HARDY INEQUALITY ON TIME SCALES AND ITS APPLICATION TO HALF-LINEAR DYNAMIC EQUATIONS

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A time-scale version of the Hardy inequality is presented, which unifies and extends well-known Hardy inequalities in the continuous and in the discrete setting. An application in the oscillation theory of half-linear dynamic equations is given.

1. Introduction and preliminaries

One gets more than two hundred papers when searching by the keywords "Hardy" and "inequality" in the review journals Zentralblatt für Mathematik or Mathematical Reviews. Almost half of these publications appeared after 1990. In the absolute majority, these papers deal with various generalizations, extensions and improvements of the well-known Hardy inequality (HI) presented in monograph [8] (both in the continuous and in the discrete setting), namely, for example, HI in several variables, weighted HI, inequalities of Hardy's type involving certain transforms and forms, HI involving higher order derivatives, HI on certain manifolds, in various spaces, and many others. Many related topics can be also found when one looks for inequalities involving functions and their integrals and derivatives. Recall that the classical HI in integral form, discovered by Hardy, reads as

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f(\xi)d\xi\right)^\alpha dt \le \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_0^\infty f^\alpha(t)dt, \tag{1.1}$$

where $\alpha > 1$ and f is a measurable nonnegative function, and its discrete version essentially takes the same form with sums instead of integrals. Let us mention at least a few papers [5, 11, 15], among many others dealing with various types of HI's, and nice monographs [12, 13, 14]. All above facts seem to prove that there is no possibility of a last word on Hardy inequality.

What we offer in our paper is unification and extension of the classical Hardy integral inequality and the discrete Hardy inequality by means of the theory of time scales. This main result is presented in Section 2, together with some comments. In Section 3, we give an application of our extension of the Hardy inequality in the oscillation theory of half-linear dynamic equations. More precisely, we examine oscillatory properties of

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a generalized Euler dynamic equation. Those results turn out to be new even in the special linear case. The questions how the graininess of the time scale affects the (non)oscillation of the equation, as well as some other related topics, are also discussed there.

Before we present our main result, let us recall some essentials about time scales. In 1988, Hilger [9] introduced the calculus on time scales which unifies continuous and discrete analysis. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the *forward jump operator* σ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the *graininess* μ of the time scale $\mathbb{T}$ by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, respectively. We denote $f^\sigma := f \circ \sigma$. For a function $f : \mathbb{T} \to \mathbb{R}$ the *delta derivative* is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}. \tag{1.2}$$

Here are some basic formulas involving delta derivatives: $f^\sigma = f + \mu f^\Delta$, $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f^\Delta g^\sigma + f g^\Delta$, $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/g g^\sigma$, where f, g are delta differentiable and $g g^\sigma \neq 0$ in the last formula. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The classes of real rd-continuous functions and real piecewise rd-continuously delta differentiable functions on an interval I will be denoted by $C_{\mathrm{rd}}(I, \mathbb{R})$ and $C_{\mathrm{p}}^1(I, \mathbb{R})$, respectively. For $a, b \in \mathbb{T}$ and a delta differentiable function f, the *Cauchy integral* is defined by $\int_a^b f^\Delta(t)\Delta t = f(b) - f(a)$. For the concept of the *Riemann delta integral* and the *Lebesgue delta integral*, see [3, Chapter 5]. Note that the definition of the Riemann delta integrability is similar to the classical one for functions of a real variable, and that the Lebesgue delta integral is the Lebesgue integral associated with the so-called Lebesgue delta measure. Every rd-continuous function is Riemann delta integrable, and every Riemann delta integrable function is Lebesgue delta integrable. Throughout, for convenience, when we speak about a delta integrability, we mean the integrability in some of the above senses. The integration by parts formula reads $\int_a^b f^\Delta(t) g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\sigma(t) g^\Delta(t)\Delta t$, and an improper integral is defined as $\int_a^\infty f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s$. Note that we have

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta = f', \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt, \quad \text{when } \mathbb{T} = \mathbb{R}, \tag{1.3}$$

while

$$\sigma(t) = t+1, \quad \mu(t) \equiv 1, \quad f^\Delta = \Delta f, \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \quad \text{when } \mathbb{T} = \mathbb{Z}. \tag{1.4}$$

Many other information concerning time scales and dynamic equations on time scales can be found in the books [2, 3].

In some of the computations below we will use the estimates

$$\int_a^\infty \frac{\Delta s}{(\sigma(s))^\alpha} \leq \int_a^\infty \frac{ds}{s^\alpha} \leq \int_a^\infty \frac{\Delta s}{s^\alpha}, \tag{1.5}$$

which are proved in the next lemma. Note that these estimates are trivial when $\mathbb{T} = \mathbb{R}$. Also, it is easy to see them when

$$\mathbb{T} = \{t_k : k \in \mathbb{N}_0\} \quad \text{with } 0 < t_0 < t_1 < t_2,\ \lim_{k\to\infty} t_k = \infty \tag{1.6}$$

(in particular, $\mathbb{T} = \mathbb{N}$), see [3, Lemma 5.55], or

$$\mathbb{T} = \bigcup_{i=0}^{\infty} [a_i, b_i] \quad \text{with } 0 < a_i < b_i < a_{i+1},\ i \in \mathbb{N}_0. \tag{1.7}$$

However, in general case, they have not been proven yet. Note that similar observations as in the next lemma can be done without difficulties when the integrals are taken over finite intervals, and also when the integrand is replaced by a nonincreasing function.

Lemma 1.1. *Let $\alpha > 1$ be a constant. Then estimates (1.5) hold on any time scale which is unbounded above and contains a positive number a.*

Proof. Denote $[a,\infty)_{\mathbb{T}} := \{t \in \mathbb{T} : t \geq a\}$, where $\mathbb{T}$ is a particular time scale, which is unbounded above. We prove only that $I \leq \tilde{I}$, where $I := \int_a^\infty s^{-\alpha} ds$ and $\tilde{I} := \int_a^\infty s^{-\alpha}\Delta s$, since the other inequality can be proven analogously. If $\tilde{I} = \infty$ (which may indeed happen), then there is nothing to prove. Otherwise, suppose by a contradiction that there exist a time scale $\tilde{\mathbb{T}}$ unbounded above and $a \in \tilde{\mathbb{T}}$ such that $I > \tilde{I}$, where $\tilde{I}$ is taken over $[a,\infty)_{\tilde{\mathbb{T}}}$, which implies $I - \varepsilon > \tilde{I}$ for a suitable positive ε. On the other hand, by virtue of the definition of the delta Riemann integrability, there exists a time scale $\mathbb{T}_D$ containing a and satisfying (1.6), such that $|\tilde{I} - I_D| < \varepsilon/2$, where $I_D := (\mathbb{T}_D)\int_a^\infty s^{-\alpha}\Delta s$ (here the delta integral is taken over $[a,\infty)_{\mathbb{T}_D}$). Thus we get $\tilde{I} + \varepsilon < I \leq I_D < \tilde{I} + \varepsilon/2$, a contradiction. □

The following statement will be useful in proving the main results. For the proof see, for example, [16]; note that the Young inequality plays a crucial role there.

Lemma 1.2 (Hölder's inequality on time scales). *Let $\alpha > 1$, β be the conjugate number of α, and f, g be delta integrable on $[a,b]$. Then*

$$\int_a^b |f(t)g(t)|\Delta t < \left(\int_a^b |f(t)|^\alpha \Delta t\right)^{1/\alpha} \left(\int_a^b |g(t)|^\beta \Delta t\right)^{1/\beta}, \tag{1.8}$$

unless either f, g are proportional, or at least one of the functions is identically zero.

2. Main result

Throughout this section we assume that $\mathbb{T}$ is unbounded above. Our main result reads as follows.

Theorem 2.1 (Hardy inequality on time scales). *Let $\alpha > 1$ be a constant, a function f be nonnegative and such that the delta integral $\int_a^\infty (f(s))^\alpha \Delta s$ exists as a finite number. Denote*

$F(t) := \int_a^t f(s)\Delta s$. Then

$$\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t)-a}\right)^\alpha \Delta t < \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_a^\infty (f(t))^\alpha \Delta t, \tag{2.1}$$

unless $f \equiv 0$. If, in addition, $\mu(t)/t \to 0$ as $t \to \infty$, then the constant is the best possible.

Proof. Without loss of generality we may suppose that $f(a)>0$. Denote $\varphi(t)=F(t)/(t-a)$. For convenience we skip the argument t sometimes in the computations. Then

$$\begin{aligned}(\varphi^\sigma)^\alpha - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1} f &= (\varphi^\sigma)^\alpha - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1}((t-a)\varphi)^\Delta \\ &= (\varphi^\sigma)^\alpha - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1}\varphi^\sigma - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1}(t-a)\varphi^\Delta \\ &= \frac{-1}{\alpha-1}(\varphi^\sigma)^\alpha - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1}(t-a)\varphi^\Delta\end{aligned} \tag{2.2}$$

at $t \geq a$. Further, there exists $\eta(t)$ between $\varphi(t)$ and $\varphi^\sigma(t)$ such that $[(\varphi(t))^\alpha]^\Delta = \alpha(\eta(t))^{\alpha-1}\varphi^\Delta(t)$. Since $\mu(t)\operatorname{sgn}\varphi^\Delta(t) = \operatorname{sgn}(\varphi^\sigma(t)-\varphi(t))$ and φ is nonnegative, we have $\alpha(\varphi^\sigma)^{\alpha-1}\varphi^\Delta \geq \alpha\eta^{\alpha-1}\varphi^\Delta = (\varphi^\alpha)^\Delta$ at $t \geq a$. Using this estimate, we obtain from (2.2)

$$\begin{aligned}(\varphi^\sigma)^\alpha - \frac{\alpha}{\alpha-1}(\varphi^\sigma)^{\alpha-1} f &\leq -\frac{1}{\alpha-1}(\varphi^\sigma)^\alpha - \frac{1}{\alpha-1}(\varphi^\alpha)^\Delta(t-a) \\ &= -\frac{1}{\alpha-1}\left[\varphi^\alpha(t-a)\right]^\Delta.\end{aligned} \tag{2.3}$$

Integrating, we get

$$\int_a^t (\varphi^\sigma(s))^\alpha \Delta s - \frac{\alpha}{\alpha-1}\int_a^t (\varphi^\sigma(s))^{\alpha-1} f(s)\Delta s \leq -\frac{1}{\alpha-1}(\varphi(t))^\alpha(t-a) \leq 0 \tag{2.4}$$

for $t \geq a$. Hence, by the Hölder inequality on time scales (Lemma 1.2),

$$\begin{aligned}\int_a^t (\varphi^\sigma(s))^\alpha \Delta s &\leq \frac{\alpha}{\alpha-1}\int_a^t (\varphi^\sigma(s))^{\alpha-1} f(s)\Delta s \\ &\leq \frac{\alpha}{\alpha-1}\left(\int_a^t (f(s))^\alpha \Delta s\right)^{1/\alpha}\left(\int_a^t (\varphi^\sigma(s))^\alpha \Delta s\right)^{1/\beta}\end{aligned} \tag{2.5}$$

for $t \geq a$. Dividing by the last factor on the right (it is positive), and raising the result to the αth power, we get

$$\int_a^t (\varphi^\sigma(s))^\alpha \Delta s \leq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \int_a^t (f(s))^\alpha \Delta s \tag{2.6}$$

for $t \geq a$. Now, let t tend to ∞ to obtain (2.1), except that we have "less than or equal to" in place of "strictly less than." In particular we see that $\int_a^\infty (\varphi^\sigma(t))^\alpha \Delta t$ is finite. Next we show that "strictly less than" in (2.1) holds unless $f \equiv 0$. Return to (2.5) and replace t by ∞ to get

$$\begin{aligned}\int_a^\infty (\varphi^\sigma(s))^\alpha \Delta s &\le \frac{\alpha}{\alpha-1}\int_a^\infty (\varphi^\sigma(s))^{\alpha-1} f(s)\Delta s \\ &\le \frac{\alpha}{\alpha-1}\left(\int_a^\infty (f(s))^\alpha \Delta s\right)^{1/\alpha}\left(\int_a^\infty (\varphi^\sigma(s))^\alpha \Delta s\right)^{1/\beta}.\end{aligned} \tag{2.7}$$

There is a strict inequality in the second place unless f^α and $(\varphi^\sigma)^\alpha$ are proportional, that is, unless $f(t) = C\varphi^\sigma(t)$ for $t \ge a$, where C is independent of t. It can be shown that $C = 1$. Indeed, if a is right-scattered, then

$$\varphi^\sigma(a) = \frac{F^\sigma(a)}{\sigma(a)-a} = \frac{\mu(a)F(a)}{\mu(a)} = f(a), \tag{2.8}$$

while if a is right-dense, we have

$$\varphi^\sigma(a) = \varphi(a) = \lim_{t\to a+}\frac{F(t)}{t-a} = \lim_{t\to a+} f(t) = f(a). \tag{2.9}$$

Since $f = C\varphi^\sigma$ and $f(a) \ne 0$, we get $C = 1$. This is possible only when f is a constant. But if f were a nonzero constant function, this would be inconsistent with the convergence of $\int_a^\infty (f(s))^\alpha \Delta s$. Hence

$$\int_a^\infty (\varphi^\sigma(s))^\alpha \Delta s < \frac{\alpha}{\alpha-1}\left(\int_a^\infty (f(s))^\alpha \Delta s\right)^{1/\alpha}\left(\int_a^\infty (\varphi^\sigma(s))^\alpha \Delta s\right)^{1/\beta}, \tag{2.10}$$

and (2.1) follows from (2.10) in the same way as (2.6) does from (2.5).

Now we prove that the constant factor is the best possible provided $\mu(t)/t \to 0$ as $t \to \infty$. Put

$$f(t) = \begin{cases} 0 & \text{for } t \in [a, a'), \\ (t-a)^{-1/\alpha} & \text{for } t \in [a', b], \\ 0 & \text{for } t \in (b, \infty), \end{cases} \tag{2.11}$$

where $a < a' < b$. Then $\int_a^\infty (f(t))^\alpha \Delta t = \int_{a'}^{\sigma(b)} (\Delta t/(t-a))$ and

$$\begin{aligned} F^\sigma(t) &= \int_a^{\sigma(t)} f(s)\Delta s = \int_{a'}^{\sigma(t)} \frac{\Delta s}{(t-a)^{1/\alpha}} \\ &\ge \int_{a'}^t \frac{ds}{(s-a)^{1/\alpha}} = \frac{\alpha}{\alpha-1}\left[(t-a)^{(\alpha-1)/\alpha} - (a'-a)^{(\alpha-1)/\alpha}\right] \end{aligned} \tag{2.12}$$

for $t \in [a', b]$. Hence

$$\frac{F^\sigma(t)}{t-a} \ge \left(\frac{\alpha}{\alpha-1}\right)\frac{1-((a'-a)/(t-a))^{(\alpha-1)/\alpha}}{(t-a)^{1/\alpha}}, \tag{2.13}$$

which implies

$$\left(\frac{F^\sigma(t)}{t-a}\right)^\alpha \geq \left(\frac{\alpha}{\alpha-1}\right)^\alpha \frac{1-\varepsilon_t}{t-a}, \tag{2.14}$$

$t \in [a', b]$, where $\varepsilon_t \to 0$ as $t \to \infty$. Consequently,

$$\begin{aligned}\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t)-a}\right)^\alpha &= \int_{a'}^{\sigma(b)} \left(\frac{F^\sigma(t)}{t+\mu(t)-a}\right)^\alpha \Delta t \\ &\geq \int_{a'}^{\sigma(b)} \left(\frac{F^\sigma(t)}{t-a}\right)^\alpha \left(\frac{t-a}{t-a+\mu(t)}\right)^\alpha \Delta t \\ &\geq \left(\frac{\alpha}{\alpha-1}\right)^\alpha (1-\delta_b) \int_a^\infty (f(t))^\alpha \Delta t,\end{aligned} \tag{2.15}$$

where $\delta_b \to 0$ as $b \to \infty$. Hence any inequality of the type

$$\int_a^\infty \left(\frac{F^\sigma(t)}{\sigma(t)-a}\right)^\alpha \Delta t < \left(\frac{\alpha}{\alpha-1}\right)^\alpha (1-\varepsilon) \int_a^\infty (f(t))^\alpha \Delta t, \tag{2.16}$$

with $\varepsilon > 0$, fails to hold if f is chosen as above and b is sufficiently large. □

Remark 2.2. (i) If one wants to have a Hardy inequality on a finite segment, then simply take a function f which is eventually trivial. However, note that, for instance, in [18] the result is presented for the classical integral Hardy inequality ($\mathbb{T} = \mathbb{R}$) showing that the constant on the right-hand side can be lowered somehow (depending on a, b) provided the integrals are taken over a real interval $[a,b]$, $0 < a < b < \infty$.

(ii) There is an open problem to find out whether the constant in Theorem 2.1 is the best possible also on other time scales than just those satisfying $\lim_{t\to\infty} \mu(t)/t = 0$. Nevertheless, the inequality itself works on any time scale. In the next section, we will see that the problem of the best possible constants can be related to the problem of oscillation of certain half-linear dynamic equation. Certain connections with a Wirtinger type inequality are also mentioned there.

3. Application to a generalized Euler dynamic equation

Throughout this section we assume that $\mathbb{T}$ is unbounded above. Consider the generalized Euler dynamic equation

$$[\Phi(y^\Delta)]^\Delta + \frac{\gamma}{(\sigma(t))^\alpha}\Phi(y^\sigma) = 0, \tag{3.1}$$

where $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$ with $\alpha > 1$. This equation is a special case of the well studied half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0, \tag{3.2}$$

where $p, r \in C_{\mathrm{rd}}([a,\infty),\mathbb{R})$ with $r(t) \neq 0$. In [1, 16, 17], it was shown that although a solution space of (3.2) is homogeneous and not generally additive, many properties (like

Sturmian theory) known from the theory of linear dynamic equations extend to (3.2). Note that (3.2) reduces to the linear Sturm-Liouville equation $(r(t)y^\Delta)+p(t)y^\sigma=0$ when $\alpha=2$.

Next we examine oscillatory properties of (3.1). Before we will do this, let us recall some useful concepts and statements. We start with the definition.

Definition 3.1. (i) We say that a solution y of (3.2) has a *generalized zero* at t in case $y(t)=0$. We say y has a *generalized zero* in $(t,\sigma(t))$ in case $r(t)y(t)y(\sigma(t))<0$ and $\mu(t)>0$. We say that (3.2) is *disconjugate* on the interval $[a,b]$, if there is no nontrivial solution of (3.2) with two (or more) generalized zeros in $[a,b]$.

(ii) Equation (3.2) is said to be *nonoscillatory* (on $[a,\infty)$) if there exists $c\in[a,\infty)$ such that this equation is disconjugate on $[c,d]$ for every $d>c$. In the opposite case (3.2) is said to be *oscillatory* (on $[a,\infty)$). Oscillation of (3.2) may be equivalently defined as follows. A nontrivial solution y of (3.2) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[a,\infty)$. By the Sturm type separation theorem, which extends to (3.2), see [16], one solution of (3.2) is (non)oscillatory if and only if every solution of (3.2) is (non)oscillatory. Hence we can speak about *oscillation* or *nonoscillation of* (3.2).

Next we present a very important tool in the oscillation theory of (3.2), namely the so-called variational principle.

Proposition 3.2 [16]. *Equation (3.2) is nonoscillatory if and only if there exists $a\in\mathbb{T}$ such that*

$$\mathcal{F}(\xi)=\int_a^\infty\left\{r\,|\,\xi^\Delta\,|^\alpha-p\,|\,\xi^\sigma\,|^\alpha\right\}(t)\Delta t>0 \tag{3.3}$$

for every nontrivial $\xi\in U(a)$ (the class of the so-called admissible functions), where

$$U(a):=\{\xi\in C_p^1([a,\infty),\mathbb{R}):\exists b>a \text{ with } \xi(t)=0 \text{ if } t\notin(a,b)\}. \tag{3.4}$$

The following statement is an extension of the well-known Sturm comparison theorem. Along with (3.2) consider

$$[R(t)\Phi(z^\Delta)]^\Delta+P(t)\Phi(z^\sigma)=0, \tag{3.5}$$

where R and P are subject to the conditions imposed on r and p, respectively.

Proposition 3.3 [16]. *Assume that $R(t)\ge r(t)$ and $p(t)\ge P(t)$ for all large t. If (3.2) is nonoscillatory, then (3.5) is nonoscillatory.*

Now we present an extension of nonoscillation criterion known from the theory of linear second-order differential equations.

Proposition 3.4 [16]. *Suppose that*

$$\int_a^\infty p(s)\Delta s \text{ is convergent,} \tag{3.6}$$

$$r(t)>0,\quad \int_a^\infty r^{1-\beta}(s)\Delta s=\infty. \tag{3.7}$$

Further assume that

$$\lim_{t\to\infty}\frac{\mu(t)r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\Delta s}=0. \tag{3.8}$$

If

$$-\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}<\liminf_{t\to\infty}\mathcal{A}(t)\le\limsup_{t\to\infty}\mathcal{A}(t)<\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.9}$$

where

$$\mathcal{A}(t):=\left(\int_a^t r^{1-\beta}(s)\Delta s\right)^{\alpha-1}\int_t^\infty p(s)\Delta s, \tag{3.10}$$

then (3.2) is nonoscillatory.

The following oscillatory criterion is of Hille-Wintner type.

Proposition 3.5 [16]. *Let (3.7) hold and $\int_a^\infty p(s)\Delta s=\infty$. Then (3.2) is oscillatory.*

If $\int_a^\infty p(s)\Delta s$ converges, then the following oscillatory criterion may be used.

Proposition 3.6 [1]. *Suppose that (3.7) and (3.6) hold with $p(t)\ge 0$. If there exists a constant $M>0$ such that*

$$\mu(t)r^{1-\beta}(t)\le M \quad \text{for all large } t, \tag{3.11}$$

$$\liminf_{t\to\infty}\mathcal{A}(t)>\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.12}$$

where $\mathcal{A}$ is defined by (3.10), then (3.2) is oscillatory.

Now we are ready to examine (3.1). Denote $\gamma_\alpha:=[(\alpha-1)/\alpha]^\alpha$.

Claim 3.7. *If $\gamma\le\gamma_\alpha$, then (3.1) is nonoscillatory.*

Proof. First assume $\gamma=\gamma_\alpha$. Let $a\in\mathbb{T}$ be positive, and f be a function such that $\xi(t)=\int_a^t f(s)\Delta s$ is admissible, which means that ξ belongs to the class $U(a)$ defined in Proposition 3.2. Clearly $\xi^\Delta(t)=f(t)$. We have

$$\begin{aligned}\mathcal{F}(\xi)&=\int_a^\infty\left\{\left|\xi^\Delta(t)\right|^\alpha-\frac{\gamma_\alpha}{(\sigma(t))^\alpha}\left|\xi^\sigma(t)\right|^\alpha\right\}(t)\Delta t\\&=\int_a^\infty\left\{|f(t)|^\alpha-\frac{\gamma_\alpha}{(\sigma(t))^\alpha}\left|\int_a^{\sigma(t)}f(s)\Delta s\right|^\alpha\right\}(t)\Delta t\\&\ge\int_a^\infty\left\{|f(t)|^\alpha-\frac{\gamma_\alpha}{(\sigma(t)-a)^\alpha}\left(\int_a^{\sigma(t)}|f(s)|\Delta s\right)^\alpha\right\}(t)\Delta t.\end{aligned} \tag{3.13}$$

Now the last expression is positive by (2.1) provided f is nontrivial, which is our case, if we assume that $\xi\not\equiv 0$. Hence, (3.1) is nonoscillatory by Proposition 3.2. To show that (3.1) is nonoscillatory when $\gamma < \gamma_\alpha$, use the Sturm type comparison theorem (Proposition 3.3) and the fact that (3.1) with $\gamma = \gamma_\alpha$ is nonoscillatory. □

Remark 3.8. (i) Note that if $0 < \gamma < \gamma_\alpha$, then nonoscillation of (3.1) follows also from Proposition 3.4 (the case $\gamma \le 0$ can be treated by using the comparison theorem since it is very easy to find a (nonoscillatory) solution of the equation $[\Phi(y^\Delta)]^\Delta = 0$, whose solution space has a linear structure). However, some additional assumptions are needed. Indeed, (3.7) is clearly fulfilled. Since (1.5) holds, $p(t) = \gamma(\sigma(t))^{-\alpha}$ satisfies (3.6). Further, (3.8) in case of (3.1) requires $\mu(t)/t \to 0$ as $t \to \infty$. Finally to show that (3.9) is satisfied, we compute

$$\begin{aligned}\mathcal{A}(t) &= (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(\sigma(s))^\alpha}\Delta s \le (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{s^\alpha}ds \\ &= \frac{\gamma}{\alpha-1}\left(\frac{t-a}{t}\right)^{\alpha-1} \le \frac{\gamma_\alpha}{\alpha-1} - \varepsilon,\end{aligned} \tag{3.14}$$

which holds for large t and suitable positive ε. Note that if $\gamma = \gamma_\alpha$, then nonoscillation of (3.1) cannot be detected by the above criterion. Comparing the result obtained by using the Hardy inequality with this one, we see that the former one does not require any additional assumptions.

(ii) Claim 3.7 can be perhaps proved by means of the fact that the existence of u such that $(ruu^\sigma)(t) > 0$ and $u^\sigma(t)\{[r(t)\Phi(u^\Delta(t))]^\Delta + p(t)\Phi(u^\sigma(t))\} \le 0$ (in a neighborhood of ∞) is equivalent to nonoscillation of (3.2), since we conjecture that the function $u(t) = t^{(\alpha-1)/\alpha}$ satisfies the inequality $[\Phi(y^\Delta)]^\Delta + (\gamma_\alpha/(\sigma(t))^\alpha)\Phi(y^\sigma) \le 0$, and this would imply nonoscillation of (3.2) with $\gamma = \gamma_\alpha$.

(iii) The proof of the Hardy inequality via the variational principle is another open problem. We conjecture that the Hardy inequality can be viewed as a necessary condition for nonoscillation of (3.1) with $\gamma = \gamma_\alpha$ (more precisely, as a necessary condition for the existence of certain positive nondecreasing solution of the above mentioned Euler type inequality).

It remains to examine (3.1) when $\gamma > \gamma_\alpha$.

Claim 3.9. *Assume that μ is bounded. If $\gamma > \gamma_\alpha$, then (3.1) is oscillatory.*

Proof. We apply Proposition 3.6. Condition (3.11) in the case of (3.1) reads as $\mu(t) \le M$, which clearly holds. To show that (3.12) is fulfilled, we use the boundedness of μ, although it suffices $\mu(t)/t \to 0$ as $t \to \infty$, and we proceed as follows:

$$\begin{aligned}\mathcal{A}(t) &= (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(\sigma(s))^\alpha}\Delta s \ge (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(s+M)^\alpha}\Delta s \\ &\ge (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(s+M)^\alpha}ds = \frac{\gamma}{\alpha-1}\left(\frac{t-a}{t+M}\right)^{\alpha-1} \ge \frac{\gamma_\alpha}{\alpha-1} + \varepsilon,\end{aligned} \tag{3.15}$$

which holds for large t and suitable positive ε. □

Remark 3.10. (i) There is an open problem to prove that (3.1) oscillates when $\gamma > \gamma_\alpha$ on any time scale unbounded above, and not only on $\mathbb{T}$ with bounded μ. In other words, we would like to know whether there exists an unbounded time scale, on which (3.2) is nonoscillatory for some $\gamma > \gamma_\alpha$; such a result is not expected from the differential/difference equations case. The related fact which we are interested in is whether γ_α is indeed a critical time-scale-invariant constant—this will be discussed in the second part of this section.

(ii) As we could see above, there are some connections between the Hardy inequality and the generalized Euler dynamic equation (via the variational principle), and so we expect that the problem with oscillation, mentioned in part (i) of this remark, is closely related to the problem of proving that the constant in (2.1) is the best possible on any time scale.

(iii) There is a criterion similar to Proposition 3.6, see [16], where (3.11) and the nonnegativity of p are not required. However, the constant on the right-hand side of (3.12) is replaced by (larger) 1.

One can ask why just $(\sigma(t))^\alpha$ appears in (3.1). Why not t^α, or something else? To discuss this question, first recall some known results on linear equations. Note that, for example, in [4, 10], oscillatory properties of the Euler type linear equation

$$y^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)} y^\sigma = 0 \tag{3.16}$$

are studied. In [4], it is shown that (3.16) is oscillatory provided $\gamma > 1/4$. In [10], the author uses the Wirtinger type inequality on time scales, to show that (3.16) is nonoscillatory provided

$$0 < \limsup_{a\to\infty} \left\{ \left(\sup_{t\ge a} \frac{\sigma(t)}{t} \right)^{1/2} + \left\{ \left(\sup_{t\ge a} \frac{\mu(t)}{t} \right) + \left(\sup_{t\ge a} \frac{\sigma(t)}{t} \right) \right\}^{1/2} \right\}^2 = \frac{1}{\gamma} =: \frac{1}{\bar{\gamma}} < \infty. \tag{3.17}$$

More precisely, the inequality

$$\int_a^b |G^\Delta(t)| (u^\sigma(t))^2 \Delta t \le \Psi \int_a^b \frac{G(t)G^\sigma(t)}{|G^\Delta(t)|} (u^\Delta(t))^2 \Delta t, \tag{3.18}$$

which holds for a positive monotone G, and an admissible u, is applied with $G(t) = 1/t$ in the variational principle. The number Ψ, depending on the interval, is defined by

$$\Psi := \left\{ \left(\sup_{t\in[a,b]^\kappa} \frac{G(t)}{G^\sigma(t)} \right)^{1/2} + \left[\left(\sup_{t\in[a,b]^\kappa} \frac{\mu(t)|G^\Delta(t)|}{G^\sigma(t)} \right) + \left(\sup_{t\in[a,b]^\kappa} \frac{G(t)}{G^\sigma(t)} \right) \right]^{1/2} \right\}^2, \tag{3.19}$$

where κ cuts a possible isolated maximum of $[a,b]$. Note that if $G(t) = 1/t$, then Ψ reduces to the expression in the brackets in (3.17) with a relevant interval, and (3.18) becomes

$$\int_a^b \frac{1}{t\sigma(t)} (u^\sigma(t))^2 \Delta t \le \Psi \int_a^b (u^\Delta(t))^2 \Delta t, \tag{3.20}$$

where $\Psi \geq 4$; Ψ may be strictly greater than 4 even when $\mathbb{T} = \mathbb{Z}$. Compare (3.20) with the Hardy inequality where $\alpha = 2$, that is, with

$$\int_a^\infty \frac{1}{(\sigma(t) - a)^2} \left(F^\sigma(t)\right)^2 \Delta t \leq 4 \int_a^\infty \left(F^\Delta(t)\right)^2 \Delta t. \tag{3.21}$$

Note also that "α-degree" extensions of a Wirtinger inequality were stated in [6] for the continuous case, and in [7] for the discrete case, together with nonoscillatory criteria—as applications, of a similar type as Proposition 3.4, for half-linear differential and difference equations, respectively. A time-scale version which would unify these inequalities is an open question so far. One can observe that $\bar{\gamma}$ in (3.17) cannot be greater than 1/4, and that (3.16) is nonoscillatory for $\gamma \leq \bar{\gamma}$. In fact, if $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ (differential or difference equations case, resp.), then $\bar{\gamma} = 1/4$, which is well-known critical constant. However, we can see that if a graininess is suitably large, then the constant $\bar{\gamma}$ is strictly less than 1/4, and we do not know how to determine the oscillatory behavior of (3.16) when $\gamma \in (\bar{\gamma}, 1/4]$, using this criterion.

Let us apply our results (Claims 3.7 and 3.9) to the linear case, that is, let us assume $\alpha = 2$. Then $\gamma_\alpha = 1/4$, and we get that

$$y^{\Delta\Delta} + \frac{\gamma}{(\sigma(t))^2} y^\sigma = 0 \tag{3.22}$$

is nonoscillatory provided $\gamma \leq 1/4$, and oscillatory for $\gamma > 1/4$ (however with μ bounded in the latter case). In contrast to the results for (3.16), here we have a problem with the case when μ is unbounded and $\gamma > 1/4$.

Now let us return to the question presented after Remark 3.10. We can see at the first sight that there is a slight but significant difference between the coefficients of the second terms of (3.16) and (3.22). The expression $1/(t\sigma(t))$ in (3.16) may come from the fact that $(1/t)^\Delta = -1/(t\sigma(t))$. However, the situation in the half-linear case is much more complicated. We do not know how to extend this approach. On the other hand, our arguments why we choose just $(\sigma(t))^2$ (or, more generally, $(\sigma(t))^\alpha$) in the Euler type equation (3.22) (in (3.1)) reflects the process of discretization. More precisely, when we use a usual discretization scheme to approximate the second derivative, then the discrete counterpart of the equation $y'' + p(t)y = 0$ is the difference equation $\Delta^2 y_k + p_k y_{k+1} = 0$. We can see that the unknown function y in the second term has an index $k+1$. This suggests to take a coefficient p with $k+1$ as well, in order to get a "real" discrete counterpart, in our sense. Consequently, we should consider equation $y^{\Delta\Delta} + p^\sigma(t) y^\sigma = 0$. This extends also to the half-linear case. Another argument for why we have chosen just $\sigma(t)$ in the coefficient of (3.22) or (3.1) is that this matches the Hardy inequality.

We conclude this paper with an example showing what may happen when we consider the equation

$$\left[\Phi(y^\Delta)\right]^\Delta + \frac{\gamma}{t^\alpha} \Phi(y^\sigma) = 0 \tag{3.23}$$

instead of (3.1), that is, $\sigma(t)$ in the coefficient of (3.1) is replaced by t. First assume that $\mathbb{T}$ is a time scale such that, for instance, $\mu(t) \leq M$, $M > 0$, for all $t \in \mathbb{T}$. Then we have, assuming $0 < \gamma < \gamma_\alpha$,

$$\begin{aligned}
\mathcal{A}(t) &= (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{s^\alpha}\Delta s = (t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(s+M)^\alpha}\left(\frac{s+M}{s}\right)^\alpha \Delta s \\
&\le (1+\varepsilon_1)(t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(s+M)^\alpha}\Delta s \\
&\le (1+\varepsilon_1)(t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{(\sigma(s))^\alpha}\Delta s \le (1+\varepsilon_1)(t-a)^{\alpha-1}\int_t^\infty \frac{\gamma}{s^\alpha}ds \\
&= (1+\varepsilon_1)\frac{\gamma}{\alpha-1}\left(\frac{t-a}{t}\right)^{\alpha-1} \le \frac{\gamma_\alpha}{\alpha-1} - \varepsilon,
\end{aligned} \tag{3.24}$$

where t is large, and ε, ε_1 are positive suitable constants. Hence (3.23) is nonoscillatory by Proposition 3.4. Now pick a time scale such that $\int_a^\infty t^{-\alpha}\Delta t = \infty$, for example, let $\mathbb{T} = \{2^{\alpha^k} : k \in \mathbb{N}_0\}$, (see [3, Chapter 5]). Let γ be the same as before. Equation (3.23) is then oscillatory by Proposition 3.5. Thus we have an example showing that oscillatory properties of (3.23) may be completely changed when one replaces a time scale by a different one, leaving the form of the equation the same. In particular, there is no "important" (time-scale-invariant) critical constant γ_α in (3.23).

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