THE INEQUALITY OF MILNE AND ITS CONVERSE II

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We prove the following let $\alpha,\beta,a>0$, and $b>0$ be real numbers, and let w_j $(j=1,\ldots,n;$ $n\geq 2)$ be positive real numbers with $w_1+\cdots+w_n=1$. The inequalities $\alpha\sum_{j=1}^{n} w_j/(1-p_j^a)\leq \sum_{j=1}^{n} w_j/(1-p_j)\sum_{j=1}^{n} w_j/(1+p_j)\leq\beta\sum_{j=1}^{n} w_j/(1-p_j^b)$ hold for all real numbers $p_j\in[0,1)$ $(j=1,\ldots,n)$ if and only if $\alpha\leq\min(1,a/2)$ and $\beta\geq\max(1,(1-\min_{1\leq j\leq n} w_j/2)b)$. Furthermore, we provide a matrix version. The first inequality (with $\alpha=1$ and $a=2$) is a discrete counterpart of an integral inequality published by E. A. Milne in 1925.

1. Introduction

Motivated by an interesting paper of Rao [8], we proved in [1] the following double-inequality for sums.

Proposition 1.1. *Let w_j $(j=1,\ldots,n;n\geq 2)$ be positive real numbers with $w_1+\cdots+w_n=1$. Then we have for all real numbers $p_j\in[0,1)$ $(j=1,\ldots,n)$,*

$$\left(\sum_{j=1}^{n}\frac{w_j}{1-p_j^2}\right)^{c_1}\leq\sum_{j=1}^{n}\frac{w_j}{1-p_j}\sum_{j=1}^{n}\frac{w_j}{1+p_j}\leq\left(\sum_{j=1}^{n}\frac{w_j}{1-p_j^2}\right)^{c_2}, \tag{1.1}$$

with the best possible exponents

$$c_1=1,\qquad c_2=2-\min_{1\leq j\leq n} w_j. \tag{1.2}$$

The left-hand side of (1.1) (with $c_1=1$) is a discrete version of an integral inequality due to Milne [7]. Rao showed that (1.1) (with $c_1=1$ and $c_2=2$) is valid for all $w_j>0$ $(j=1,\ldots,n)$ with $w_1+\cdots+w_n=1$ and all $p_j\in(-1,1)$ $(j=1,\ldots,n)$.

Double-inequality (1.1) admits the following matrix version; see [1, 8].

Proposition 1.2. *Let w_j $(j=1,\ldots,n;n\geq 2)$ be positive real numbers with $w_1+\cdots+w_n=1$ and let I be the unit matrix. Then we have for all families of commuting Hermitian* *matrices* $P_1,\dots,P_n$ *with* $0 \le P_j < I$ $(j = 1,\dots,n)$,

$$\left(\sum_{j=1}^{n} w_j (I^2 - P_j^2)^{-1}\right)^{c_1} \le \sum_{j=1}^{n} w_j (I - P_j)^{-1} \sum_{j=1}^{n} w_j (I + P_j)^{-1} \le \left(\sum_{j=1}^{n} w_j (I^2 - P_j^2)^{-1}\right)^{c_2}, \tag{1.3}$$

with the best possible exponents

$$c_1 = 1, \qquad c_2 = 2 - \min_{1\le j\le n} w_j. \tag{1.4}$$

In Section 2 we provide new bounds for $\sum_{j=1}^{n} w_j/(1-p_j) \sum_{j=1}^{n} w_j/(1+p_j)$, which are closely related to those given in (1.1). It turns out that the new upper bound and the upper bound in (1.1) cannot be compared. And in Section 3 we present a matrix analogue of our discrete double-inequality.

2. Inequalities for sums

The following counterpart of Proposition 1.1 holds.

Theorem 2.1. *Let* $\alpha, \beta, a > 0$, *and* $b > 0$ *be real numbers. Further, let* w_j $(j = 1,\dots,n;\ n \ge 2)$ *be positive real numbers with* $w_1 + \cdots + w_n = 1$. *The inequalities*

$$\alpha \sum_{j=1}^{n} \frac{w_j}{1 - p_j^a} \le \sum_{j=1}^{n} \frac{w_j}{1 - p_j} \sum_{j=1}^{n} \frac{w_j}{1 + p_j} \le \beta \sum_{j=1}^{n} \frac{w_j}{1 - p_j^b} \tag{2.1}$$

hold for all real numbers $p_j \in [0,1)$ $(j = 1,\dots,n)$ *if and only if*

$$\alpha \le \min(1, a/2), \qquad \beta \ge \max\left(1, \left(1 - \min_{1\le j\le n} w_j/2\right) b\right). \tag{2.2}$$

Proof. Let $w = \min_{1\le j\le n} w_j$ and $c = 2/(2-w)$. First, we suppose that $\beta \ge \max(1, b/c)$. Since

$$\max(1, b/c) \ge \frac{1 - p^b}{1 - p^c} \quad (0 \le p < 1), \tag{2.3}$$

we obtain

$$\beta \sum_{j=1}^{n} \frac{w_j}{1 - p_j^b} \ge \sum_{j=1}^{n} \frac{w_j}{1 - p_j^c}. \tag{2.4}$$

To prove the right-hand side of (2.1) we may assume that

$$0 \le p_n \le p_{n-1} \le \cdots \le p_1 < 1. \tag{2.5}$$

We define

$$\begin{aligned} F(p_1,\dots,p_n) &= \sum_{j=1}^{n} \frac{w_j}{1-p_j^c} - \sum_{j=1}^{n} \frac{w_j}{1-p_j} \sum_{j=1}^{n} \frac{w_j}{1+p_j}, \\ F_q(p) &= F(p,\dots,p,p_{q+1},\dots,p_n), \quad 1 \le q \le n-1,\ p_{q+1} < p < 1. \end{aligned} \tag{2.6}$$

Differentiation leads to

$$\frac{(1-p^2)^2}{W_q} F_q'(p) = cp^{c-1}\left(\frac{1-p^2}{1-p^c}\right)^2 - 2pW_q + \sum_{j=q+1}^{n} w_j \left(\frac{(1-p)^2}{1-p_j} - \frac{(1+p)^2}{1+p_j}\right), \tag{2.7}$$

where $W_q = w_1 + \cdots + w_q$. Using

$$\frac{(1-p)^2}{1-p_j} - \frac{(1+p)^2}{1+p_j} \ge (1-p)^2 - (1+p)^2 \quad \text{for } j = q+1,\dots,n, \tag{2.8}$$

we get

$$\begin{aligned} \frac{(1-p^2)^2}{W_q} F_q'(p) &\ge cp^{c-1}\left(\frac{1-p^2}{1-p^c}\right)^2 - 4p + 2pW_q \\ &\ge cp^{c-1}\left(\frac{1-p^2}{1-p^c}\right)^2 - 4c^{-1}p = G(c,p), \quad \text{say.} \end{aligned} \tag{2.9}$$

Let

$$E(r,s;x,y) = \left(\frac{s}{r}\frac{x^r - y^r}{x^s - y^s}\right)^{1/(r-s)} \tag{2.10}$$

be the extended mean of order (r,s) of $x, y > 0$. Then we have

$$G(c,p) = 4c^{-1}p^{c-1}\left(E(2,c;p,1)\right)^{4-2c} - 4c^{-1}p. \tag{2.11}$$

Since $1 < c < 2$ and $E(r,s;x,y)$ increases with increase in either r or s (see [4]), we obtain

$$E(2,c;p,1) \ge E(2,1;p,1) = \frac{p+1}{2} > p^{1/2}. \tag{2.12}$$

From (2.11) and (2.12) we conclude that $G(c,p) > 0$. This implies that F_q is strictly increasing on $[p_{q+1}, 1)$. Hence, we get

$$\begin{aligned} F(p_1,\dots,p_n) &= F_1(p_1) \ge F_1(p_2) = F_2(p_2) \ge F_2(p_3) \\ &\ge \cdots \ge F_{n-1}(p_{n-1}) \ge F_{n-1}(p_n) = \frac{1}{1-p_n^c} - \frac{1}{1-p_n^2} \ge 0. \end{aligned} \tag{2.13}$$

Combining (2.4) and (2.13) it follows that the inequality on the right-hand side of (2.1) is valid.

Next, let $\alpha \le \min(1, a/2)$. Applying

$$\min(1, a/2) \le \frac{1-p^a}{1-p^2} \quad (0 \le p < 1) \tag{2.14}$$

and the first inequality of (1.1) (with $c_1 = 1$) we conclude that the left-hand side of (2.1) holds for all real numbers $p_j \in [0,1)$ $(j = 1,\ldots,n)$.

It remains to show that the validity of (2.1) implies (2.2). We set $p_1 = \cdots = p_n = p \in (0,1)$. Then the left-hand side of (2.1) leads to

$$\alpha \le \frac{1-p^a}{1-p^2}. \tag{2.15}$$

We let p tend to 0 and obtain $\alpha \le 1$. And, if p tends to 1, then (2.15) yields $\alpha \le a/2$. Let $w = w_k$ with $k \in \{1,\ldots,n\}$. We set $p_j = 0$ $(1 \le j \le n;\ j \ne k)$ and $p_k = p \in (0,1)$. Then the right-hand side of (2.1) is equivalent to

$$\frac{\big(1-w+w/(1-p)\big)\big(1-w+w/(1+p)\big)}{1-w+w/(1-p^b)} \le \beta. \tag{2.16}$$

If p tends to 0, then $1 \le \beta$. And, if p tends to 1, then we get $(1-w/2)b \le \beta$. □

Remarks 2.2. (i) We define for $b > 0$,

$$H(b) = \max\big(1, (1-w/2)b\big) \sum_{j=1}^{n} \frac{w_j}{1-p_j^b}, \tag{2.17}$$

where $w_j > 0$ $(j = 1,\ldots,n)$, $w_1 + \cdots + w_n = 1$, $w = \min_{1\le j\le n} w_j$, and $p_j \in [0,1)$ $(j = 1,\ldots,n)$. If $0 < b < 2/(2-w)$, then

$$H'(b) = \sum_{j=1}^{n} \frac{w_j p_j^b \log(p_j)}{\big(1-p_j^b\big)^2} \le 0. \tag{2.18}$$

And, if $b > 2/(2-w)$, then

$$H'(b) = (1-w/2) \sum_{j=1}^{n} \frac{w_j}{\big(1-p_j^b\big)^2} \big(1-p_j^b+p_j^b \log(p_j^b)\big) \ge 0. \tag{2.19}$$

This implies that H is decreasing on $(0, 2/(2-w)]$ and increasing on $[2/(2-w), \infty)$. Hence: if (2.2) holds, then the function

$$H^*(\beta, b) = \beta \sum_{j=1}^{n} \frac{w_j}{1-p_j^b} \tag{2.20}$$

satisfies $H^*(\beta,b) \ge H^*(1, 2/(2-w))$. This means that the expression on the right-hand side of (2.1) attains its smallest value if $\beta = 1$ and $b = 2/(2-w)$. Similarly, we obtain: if (2.2) holds, then the expression on the left-hand side of (2.1) attains its largest value if $\alpha = 1$ and $a = 2$.

(ii) The upper bounds given in (1.1) with $c_2 = 2 - w$ and (2.1) with $\beta = 1$, $b = 2/(2 - w)$ cannot be compared. To prove this we set $p_1 = \cdots = p_n = p \in (0,1)$ and denote by $R_1(p)$ and $R_2(p)$ the expressions on the right-hand side of (1.1) and (2.1), respectively. Then we get

$$R_1(p) = \left(\frac{1}{1-p^2}\right)^{c_2}, \qquad R_2(p) = \frac{1}{1-p^b}. \tag{2.21}$$

First, we show that $R_1(p) > R_2(p)$ in the neighbourhood of 1. Let

$$\Delta(p) = R_1(p) - R_2(p), \qquad \varphi(p) = (1-p^b)\Delta(p). \tag{2.22}$$

Since $c_2 > 1$, $b > 1$ we have

$$\lim_{p\to 1}\varphi(p) = \lim_{p\to 1}\frac{bp^{b-1}}{2pc_2(1-p^2)^{c_2-1}} - 1 = \infty. \tag{2.23}$$

This implies that φ and Δ are positive in the neighbourhood of 1.

Next, we show that $R_1(p) < R_2(p)$ in the neighbourhood of 0. Let

$$\sigma(p) = \Delta(p^{1/2}). \tag{2.24}$$

We obtain $\sigma(0) = 0$ and since $0 < b/2 < 1$ we get

$$\lim_{p\to 0}\sigma'(p) = \lim_{p\to 0}\left(\frac{c_2}{(1-p)^{c_2+1}} - \frac{b}{2}p^{b/2-1}\frac{1}{(1-p^{b/2})^2}\right) = -\infty. \tag{2.25}$$

This implies that σ and Δ attain negative values in the neighbourhood of 0.

(iii) The two-parameter mean value family defined in (2.10) has been the subject of intensive research. The main properties are studied in [4–6], where also historical remarks and references can be found.

3. Matrix inequalities

We now provide a matrix analogue of Theorem 2.1. The reader who wants to have a proper understanding of the following theorem and its proof needs a general knowledge of matrix theory. We refer to the monographs [2, 3].

Theorem 3.1. *Let $\alpha,\beta,a > 0$, and $b > 0$ be real numbers. Further, let w_j ($j = 1,\dots,n$; $n \ge 2$) be positive real numbers with $w_1 + \cdots + w_n = 1$. The inequalities*

$$\alpha\sum_{j=1}^n w_j(I-P_j^a)^{-1} \le \sum_{j=1}^n w_j(I-P_j)^{-1}\sum_{j=1}^n w_j(I+P_j)^{-1} \le \beta\sum_{j=1}^n w_j(I-P_j^b)^{-1} \tag{3.1}$$

hold for all families of commuting Hermitian matrices $P_1,\dots,P_n$, satisfying $0 \le P_j < I$ in the Löwner ordering, if and only if

$$\alpha \le \min(1,a/2), \qquad \beta \ge \max\left(1,\left(1-\min_{1\le j\le n} w_j/2\right)b\right). \tag{3.2}$$

Proof. First, we assume that (3.2) is valid. Since the P_j commute, there exists a nonsingular matrix S such that $S^{-1}P_jS = \operatorname{diag}(\dots,\lambda_{lj},\dots)$, where $\lambda_{1j},\dots,\lambda_{nj}$ are the eigenvalues of P_j. By definition of the positive semidefinite ordering (Löwner ordering) it follows that $P_j < I$ implies $0 \le \lambda_{lj} < 1$ for $l = 1,\dots,n$. So the expressions given in (3.1) make sense. Denoting by L, M, and R the matrices on the left-hand side, in the middle, and on the right-hand side of (3.1), respectively, we get

$$S^{-1}LS = \operatorname{diag}\left(\dots,\alpha\sum_{j=1}^{n}\frac{w_j}{1-\lambda_{lj}^{a}},\dots\right), \qquad S^{-1}MS = \operatorname{diag}\left(\dots,\sum_{j=1}^{n}\frac{w_j}{1-\lambda_{lj}}\sum_{j=1}^{n}\frac{w_j}{1+\lambda_{lj}},\dots\right),$$
$$S^{-1}RS = \operatorname{diag}\left(\dots,\beta\sum_{j=1}^{n}\frac{w_j}{1-\lambda_{lj}^{b}},\dots\right). \tag{3.3}$$

Applying Theorem 2.1 we obtain $S^{-1}LS \le S^{-1}MS \le S^{-1}RS$, and hence $L \le M \le R$.

Next, we suppose that (3.1) holds for all families of commuting Hermitian matrices $P_1,\dots,P_n$, satisfying $0 \le P_j < I$. We proceed in analogy with the proof of Theorem 2.1: put $P_1 = \cdots = P_n = \operatorname{diag}(p,\dots,p)$ with $p \in (0,1)$. Then the left-hand side of (3.1) leads to an inequality for scalar matrices (i.e., multiples of the identity I), namely,

$$\alpha\frac{1}{1-p^a}I \le \frac{1}{1-p}I\cdot\frac{1}{1+p}I. \tag{3.4}$$

Considering a pair of corresponding diagonal entries we conclude that this inequality is equivalent to (2.15). Tending with p to 0 and 1, respectively, we get $\alpha \le \min(1,a/2)$. Next, let $w = w_k$, where $k \in \{1,\dots,n\}$. We set $P_j = 0$ for $j \ne k$ and $P_k = pI$. Then the right-hand side of (3.1) yields

$$\big((1-w)I + (w/(1-p))I\big)\cdot\big((1-w)I + (w/(1+p))I\big) \le \beta\big((1-w)I + (w/(1-p^b))I\big). \tag{3.5}$$

Again, this is an inequality for scalar matrices and it suffices to consider diagonal entries. This leads to (2.16). We let p tend to 0 and 1, respectively, and obtain the second of the inequalities (3.2). □

References

[1] H. Alzer and A. Kovačec, *The inequality of Milne and its converse*, Journal of Inequalities and Applications **7** (2002), no. 4, 603–611.
[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[3] ______, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[4] E. B. Leach and M. C. Sholander, *Extended mean values*, The American Mathematical Monthly **85** (1978), no. 2, 84–90.
[5] ______, *Extended mean values. II*, Journal of Mathematical Analysis and Applications **92** (1983), no. 1, 207–223.
[6] ______, *Multi-variable extended mean values*, Journal of Mathematical Analysis and Applications **104** (1984), no. 2, 390–407.

[7] E. A. Milne, *Note on Rosseland's integral for the stellar absorption coefficient*, Monthly Notices of the Royal Astronomical Society **85** (1925), 979–984.

[8] C. R. Rao, *Statistical proofs of some matrix inequalities*, Linear Algebra and its Applications **321** (2000), no. 1-3, 307–320.

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