-1}, \tag{42}$$

or equivalently,

$$4A(A+B)^{-1}B \leqslant A+B \leqslant \frac{(m+M)^2}{mM}A(A+B)^{-1}B, \tag{43}$$

where M and m are, respectively, the maximum and minimum positive eigenvalues of A and B.

The product $A(A+B)^{-1}B$ is well known in the literature as the parallel sum of A and B. Thus (43) is in fact a two-side inequality between the sum and parallel sum of two Hermitian positive definite matrices over $\mathbb{H}$.

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Yongge Tian: School of Economics, Shanghai University of Finance and Economics, 777 Guoding Road, Shanghai, 200433 China
E-mail address: yongge@mail.shufe.edu.cn

George P. H. Styan: Department of Mathematics and Statistics, McGill University, Montréal, Québec, Canada H3A 2K6
E-mail address: styan@math.mcgill.ca