OSCILLATION AND NONOSCILLATION THEOREMS FOR A CLASS OF EVEN-ORDER QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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We are concerned with the oscillatory and nonoscillatory behavior of solutions of even-order quasilinear functional differential equations of the type $(|y^{(n)}(t)|^{\alpha}\operatorname{sgn} y^{(n)}(t))^{(n)} + q(t)|y(g(t))|^{\beta}\operatorname{sgn} y(g(t)) = 0$, where α and β are positive constants, $g(t)$ and $q(t)$ are positive continuous functions on $[0,\infty)$, and $g(t)$ is a continuously differentiable function such that $g'(t) > 0$, $\lim_{t\to\infty} g(t) = \infty$. We first give criteria for the existence of nonoscillatory solutions with specific asymptotic behavior, and then derive conditions (sufficient as well as necessary and sufficient) for all solutions to be oscillatory by comparing the above equation with the related differential equation without deviating argument.

1. Introduction

We consider even-order quasilinear functional differential equations of the form

$$\left(\left|y^{(n)}(t)\right|^{\alpha}\operatorname{sgn} y^{(n)}(t)\right)^{(n)} + q(t)\left|y(g(t))\right|^{\beta}\operatorname{sgn} y(g(t)) = 0, \tag{A}$$

where

(a) α and β are positive constants;

(b) $q : [0,\infty) \to (0,\infty)$ is a continuous function;

(c) $g : [0,\infty) \to (0,\infty)$ is a continuously differentiable function such that $g'(t) > 0$, $t \ge 0$, and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of (A) we mean a function $y : [T_y,\infty) \to \mathbb{R}$ which is n times continuously differentiable together with $|y^{(n)}|^{\alpha}\operatorname{sgn} y^{(n)}$ and satisfies (A) at all sufficiently large t. Those solutions which vanish in a neighborhood of infinity will be excluded from our consideration. A solution is said to be oscillatory if it has a sequence of zeros clustering around ∞, and nonoscillatory otherwise.

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The objective of this paper is to study the oscillatory and nonoscillatory behavior of solutions of (A). In Section 2 we begin with the classification of nonoscillatory solutions of (A) according to their asymptotic behavior as $t \to \infty$. It suffices to restrict our consideration to eventually positive solutions of (A), since if $y(t)$ is a solution of (A), then so is $-y(t)$. Let P denote the totally of eventually positive solutions of (A). It will be shown that it is natural to divide P into the following two classes:

$$\begin{gathered} P(\mathrm{I}) = P(\mathrm{I}_0) \cup P(\mathrm{I}_1) \cup \cdots \cup P(\mathrm{I}_{2n-1}), \\ P(\mathrm{II}) = P(\mathrm{II}_1) \cup P(\mathrm{II}_3) \cup \cdots \cup P(\mathrm{II}_{2n-1}), \end{gathered} \tag{1.1}$$

where $P(\mathrm{I}_j)$, $j \in \{0,1,\dots,2n-1\}$, and $P(\mathrm{II}_k)$, $k \in \{1,3,\dots,2n-1\}$, consist of solutions $y(t)$ satisfying

$$\begin{gathered} \lim_{t\to\infty} \frac{y(t)}{\varphi_j(t)} = \text{const} > 0, \\ \lim_{t\to\infty} \frac{y(t)}{\varphi_{k-1}(t)} = \infty, \qquad \lim_{t\to\infty} \frac{y(t)}{\varphi_k(t)} = 0, \end{gathered} \tag{1.2}$$

respectively. Here the functions $\varphi_i(t)$, $i = 0,1,\dots,2n-1$, are defined by

$$\varphi_i(t) = t^i \quad (i = 0,1,\dots,n-1), \qquad \varphi_i(t) = t^{n+(i-n)/\alpha} \quad (i = n, n+1,\dots,2n-1). \tag{1.3}$$

Moreover, we will give the integral representations for positive solutions belonging to each of these two classes. Next, In Section 3 we will give necessary and sufficient conditions for the existence of positive solutions belonging to the class $P(\mathrm{I})$ as well as sufficient conditions for the existence of positive solutions belonging to the class $P(\mathrm{II})$.

In Section 5 we derive criteria for all solutions of (A) to be oscillatory. Our derivations depend heavily on oscillation theory of even-order nonlinear differential equations

$$\left(|y^{(n)}(t)|^{\alpha} \operatorname{sgn} y^{(n)}(t)\right)^{(n)} + q(t)|y(t)|^{\beta} \operatorname{sgn} y(t) = 0 \tag{B}$$

recently developed by Tanigawa in [7]. Comparison theorems which will be established in Section 4 enable us to deduce oscillation of an equation of the form (A) from that of a similar equation with a different functional argument.

We note that oscillation properties of second-order functional differential equations involving nonlinear Sturm-Liouville-type differential operators have been investigated by Kusano and Lalli [2], Kusano and Wang [4], and Wang [9]. Moreover, in a recent paper by Tanigawa [6] oscillation criteria for fourth-order functional differential equations

$$\left(|y''(t)|^{\alpha} \operatorname{sgn} y''(t)\right)'' + q(t)|y(g(t))|^{\beta} \operatorname{sgn} y(g(t)) = 0 \tag{C}$$

have been presented.

2. Classification and integral representations of positive solutions

Our purpose here is to make a detailed analysis of the structure of the set P of all possible positive solutions of (A).

Classification of positive solutions. Let $y(t)$ be an eventually positive solution of (A) on $[t_0,\infty)$, $t_0 \geq 0$. Then, we have the following lemma which was proved by Tanigawa and Fentao in [8] and which is a natural generalization of the well-known Kiguradze lemma [1].

It will be convenient to make use of the symbols L_i, $i = 1,2,\dots,2n-1$, to denote the "quasiderivatives" generating the differential operator $L_{2n}y = (|y^{(n)}|^\alpha \operatorname{sgn} y^{(n)})^{(n)}$:

$$\begin{gathered}
L_i y = y^{(i)}, \quad i = 1,2,\dots,n-1, \\
L_i y = \left(\left| y^{(n)} \right|^{\alpha} \operatorname{sgn} y^{(n)}\right)^{(i-n)}, \quad i = n,n+1,\dots,2n, \\
L_{i+1} y = \left(L_i y\right)', \quad i = 1,2,\dots,n-2,\ n,n+1,\dots,2n-1, \\
L_n y = \left| (L_{n-1}y)' \right|^{\alpha} \operatorname{sgn}\left(L_{n-1}y\right)', \qquad L_0 y = y.
\end{gathered} \tag{2.1}$$

Lemma 2.1. *If $y(t)$ is a positive solution of (A) on $[t_0,\infty)$, then there exist an odd integer $k \in \{1,3,\dots,2n-1\}$ and a $t_1 > t_0$ such that*

$$\begin{gathered}
L_i y(t) > 0, \quad t \geq t_1, \text{ for } i = 0,1,\dots,k-1, \\
(-1)^{i-k} L_i y(t) > 0, \quad t \geq t_1, \text{ for } i = k,k+1,\dots,2n-1.
\end{gathered} \tag{2.2}$$

We denote by P_k the subset of P consisting of all positive solutions $y(t)$ of (A) satisfying (2.2). The above lemma shows that P has the decomposition

$$P = P_1 \cup P_3 \cup \cdots \cup P_{2n-1}. \tag{2.3}$$

Since $L_i y(t)$, $i \in \{0,1,\dots,2n-1\}$, are eventually monotone, they tend to finite or infinite limits as $t \to \infty$, that is,

$$\lim_{t\to\infty} L_i y(t) = \omega_i, \quad i \in \{0,1,\dots,2n-1\}. \tag{2.4}$$

One can easily show that if $y \in P_k$ for $k \in \{1,3,\dots,2n-1\}$, then ω_k is a finite nonnegative number and the set of its asymptotic values $\{\omega_i\}$ falls into one of the following three cases:

$$\begin{gathered}
\omega_0 = \omega_1 = \cdots = \omega_{k-1} = \infty, \qquad \omega_k \in (0,\infty), \qquad \omega_{k+1} = \omega_{k+2} = \cdots = \omega_{2n-1} = 0, \\
\omega_0 = \omega_1 = \cdots = \omega_{k-1} = \infty, \qquad \omega_k = \omega_{k+1} = \cdots = \omega_{2n-1} = 0, \\
\omega_0 = \omega_1 = \cdots = \omega_{k-2} = \infty, \qquad \omega_{k-1} \in (0,\infty), \qquad \omega_k = \omega_{k+1} = \cdots = \omega_{2n-1} = 0.
\end{gathered} \tag{2.5}$$

Observing that by L'Hospital's rule, we have, for every $j \in \{1,2,\dots,2n-1\}$, that

$$\lim_{t\to\infty}\frac{y(t)}{\varphi_j(t)} = \text{const} \geq 0 \text{ or } \infty \Longleftrightarrow \lim_{t\to\infty} L_j y(t) = \text{const} \geq 0 \text{ or } \infty, \tag{2.6}$$

equivalent expressions for these classes of positive solutions of (A) are the following:

$$\begin{aligned}
&\text{(i)} \quad \lim_{t\to\infty}\frac{y(t)}{\varphi_k(t)} = \text{const} > 0,\\
&\text{(ii)} \quad \lim_{t\to\infty}\frac{y(t)}{\varphi_k(t)} = 0, \qquad \lim_{t\to\infty}\frac{y(t)}{\varphi_{k-1}(t)} = \infty,\\
&\text{(iii)} \quad \lim_{t\to\infty}\frac{y(t)}{\varphi_{k-1}(t)} = \text{const} > 0,
\end{aligned} \tag{2.7}$$

where $\varphi_0(t),\dots,\varphi_{2n-1}(t)$ are defined by (1.3). Note that these functions are particular solutions of the unperturbed equation $L_{2n}y(t) = 0$. Observing that cases (i) and (iii) are of the same category, it is natural to classify P broadly into the two classes $P(\mathrm{I}) = P(\mathrm{I}_0) \cup P(\mathrm{I}_1) \cup \cdots \cup P(\mathrm{I}_{2n-1})$ and $P(\mathrm{II}) = P(\mathrm{II}_1) \cup P(\mathrm{II}_3) \cup \cdots \cup P(\mathrm{II}_{2n-1})$ consisting, respectively, of

$$\begin{aligned}
P(\mathrm{I}_j) &= \left\{ y \in P : \lim_{t\to\infty}\frac{y(t)}{\varphi_j(t)} = \text{const} > 0 \right\},\\
P(\mathrm{II}_k) &= \left\{ y \in P : \lim_{t\to\infty}\frac{y(t)}{\varphi_{k-1}(t)} = \infty,\ \lim_{t\to\infty}\frac{y(t)}{\varphi_k(t)} = 0 \right\}.
\end{aligned} \tag{2.8}$$

Integral representations for positive solutions. We will establish the existence of eventually positive solutions for each of the above classes $P(\mathrm{I})$ and $P(\mathrm{II})$. For this purpose a crucial role will be played by integral representations for $P(\mathrm{I}_j)$ and $P(\mathrm{II}_k)$ types of solutions of (A) established below.

Let $y(t)$ be a positive solution of (A) such that $y(t) > 0$, $y(g(t)) > 0$ on $[t_0,\infty)$. Let us first derive an integral representation of the solution $y(t)$ from the class $P(\mathrm{I}_j)$, $j \in \{0, 1,\dots,2n-1\}$.

If $j \in \{n,n+1,\dots,2n-1\}$, then we integrate (A) $2n-j$ times from t to ∞ and then integrate the resulting equation j times from t_0 to t to obtain

(i) for $j \in \{n+1,n+2,\dots,2n-1\}$,

$$\begin{aligned}
y(t) = \zeta(t) + \int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\Bigg[\xi_j(s) + (-1)^{2n-j-1}\int_{t_0}^{s}\frac{(s-r)^{j-n-1}}{(j-n-1)!}&\\
\times\int_r^\infty\frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!}q(\sigma)y\big(g(\sigma)\big)^{\beta}d\sigma\, dr\Bigg]^{1/\alpha}&ds;
\end{aligned} \tag{2.9}$$

(ii) for $j = n$,

$$y(t) = \zeta(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\left[\omega_n + (-1)^{n-1}\int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} q(r)\left(y(g(r))\right)^{\beta} dr\right]^{1/\alpha} ds, \tag{2.10}$$

where

$$\begin{gathered}\xi_j(t) = \sum_{i=n}^{j-1} L_i y(t_0)\frac{\left(t-t_0\right)^{i-n}}{(i-n)!} + \omega_j \frac{\left(t-t_0\right)^{j-n}}{(j-n)!} \quad (n+1 \le j \le 2n-1), \\ \zeta(t) = \sum_{i=0}^{n-1} L_i y(t_0)\frac{\left(t-t_0\right)^{i}}{i!}.\end{gathered} \tag{2.11}$$

If $j \in \{0,1,\dots,n-1\}$, then first integrating (A) $2n-j(=n+(n-j))$ times from t to ∞ and then integrating j times from t_0 to t, we have

(i) for $j \in \{1,2,\dots,n-1\}$,

$$\begin{aligned} y(t) = {} & \zeta_j^*(t) \\ & + (-1)^{2n-j-1}\int_{t_0}^{t} \frac{(t-s)^{j-1}}{(j-1)!}\int_s^{\infty} \frac{(r-s)^{n-j-1}}{(n-j-1)!}\left[\int_r^{\infty} \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma)\left(y(g(\sigma))\right)^{\beta} d\sigma\right]^{1/\alpha} dr\, ds;\end{aligned} \tag{2.12}$$

(ii) for $j = 0$,

$$y(t) = \omega_0 + (-1)^{2n-1}\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} q(r)\left(y(g(r))\right)^{\beta} dr\right]^{1/\alpha} ds, \tag{2.13}$$

where

$$\zeta_j^*(t) = \sum_{i=0}^{j-1} L_i y(t_0)\frac{\left(t-t_0\right)^{i}}{i!} + \omega_j \frac{\left(t-t_0\right)^{j}}{j!} \quad (1 \le j \le n-1). \tag{2.14}$$

As regards $y \in P(\mathrm{II}_k)$, $k \in \{1,3,\dots,2n-1\}$, an integral representation is expressed by (2.9)–(2.13) with $\omega_j = 0$ for $j = k$.

3. Nonoscillation criteria

It will be shown that necessary and sufficient conditions can be established for the existence of positive solutions from class $P(\mathrm{I})$.

THEOREM 3.1. *Let $j \in \{0,1,\dots,2n-1\}$. There exists a positive solutions of (A) belonging to $P(I_j)$ if and only if*

$$\int_0^{\infty} t^{n-j-1}\left[\int_t^{\infty} s^{n-1} q(s)\left(\varphi_j(g(s))\right)^{\beta} ds\right]^{1/\alpha} dt < \infty, \quad j = 0,1,\dots,n-1, \tag{3.1}$$

$$\int_0^{\infty} t^{2n-j-1} q(t)\left(\varphi_j(g(t))\right)^{\beta} dt < \infty, \quad j = n, n+1,\dots,2n-1. \tag{3.2}$$

Proof (*the "only if" part*). Suppose that (A) has a positive solution $y(t)$ of class $P(\mathrm{I}_j)$. Notice that since $y(t)$ satisfies asymptotic relations (2.7)(i) and (iii), there exist positive constants c_j, C_j such that

$$c_j\varphi_j(t) \le y(t) \le C_j\varphi_j(t), \quad t \ge t_0. \tag{3.3}$$

In deriving (2.9)–(2.13) we found the convergence of the integrals

$$\begin{gathered}\int_{t_0}^{\infty} t^{2n-j-1}q(t)\big(y(g(t))\big)^{\beta}dt < \infty, \quad \text{for } j = n, n+1, \dots, 2n-1,\\ \int_{t_0}^{\infty} t^{n-j-1}\bigg[\int_t^{\infty} s^{n-1}q(s)\big(y(g(s))\big)^{\beta}ds\bigg]^{1/\alpha}dt < \infty, \quad \text{for } j = 0, 1, \dots, n-1.\end{gathered} \tag{3.4}$$

These together with (3.3), show that the conditions (3.1) and (3.2) are satisfied.

(The "if" part.) We will distinguish two cases for $j \in \{0, 1, \dots, n-1\}$ and for $j \in \{n, n+1, \dots, 2n-1\}$.

Case 1. Let $j \in \{n, n+1, \dots, 2n-1\}$ and suppose that (3.2) is satisfied. Let $c > 0$ be an arbitrarity fixed constant and choose $t_0 > 0$ such that

$$\int_{t_0}^{\infty} \frac{t^{2n-j-1}}{(2n-j-1)!}q(t)\big(\varphi_j(g(t))\big)^{\beta}dt \le A\big[(j-n)!\big]^{\beta/\alpha}\bigg[\bigg(1+\frac{j-n}{\alpha}\bigg)\cdots\bigg(n+\frac{j-n}{\alpha}\bigg)\bigg]^{\beta}c^{1-\beta/\alpha}, \tag{3.5}$$

where

$$A = 2^{-\beta/\alpha} \quad \text{if } 2n-j-1 \text{ is even}, \qquad A = 2^{-1} \quad \text{if } 2n-j-1 \text{ is odd}. \tag{3.6}$$

Define the constants k_1 and k_2 by

$$k_i = \frac{c_i}{\big[(j-n)!\big]^{1/\alpha}(1+(j-n)/\alpha)\cdots(n+(j-n)/\alpha)}, \quad i = 1, 2, \dots, \tag{3.7}$$

where

$$\begin{gathered}c_1 = c^{1/\alpha}, \quad c_2 = (2c)^{1/\alpha} \quad \text{if } 2n-j-1 \text{ is even},\\ c_1 = \Big(\frac{c}{2}\Big)^{1/\alpha}, \quad c_2 = c^{1/\alpha} \quad \text{if } 2n-j-1 \text{ is odd}.\end{gathered} \tag{3.8}$$

Put $t_* = \min\{t_0, \inf_{t \ge t_0} g(t)\}$, and define

$$\tilde{\varphi}_j(t) = \begin{cases}\varphi_j(t-t_0), & t \ge t_0\\ 0, & t \le t_0.\end{cases} \tag{3.9}$$

Let Y denote the set

$$Y=\{y\in C[t_*,\infty):k_1\widetilde{\varphi}_j(t)\le y(t)\le k_2\widetilde{\varphi}_j(t),\ t\ge t_*\},\tag{3.10}$$

and define the mapping $\mathcal{F}_j:Y\to C[t_*,\infty)$ as follows: for $j\in\{n+1,n+2,\dots,2n-1\}$,

$$\begin{aligned}\mathcal{F}_jy(t)=\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}&\\ \times\Bigg[\frac{c(s-t_0)^{j-n}}{(j-n)!}+(-1)^{2n-j-1}\int_{t_0}^{s}\frac{(s-r)^{j-n-1}}{(j-n-1)!}&\\ \times\int_{r}^{\infty}\frac{(\sigma-r)^{2n-j-1}}{(2n-j-1)!}q(\sigma)\big(y(g(\sigma))\big)^{\beta}d\sigma\,dr\Bigg]^{1/\alpha}ds,&\\ t\ge t_0,&\\ \mathcal{F}_jy(t)=0,\quad t_*\le t\le t_0,&\end{aligned}\tag{3.11}$$

and for $j=n$,

$$\begin{aligned}\mathcal{F}_ny(t)=\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[c+(-1)^{n-1}\int_{s}^{\infty}\frac{(r-s)^{n-1}}{(n-1)!}q(r)\big(y(g(r))\big)^{\beta}dr\right]^{1/\alpha}ds,\quad t\ge t_0,&\\ \mathcal{F}_ny(t)=0,\quad t_*\le t\le t_0.&\end{aligned}\tag{3.12}$$

It can be verified that $\mathcal{F}_j$ maps Y continuously into a relatively compact subset of Y. First, we can show that $\mathcal{F}_j(Y)\subset Y$ by using the expression

$$\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}(s-t_0)^{(j-n)/\alpha}ds=\frac{\varphi_j(t-t_0)}{(1+(j-n)/\alpha)\cdots(n+(j-n)/\alpha)}.\tag{3.13}$$

Next, let $\{y_m(t)\}$ be a sequence of functions in Y converging to $y_0(t)$ on any compact subinterval of $[t_*,\infty)$. Then, by virtue of the Lebesgue convergence theorem it follows that the sequence $\{\mathcal{F}_jy_m(t)\}$ converges to $\mathcal{F}_jy_0(t)$ on compact subintervals of $[t_*,\infty)$, which implies the continuity of the mapping $\mathcal{F}_j$. Finally, since the sets $\mathcal{F}_j(Y)$ and $\mathcal{F}_j'(Y)=\{(\mathcal{F}_jy)':y\in Y\}$ are locally bounded on $[t_*,\infty)$, the Arzelá theorem implies that $\mathcal{F}_j(Y)$ is relatively compact in $C[t_*,\infty)$. Thus, all the hypotheses of the Schauder-Tychonoff fixed point theorem are satisfied, and so there exists a $y\in Y$ such that $y=\mathcal{F}_jy$. In view of (3.11) and (3.12) the fixed element $y=y(t)$ is a solution of the integral equation which is a special case of (2.9) with $\zeta(t)=0$, $\xi_j(t)=(c/(j-n)!)(t-t_0)^{j-n}$ as well as it is a special case as of (2.10) with $\zeta(t)=0$, $\omega_n=c$. By differentiation of these integral equations $2n$ times, we see that $y(t)$ is a solution of the differential equation (A) on $[t_*,\infty)$ satisfying $L_jy(\infty)=c$, that is, $y\in P(\mathrm{I}_j)$.

Case 2. Let $j \in \{0,1,\dots,n-1\}$ and suppose that (3.1) is satisfied. Let $c > 0$ be any given constant and choose $t_0 > 0$ so that

$$\int_{t_0}^{\infty} \frac{t^{n-j-1}}{(n-j-1)!}\left[\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} q(s)\left(\varphi_j(s)\right)^{\beta} ds\right]^{1/\alpha} dt \le B(j!)^{\beta/\alpha} c^{1-\beta/\alpha}, \tag{3.14}$$

where

$$B = 2^{-\beta/\alpha} \quad \text{if } 2n-j-1 \text{ is even}, \qquad B = 2^{-1} \quad \text{if } 2n-j-1 \text{ is odd}. \tag{3.15}$$

Define the constants k_1 and k_2 as follows:

$$\begin{aligned} k_1 &= \frac{c}{j!}, \quad k_2 = \frac{2c}{j!} \quad \text{if } 2n-j-1 \text{ is even}, \\ k_1 &= \frac{c}{2j!}, \quad k_2 = \frac{c}{j!} \quad \text{if } 2n-j-1 \text{ is odd}, \end{aligned} \tag{3.16}$$

and define the set Y by (3.10) with these k_1, k_2. We define the mapping $\mathcal{F}_j : Y \to C[t_*,\infty)$ in the following manner: for $j \in \{1,2,\dots,n-1\}$,

$$\begin{aligned} \mathcal{F}_j y(t) = \frac{c(t-t_0)^j}{j!} &+ (-1)^{2n-j-1}\int_{t_0}^{t} \frac{(t-s)^{j-1}}{(j-1)!}\int_s^{\infty} \frac{(r-s)^{n-j-1}}{(n-j-1)!} \\ &\times \left[\int_r^{\infty} \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma)\left(y(g(\sigma))\right)^{\beta} d\sigma\right]^{1/\alpha} dr\, ds, \\ &\qquad t \ge t_0, \\ \mathcal{F}_j y(t) &= 0, \quad t_* \le t \le t_0 \end{aligned} \tag{3.17}$$

and for $j = 0$,

$$\begin{aligned} \mathcal{F}_0 y(t) = c + (-1)^{2n-1}\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\left[\int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} q(r)\left(y(g(r))\right)^{\beta} dr\right]^{1/\alpha} ds, \quad t \ge t_0, \\ \mathcal{F}_0 y(t) = 0, \quad t_* \le t \le t_0. \end{aligned} \tag{3.18}$$

Then it is routinely verified that $\mathcal{F}_j(Y) \subset Y$, that $\mathcal{F}_j$ is continuous, and that $\mathcal{F}_j(Y)$ is relatively compact in $C[t_*,\infty)$. Consequently, there exists a fixed element $y \in Y$ such that $y = \mathcal{F}_j y$, which is the integral equation (2.13) with $\omega_0 = c$ for $j = 0$ as well as it is the integral equation (2.12) with $\zeta_j^*(t) = (c/j!)(t-t_0)^j$ for $j \in \{1,2,\dots,n-1\}$. It is clear that the fixed element $y = y(t)$ is a solution of (A) belonging to $P(\mathrm{I}_j)$. This completes the proof. □

Unlike the solutions of class $P(\mathrm{I})$ it seems to be very difficult (or impossible) to characterize the existence of solutions of class $P(\mathrm{II})$, and we will be content to give sufficient conditions under which (A) possesses such solutions.

THEOREM 3.2. (i) *Let k be an odd integer less than n. Equation (A) has a solution of class $P(\mathrm{II}_k)$ if*

$$\int_0^\infty t^{n-k-1}\left[\int_t^\infty s^{n-1}q(s)\big(\varphi_k(g(s))\big)^\beta ds\right]^{1/\alpha} dt < \infty, \tag{3.19}$$

$$\int_0^\infty t^{n-k}\left[\int_t^\infty s^{n-1}q(s)\big(\varphi_{k-1}(g(s))\big)^\beta ds\right]^{1/\alpha} dt = \infty. \tag{3.20}$$

(ii) *Let n be odd and let $k = n$. Equation (A) has a solution of class $P(\mathrm{II}_k)$ if*

$$\begin{gathered}\int_0^\infty t^{n-1}q(t)\big(\varphi_n(g(t))\big)^\beta dt < \infty,\\ \int_0^\infty \left[\int_t^\infty s^{n-1}q(s)\big(\varphi_{n-1}(g(s))\big)^\beta ds\right]^{1/\alpha} dt = \infty.\end{gathered} \tag{3.21}$$

(iii) *Let k be an odd integer greater than n and less than $2n$. Equation (A) has a solution of class $P(\mathrm{II}_k)$ if*

$$\begin{gathered}\int_0^\infty t^{2n-k-1}q(t)\big(\varphi_k(g(t))\big)^\beta dt < \infty,\\ \int_0^\infty t^{2n-k}q(t)\big(\varphi_{k-1}(g(t))\big)^\beta dt = \infty.\end{gathered} \tag{3.22}$$

Proof. (i) Let k be an odd integer less than n. The desired solution $y(t)$ will be obtained as a solution of the integral equation

$$\begin{aligned}y(t) &= c\varphi_{k-1}(t)\\ &\quad+\int_{t_0}^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty \frac{(r-s)^{n-1-k}}{(n-1-k)!}\left[\int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!}q(\sigma)\big(y(g(\sigma))\big)^\beta d\sigma\right]^{1/\alpha} dr\,ds, \quad t\ge t_0,\end{aligned} \tag{3.23}$$

where $c > 0$ is fixed and $t_0 > 0$ is chosen so large that $t_* = \min\{t_0, \inf_{t\ge t_0} g(t)\} \ge 1$ and

$$\int_{t_0}^\infty \frac{t^{n-1-k}}{(n-1-k)!}\left[\int_t^\infty \frac{s^{n-1}}{(n-1)!}q(s)\big(\varphi_k(g(s))\big)^\beta ds\right]^{1/\alpha} dt \le 2^{-\beta/\alpha}c^{1-\beta/\alpha}. \tag{3.24}$$

In order to show the existence of solution $y(t)$ of the integral equation (3.23) we will show that mapping $\mathcal{G}_k y(t)$ defined on the set

$$Y = \{y \in C[t_*, \infty) : c\varphi_{k-1}(t) \le y(t) \le 2c\varphi_k(t),\ t \ge t_*\} \tag{3.25}$$

by

$$\mathcal{G}_k y(t) = c\varphi_{k-1}(t) + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-1-k}}{(n-1-k)!} \left[\int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma)\big(y(g(\sigma))\big)^\beta d\sigma \right]^{1/\alpha} dr\, ds, \quad t \geq t_0,$$

$$\mathcal{G}_k y(t) = 0, \quad t_* \leq t \leq t_0 \tag{3.26}$$

has a fixed element in Y. If $y \in Y$, then, using (3.24), we have

$$c\varphi_{k-1}(t) \leq \mathcal{G}_k y(t) \leq c\varphi_{k-1}(t) + c\int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} ds = c\varphi_{k-1}(t) + c\varphi_k(t) \leq 2c\varphi_k(t), \quad t \geq t_*, \tag{3.27}$$

which implies that $\mathcal{G}_k$ maps Y into itself. Since it could be shown without difficulty that $\mathcal{G}_k$ is continuous in the topology of $C[t_*, \infty)$ and that $\mathcal{G}_k(Y)$ is relatively compact in $C[t_*, \infty)$, there exists a fixed element y of $\mathcal{G}_k$ in Y. Repeated differentiation of (3.26) shows that

$$L_{k-1} y(t) = c(k-1)! + \int_{t_0}^{t} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} \left[\int_r^\infty \frac{(\sigma-r)^{n-1}}{(n-1)!} q(\sigma)\big(y(g(\sigma))\big)^\beta d\sigma \right]^{1/\alpha} dr\, ds, \tag{3.28}$$

$$L_k y(t) = \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} \left[\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r)\big(y(g(r))\big)^\beta dr \right]^{1/\alpha} ds, \tag{3.29}$$

for $t \geq t_0$. It is obvious that $L_k y(\infty) = 0$. Evaluating the right-hand side of (3.28), we see that it is bounded from below by

$$\begin{aligned}
&\int_{t_0}^{t} \frac{(s-t_0)^{n-k}}{(n-k)!} \left[\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r)\big(y(g(r))\big)^\beta dr \right]^{1/\alpha} ds \\
&\quad \geq c^{\beta/\alpha} \int_{t_0}^{t} \frac{(s-t_0)^{n-k}}{(n-k)!} \left[\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r)\big(\varphi_{k-1}(g(r))\big)^\beta dr \right]^{1/\alpha} ds,
\end{aligned} \tag{3.30}$$

from which, in view of (3.20), it follows that $L_{k-1}y(\infty) = \infty$. This shows that $y(t)$ belongs to $P(\mathrm{II}_k)$.

(ii) Let n be odd and let $k = n$. Choose $t_0 > 0$ large enough so that $t_* = \min\{t_0, \inf_{t \geq t_0} g(t)\} \geq 1$ and

$$\int_{t_0}^\infty t^{n-1} q(t)\big(\varphi_n(g(t))\big)^\beta dt \leq 2^{-\beta} c^{\alpha-\beta}(n-1)!, \tag{3.31}$$

where $c > 0$ is an arbitrary fixed constant. Define the mapping $\mathcal{G}_n : Y \to C[t_*, \infty)$, with the set Y defined by (3.25), in the following way:

$$\mathcal{G}_n y(t) = c\varphi_{n-1}(t) + \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\int_s^\infty \frac{(r-s)^{n-1}}{(n-1)!} q(r) \big(y(g(r))\big)^\beta dr \right]^{1/\alpha} ds, \quad t \geq t_0,$$
$$\mathcal{G}_n y(t) = 0, \quad t_* \leq t \leq t_0. \tag{3.32}$$

Proceeding as in case (i), we can prove that there exists a fixed element $y = y(t)$ of the mapping $\mathcal{G}_n$, which clearly satisfies $c\varphi_{n-1}(t) \leq y(t) \leq 2c\varphi_n(t)$ for $t \geq t_*$. Likewise we can show that $L_{n-1}y(\infty) = \infty$ and $L_n y(\infty) = 0$, which implies that $y(t) \in P(\mathrm{II}_k)$.

(iii) Let k be an odd integer greater than n and less than $2n$. In this case, we let $c > 0$ and choose $t_0 \geq 0$ large enough so that $t_* = \min\{t_0, \inf_{t \geq t_0} g(t)\} \geq 1$ and

$$\begin{aligned} &\int_{t_0}^\infty \frac{t^{2n-1-k}}{(2n-1-k)!} q(t) \big(\varphi_k(g(t))\big)^\beta dt \\ &\quad \leq 2^{-\beta} c^{\alpha-\beta} (k-n)! \left[\left(1 + \frac{k-n}{\alpha}\right) \cdots \left(n + \frac{k-n}{\alpha}\right) \right]^\alpha. \end{aligned} \tag{3.33}$$

Define the mapping $\mathcal{G}_k : Y \to C[t_*, \infty)$ by

$$\begin{aligned} \mathcal{G}_k y(t) = {} & c\varphi_{k-1}(t) \\ & + \int_{t_0}^t \frac{(t-s)^{n-1}}{(n-1)!} \left[\int_{t_0}^s \frac{(s-r)^{k-1-n}}{(k-1-n)!} \int_r^\infty \frac{(\sigma-r)^{2n-1-k}}{(2n-1-k)!} q(\sigma) \big(y(g(\sigma))\big)^\beta d\sigma\, dr \right]^{1/\alpha} ds, \\ & \qquad t \geq t_0, \\ \mathcal{G}_k y(t) = {} & 0, \quad t_* \leq t \leq t_0. \end{aligned} \tag{3.34}$$

It is easy to verify that the mapping $\mathcal{G}_k y(t)$ maps the set Y defined by (3.25) into a relatively compact subset of Y. Therefore, $\mathcal{G}_k$ has a fixed element $y = y(t)$ in Y. That $y(t)$ is a solution of class $P(\mathrm{II}_k)$ follows from differentiation of (3.34) combined with the observation below:

$$\begin{aligned} L_{k-1} y(t) &\geq \int_{t_0}^t \int_s^\infty \frac{(r-s)^{2n-k-1}}{(2n-k-1)!} q(r) \big(y(g(r))\big)^\beta dr ds \\ &\geq \int_{t_0}^t \frac{(s-t_0)^{2n-k}}{(2n-k)!} q(s) \big(y(g(s))\big)^\beta ds \\ &\geq c^\beta \int_{t_0}^t \frac{(s-t_0)^{2n-k}}{(2n-k)!} q(s) \big(\varphi_{k-1}(g(s))\big)^\beta ds \longrightarrow \infty, \quad \text{as } t \longrightarrow \infty, \\ L_k y(t) &= \int_t^\infty \frac{(s-t)^{2n-k-1}}{(2n-k-1)!} q(s) \big(y(g(s))\big)^\beta ds \longrightarrow 0, \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{3.35}$$

This completes the proof of Theorem 3.2. □

4. Comparison theorems

In order to establish criteria (preferably sharp) for all solutions of (A) to be oscillatory, we are essentially based on the following oscillation result of Tanigawa [7] for the even-order nonlinear differential equation (B).

Theorem 4.1. (i) *Let $\alpha > \beta$. All solutions of (B) are oscillatory if and only if*

$$\int_0^\infty (\varphi_{2n-1}(t))^\beta q(t)\,dt = \int_0^\infty t^{(n+(n-1)/\alpha)\beta} q(t)\,dt = \infty. \tag{4.1}$$

(ii) *Let $\alpha < \beta$. All solutions of (B) are oscillatory if and only if*

$$\int_0^\infty t^{n-1} q(t)\,dt = \infty \tag{4.2}$$

or

$$\int_0^\infty t^{n-1} q(t)\,dt < \infty, \qquad \int_0^\infty t^{n-1}\left[\int_t^\infty s^{n-1} q(s)\,ds\right]^{1/\alpha} dt = \infty. \tag{4.3}$$

Our idea is to deduce oscillation criteria for (A) from Theorem 4.1 by using two comparison theorems which relate oscillation (nonoscillation) of the equation

$$\left(|u^{(n)}(t)|^\alpha \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + F(t, u(h(t))) = 0 \tag{4.4}$$

to that of the equations

$$\left(|v^{(n)}(t)|^\alpha \operatorname{sgn} v^{(n)}(t)\right)^{(n)} + G(t, v(k(t))) = 0, \tag{4.5}$$

$$\left(|w^{(n)}(t)|^\alpha \operatorname{sgn} w^{(n)}(t)\right)^{(n)} + \frac{l'(t)}{h'(h^{-1}(l(t)))} F(h^{-1}(l(t)), w(l(t))) = 0. \tag{4.6}$$

Accordingly, the aim of this section is to establish such comparison theorems.

With regard to (4.4)–(4.6) it is assumed that

(i) $\alpha > 0$ is a constant;

(ii) h, k, and l are continuously differentiable functions on $[0,\infty)$ such that $h'(t) > 0$, $k'(t) > 0$, $l'(t) > 0$, $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} k(t) = \lim_{t\to\infty} l(t) = \infty$;

(iii) F and G are continuous functions on $[0,\infty) \times \mathbb{R}$ such that $uF(t,u) \ge 0$, $uG(t,u) \ge 0$ and $F(t,u)$, $G(t,u)$ are nondecreasing in u for any fixed $t \ge 0$.

Theorem 4.2. *Suppose that*

$$\begin{gathered} h(t) \ge k(t), \quad t \ge 0, \\ F(t,x)\operatorname{sgn} x \ge G(t,x)\operatorname{sgn} x, \quad (t,x) \in [0,\infty) \times \mathbb{R}. \end{gathered} \tag{4.7}$$

If all the solutions of (4.5) are oscillatory, then so are all the solutions of (4.4).

Theorem 4.3. *Suppose that $l(t) \ge h(t)$ for $t \ge 0$. If all the solutions of (4.6) are oscillatory, then so are all the solutions of (4.4).*

These theorems can be regarded as generalizations of the main comparison principles developed in the papers [3, 5] to differential equations involving higher-order nonlinear differential operators. To prove these theorems we need the following lemma which compares the differential equation (4.4) with the differential inequality

$$\left(|z^{(n)}(t)|^{\alpha}\operatorname{sgn} z^{(n)}(t)\right)^{(n)}+F(t,z(h(t)))\leq 0. \tag{4.8}$$

Lemma 4.4. *If there exists an eventually positive function satisfying (4.8), then (4.4) has an eventually positive solution.*

Proof of Lemma 4.4. Let $z(t)$ be an eventually positive solution of (4.8). It is easy to see that $z(t)$ satisfies Lemma 2.1, that is,

$$\begin{aligned} L_i z(t)>0,\quad & t\geq t_1,\ \text{for } i=0,1,\dots,k-1,\\ (-1)^{i-k}L_i z(t)>0,\quad & t\geq t_1,\ \text{for } i=k,k+1,\dots,2n-1, \end{aligned} \tag{4.9}$$

provided $t_1>0$ is sufficiently large. Put $t_*=\min\{t_1,\inf_{t\geq t_1}h(t)\}$. Let us now consider the set

$$U=\{u\in C[t_*,\infty):0\leq u(t)\leq z(t),\ t\geq t_*\}, \tag{4.10}$$

and the mapping $\mathcal{H}_k:U\to C[t_*,\infty)$ defined in the appropriate way corresponding to the cases $k\in\{n+1,\dots,2n-1\}$, $k=n$, and $k\in\{1,2,\dots,n\}$.

If $n<k\leq 2n-1$, then, integrating (4.8) $2n-k$ times from t to ∞, we have

$$\left(|z^{(n)}(t)|^{\alpha}\operatorname{sgn} z^{(n)}(t)\right)^{(k-n)}\geq \omega_k+\int_t^{\infty}\frac{(s-t)^{2n-k-1}}{(2n-k-1)!}F(s,z(h(s)))\,ds,\quad t\geq t_1, \tag{4.11}$$

where $\omega_k=\lim_{t\to\infty}L_k z(t)\geq 0$. Further integrations of (4.11) k times from t_1 to t yields the inequality

$$\begin{aligned} z(t)\geq{}& z(t_1)\\ &+\int_{t_1}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\Bigg[\omega_k\frac{(s-t_1)^{k-n}}{(k-n)!}\\ &\qquad+\int_{t_1}^{s}\frac{(s-r)^{k-n-1}}{(k-n-1)!}\int_r^{\infty}\frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!}F(\sigma,z(h(\sigma)))\,d\sigma\,dr\Bigg]^{1/\alpha}ds,\quad t\geq t_1. \end{aligned} \tag{4.12}$$

Define the mapping $\mathcal{H}_k$ by

$$\begin{aligned}\mathcal{H}_k u(t) = z(t_1) &+ \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\Big[\omega_k \frac{(s-t_1)^{k-n}}{(k-n)!} \\ &+ \int_{t_1}^{s} \frac{(s-r)^{k-n-1}}{(k-n-1)!}\int_r^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} F(\sigma,u(h(\sigma)))\,d\sigma\,dr\Big]^{1/\alpha} ds, \quad t \ge t_1, \\ \mathcal{H}_k u(t) &= z(t), \quad t_* \le t \le t_1.\end{aligned} \tag{4.13}$$

If $k = n$, then, integrating (4.8) n times from t to ∞, we have

$$|z^{(n)}(t)|^{\alpha} \operatorname{sgn} z^{(n)}(t) \ge \omega_n + \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} F(s,z(h(s)))\,ds. \tag{4.14}$$

Moreover, n times integration of (4.14) on $[t_1,t]$ yields the following integral inequality:

$$z(t) \ge z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\Big[\omega_n + \int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(r,z(h(r)))\,dr\Big]^{1/\alpha} ds. \tag{4.15}$$

Define the mapping $\mathcal{H}_n$ by

$$\begin{aligned}\mathcal{H}_n u(t) &= z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!}\Big[\omega_n + \int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(r,u(h(r)))\,dr\Big]^{1/\alpha} ds, \quad t \ge t_1, \\ \mathcal{H}_n u(t) &= z(t), \quad t_* \le t \le t_1.\end{aligned} \tag{4.16}$$

If $1 \le k < n$, then, integrating (4.8) $2n-k(=n+(n-k))$ times from t to ∞, we have

$$\begin{aligned}z^{(k)}(t) &\ge \omega_k + \int_t^{\infty} \frac{(s-t)^{n-k-1}}{(n-k-1)!}\Big[\int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(r,z(h(r)))\,dr\Big]^{1/\alpha} ds \\ &\ge \int_t^{\infty} \frac{(s-t)^{n-k-1}}{(n-k-1)!}\Big[\int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(r,z(h(r)))\,dr\Big]^{1/\alpha} ds.\end{aligned} \tag{4.17}$$

Futhermore, integrating (4.17) k times t_1 to t, we obtain

$$\begin{aligned}z(t) &\ge z(t_1) \\ &+ \int_{t_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!}\Big[\int_r^{\infty} \frac{(\sigma-r)^{n-1}}{(n-1)!} F(\sigma,z(h(\sigma)))\,d\sigma\Big]^{1/\alpha} dr\,ds.\end{aligned} \tag{4.18}$$

Define the mapping $\mathcal{H}_k$ by

$$\mathcal{H}_k u(t) = z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \left[\int_r^{\infty} \frac{(\sigma-r)^{n-1}}{(n-1)!} F(\sigma, u(h(\sigma)))\, d\sigma \right]^{1/\alpha} dr\, ds, \quad t \geq t_1,$$
$$\mathcal{H}_k u(t) = z(t), \quad t_* \leq t \leq t_1. \tag{4.19}$$

Then, it is easily verified that (i) $\mathcal{H}_k$ maps U into itself, (ii) $\mathcal{H}_k$ is a continuous mapping, and (iii) $\mathcal{H}_k(U)$ is a relatively compact subset of $C[t_*, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, $\mathcal{H}_k$ has a fixed element $u \in U$ such that $u = \mathcal{H}_k u$, which clearly satisfies the integral equations (4.13), (4.16), and (4.19) on $[t_*, \infty)$, respectively, that is,

$$u(t) = z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\omega_k \frac{(s-t_1)^{k-n}}{(k-n)!} + \int_{t_1}^{s} \frac{(s-r)^{k-n-1}}{(n-k-1)!} \int_r^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} F(\sigma, u(h(\sigma)))\, d\sigma\, dr \right]^{1/\alpha} ds, \quad t \geq t_1, \tag{4.20}$$

for $n < k \leq 2n-1$,

$$u(t) = z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\omega_n + \int_s^{\infty} \frac{(r-s)^{n-1}}{(n-1)!} F(r, u(h(r)))\, dr \right]^{1/\alpha} ds, \quad t \geq t_1, \tag{4.21}$$

for $k = n$, and

$$u(t) = z(t_1) + \int_{t_1}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} \left[\int_r^{\infty} \frac{(\sigma-r)^{n-1}}{(n-1)!} F(\sigma, u(h(\sigma)))\, d\sigma \right]^{1/\alpha} dr\, ds, \quad t \geq t_1, \tag{4.22}$$

for $1 \leq k < n$. Differentiation of (4.20), (4.21), and (4.22), respectively, shows that $u(t)$ is a positive solution of (4.4). This completes the proof of Lemma 4.4. □

Proof of Theorem 4.2. It is sufficient to prove that if (4.4) has an eventually positive solution, then so does (4.5).

Let $u(t)$ be an eventually positive solution of (4.4). Note that $u(t)$ is monotone increasing for all sufficiently large t. In view of (4.7), we see that there exists $t_0 > 0$ such that $u(h(t)) \geq u(k(t))$, $t \geq t_0$, and

$$F(t, u(h(t))) \geq G(t, u(k(t))), \quad t \geq t_0. \tag{4.23}$$

This together yields

$$\left(\left| u^{(n)}(t) \right|^{\alpha} \operatorname{sgn} u^{(n)}(t) \right)^{(n)} + G(t, u(k(t))) \leq 0, \quad t \geq t_0, \tag{4.24}$$

and application of Lemma 4.4 then shows that (4.5) has an eventually positive solution $v(t)$. This completes the proof. □

Proof of Theorem 4.3. The statement of the theorem is equivalent to the statement that if there exists an eventually positive solution of (4.4) then the same is true of (4.6).

Let $u(t)$ be an eventually positive solution of (4.4). The following inequalities are possible for some odd $k \in \{1, 3, \dots, 2n-1\}$:

$$\begin{gathered} L_i u(t) > 0, \quad i = 0, 1, \dots, k-1 \ \forall \text{ large } t, \\ (-1)^{i-k} L_i u(t) > 0, \quad i = k, k+1, \dots, 2n-1 \ \forall \text{ large } t. \end{gathered} \tag{4.25}$$

If $n < k \leq 2n-1$, then we have

$$\begin{aligned} u(t) \geq{} & u(t_1) \\ & + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\omega_k \frac{(s-t_1)^{k-n}}{(k-n)!} \right. \\ & \left. + \int_{t_1}^{s} \frac{(s-r)^{k-n-1}}{(n-k-1)!} \int_{r}^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} F(\sigma, u(h(\sigma))) \, d\sigma \, dr \right]^{1/\alpha} ds, \quad t \geq t_1, \end{aligned} \tag{4.26}$$

where $\omega_k = \lim_{t \to \infty} L_k u(t) \geq 0$. Combining (4.26) with the following inequality:

$$\begin{aligned} & \int_{r}^{\infty} \frac{(\sigma-r)^{2n-k-1}}{(2n-k-1)!} F(\sigma, u(h(\sigma))) \, d\sigma \\ & \quad \geq \int_{l^{-1}(h(r))}^{\infty} \frac{(\rho-r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho \\ & \quad \geq \int_{r}^{\infty} \frac{(\rho-r)^{2n-k-1}}{(2n-k-1)!} F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho, \end{aligned} \tag{4.27}$$

we get

$$\begin{aligned} u(t) \geq{} & u(t_1) \\ & + \int_{t_1}^{t} \frac{(t-s)^{n-1}}{(n-1)!} \left[\omega_k \frac{(s-t_1)^{k-n}}{(k-n)!} \right. \\ & + \int_{t_1}^{s} \frac{(s-r)^{k-n-1}}{(k-n-1)!} \int_{r}^{\infty} \frac{(\rho-r)^{2n-k-1}}{(2n-k-1)!} \\ & \left. \times F(h^{-1}(l(\rho)), u(l(\rho))) \frac{l'(\rho)}{h'(h^{-1}(l(\rho)))} \, d\rho \, dr \right]^{1/\alpha} ds, \quad t \geq t_1. \end{aligned} \tag{4.28}$$

If $k=n$, then $u(t)$ satisfies the inequality

$$u(t)\geq u(t_1)+\int_{t_1}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\omega_n+\int_s^\infty\frac{(r-s)^{n-1}}{(n-1)!}F(r,u(h(r)))\,dr\right]^{1/\alpha}ds,\quad t\geq t_1, \tag{4.29}$$

where $\omega_n=\lim_{t\to\infty}L_nu(t)\geq 0$.

If $1\leq k<n$, then $u(t)$ satisfies the inequality

$$\begin{aligned}u(t)&\geq u(t_1)\\&\quad+\int_{t_1}^{t}\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)}!\left[\int_r^\infty\frac{(\sigma-r)^{n-1}}{(n-1)!}F(\sigma,u(h(\sigma)))\,d\sigma\right]^{1/\alpha}dr\,ds,\quad t\geq t_1.\end{aligned} \tag{4.30}$$

We now observe that an essential part of the proof of Lemma 4.4 has been proving the existence of the solution for each of the integral equations (4.20), (4.21), and (4.22). That has been done by the application of Schauder-Tychonoff fixed point theorem on the basis of the corresponding integral inequalities (4.12), (4.15), and (4.18). Proceeding here in a similar way, on the basis that $u(t)$ satisfies (4.28), (4.29), and (4.30), respectively, we conclude that there exists a positive solution for each of the following equations:

$$\begin{aligned}w(t)&=u(t_1)\\&\quad+\int_{t_1}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\left[\omega_k\frac{(s-t_1)^{k-n}}{(k-n)!}\right.\\&\qquad+\int_{t_1}^{s}\frac{(s-r)^{k-n-1}}{(k-n-1)!}\int_r^\infty\frac{(\rho-r)^{2n-k-1}}{(2n-k-1)!}\\&\qquad\left.\times F(h^{-1}(l(\rho)),\omega(l(\rho)))\frac{l'(\rho)}{h'(h^{-1}(l(\rho)))}\,d\rho\,dr\right]^{1/\alpha}ds,\quad t\geq t_1\end{aligned} \tag{4.31}$$

for $n<k\leq 2n-1$,

$$\begin{aligned}w(t)=u(t_1)+\int_{t_1}^{t}\frac{(t-s)^{n-1}}{(n-1)!}&\left[\omega_n+\int_s^\infty\int_r^\infty\frac{(\rho-r)^{n-1}}{(n-1)!}\right.\\&\left.\times F(h^{-1}(l(\rho)),\omega(l(\rho)))\frac{l'(\rho)}{h'(h^{-1}(l(\rho)))}\,d\rho\right]^{1/\alpha}ds,\quad t\geq t_1\end{aligned} \tag{4.32}$$

for $k=n$, and

$$\begin{aligned}w(t)&=u(t_1)\\&\quad+\int_{t_1}^{t}\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)!}\\&\quad\times\left[\int_r^\infty\frac{(\rho-r)^{n-1}}{(n-1)!}F(h^{-1}(l(\rho)),\omega(l(\rho)))\frac{l'(\rho)}{h'(h^{-1}(l(\rho)))}\,d\rho\right]^{1/\alpha}dr\,ds,\quad t\geq t_1\end{aligned} \tag{4.33}$$

for $1 \le k < n$. It can be checked by differentiation that $w(t)$ is a positive solution of the differential equation (4.6) in each of the three cases. This completes the proof of Theorem 4.3. □

5. Oscillation criteria

The aim of this section is to establish criteria (preferably sharp) for all solutions of (A) to be oscillatory. Oscillation theorems will be established first in the sublinear case of (A) for $\alpha > \beta$ as well as in the superlinear case for $\alpha < \beta$. We first give the sufficient condition for all of solutions of sublinear equation (A) to be oscillatory.

Theorem 5.1. *Let $\alpha > \beta$. Suppose that there exists a continuously differentiable function $h : [0,\infty) \to (0,\infty)$ such that $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, and*

$$\min\{t, g(t)\} \ge h(t) \quad \forall \text{ large } t. \tag{5.1}$$

If

$$\int_0^\infty (h(t))^{(n+(n-1)/\alpha)\beta} q(t)\,dt = \infty, \tag{5.2}$$

then all solutions of (A) are oscillatory.

Proof. Let us consider the equations

$$\left(\left|z^{(n)}(t)\right|^{\alpha} \operatorname{sgn} z^{(n)}(t)\right)^{(n)} + q(t)\left|z(h(t))\right|^{\beta} \operatorname{sgn} z(h(t)) = 0, \tag{5.3}$$

$$\left(\left|w^{(n)}(t)\right|^{\alpha} \operatorname{sgn} w^{(n)}(t)\right)^{(n)} + \frac{q(h^{-1}(t))}{h'(h^{-1}(t))}\left|w(t)\right|^{\beta} \operatorname{sgn} w(t) = 0. \tag{5.4}$$

Since by (5.2),

$$\int^\infty t^{(n+(n-1)/\alpha)\beta} \frac{q(h^{-1}(t))}{h'(h^{-1}(t))}\,dt = \int^\infty (h(\tau))^{(n+(n-1)/\alpha)\beta} q(\tau)\,d\tau = \infty, \tag{5.5}$$

Theorem 4.1(i) implies that all solutions of (5.4) are oscillatory. Application of Theorem 4.3 then shows that all solutions of (5.3) are oscillatory, and the conclusion of the theorem follows from comparison of (A) with (5.3) by means of Theorem 4.2. □

It will be shown below that there is a class of sublinear equations of the type (A) for which the oscillation situation can be completely characterized.

Theorem 5.2. *Let $\alpha > \beta$ and suppose that*

$$\limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{5.6}$$

Then, all solutions of (A) are oscillatory if and only if

$$\int_0^\infty (g(t))^{(n+(n-1)/\alpha)\beta} q(t)\,dt = \infty. \tag{5.7}$$

Proof. That the oscillation of (A) implies (5.7) is an immediate consequence of Theorem 3.1.

Assume now that (5.7) is satisfied. The condition (5.6) means that there exists a constant $c > 1$ such that

$$g(t) \leq ct \quad \forall \text{ sufficiently large } t. \tag{5.8}$$

Consider the ordinary differential equation

$$\left(|z^{(n)}(t)|^{\alpha} \operatorname{sgn} z^{(n)}(t)\right)^{(n)} + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))} |z(t)|^{\beta} \operatorname{sgn} z(t) = 0. \tag{5.9}$$

Since by (5.7),

$$\int^{\infty} t^{(n+(n-1)/\alpha)\beta} \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))} dt = \int^{\infty} \left(\frac{g(t)}{c}\right)^{(n+(n-1)/\alpha)\beta} q(t)dt = \infty, \tag{5.10}$$

all solutions of (5.9) are oscillatory according to Theorem 4.1(i). From Theorem 5.1 it follows that the equation

$$\left(|u^{(n)}(t)|^{\alpha} \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + \frac{cq(g^{-1}(ct))}{g'(g^{-1}(ct))} |u(ct)|^{\beta} \operatorname{sgn} u(ct) = 0 \tag{5.11}$$

has only oscillatory solutions. Comparison of (A) with (5.11) via Theorem 5.2 then leads to the desired conclusion of the theorem. □

Oscillation criteria for (A) in the superlinear case are given in the following theorems.

Theorem 5.3. *Let $\alpha < \beta$. Suppose that there exists a continuously differentiable function $h : [0,\infty) \to (0,\infty)$ such that $h'(t) > 0$, $\lim_{t\to\infty} h(t) = \infty$, and (5.1) is satisfied. If*

$$\int_0^{\infty} (h(t))^{n-1} q(t)dt = \infty \tag{5.12}$$

or

$$\int_0^{\infty} (h(t))^{n-1} q(t)dt < \infty, \qquad \int_0^{\infty} t^{n-1} \left[\int_{h^{-1}(t)}^{\infty} (h(s))^{n-1} q(s)ds\right]^{1/\alpha} dt = \infty, \tag{5.13}$$

then all solutions of (A) are oscillatory.

The proof of Theorem 5.3 is similar to the proof of Theorem 5.1, so it will be omitted.

Theorem 5.4. *Let $\alpha < \beta$ and suppose that*

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0. \tag{5.14}$$

Then, all solutions of (A) are oscillatory if and only if either (4.2) or (4.3) holds.

Proof. We need only to prove the "if" part of the theorem, since the "only if" part follows immediately from Theorem 3.1.

In view of (5.14) there exists a positive constant $c<1$ such that

$$g(t) \geq ct \quad \forall \text{ sufficiently large } t. \tag{5.15}$$

Consider the ordinary differential equation

$$\left(\left|z^{(n)}(t)\right|^{\alpha} \operatorname{sgn} z^{(n)}(t)\right)^{(n)} + \frac{1}{c} q\left(\frac{t}{c}\right)\left|z(t)\right|^{\beta} \operatorname{sgn} z(t) = 0. \tag{5.16}$$

Using the assumptions on $q(t)$, we see that either

$$\int_0^{\infty} \frac{t^{n-1}}{c} q\left(\frac{t}{c}\right) dt = c^{n-1} \int_0^{\infty} \xi^{n-1} q(\xi) d\xi = \infty \tag{5.17}$$

or

$$\int_0^{\infty} t^{n-1}\left[\int_t^{\infty} s^{n-1}\frac{1}{c} q\left(\frac{s}{c}\right) ds\right]^{1/\alpha} dt = c^{n+(n-1)/\alpha} \int_0^{\infty} \eta^{n-1}\left[\int_{\eta}^{\infty} \xi^{n-1} q(\xi) d\xi\right]^{1/\alpha} d\eta = \infty, \tag{5.18}$$

which implies that all the solutions of (5.16) are oscillatory. We now apply one of the comparison principles, Theorem 5.2, to compare (5.16) with the equation

$$\left(\left|u^{(n)}(t)\right|^{\alpha} \operatorname{sgn} u^{(n)}(t)\right)^{(n)} + q(t)\left|u(ct)\right|^{\beta} \operatorname{sgn} u(ct) = 0, \tag{5.19}$$

and to conclude that (5.19) has the same oscillatory behavior as (5.16). Since (5.15) holds, applying another comparison principle, Theorem 5.1, we conclude that all the solutions of (A) are necessarily oscillatory. This completes the proof. □

From the proofs of Theorems 5.2 and 5.4 we see that in case $\alpha>\beta$ or $\alpha<\beta$, the oscillation of the functional differential equation

$$\left(\left|y^{(n)}(t)\right|^{\alpha} \operatorname{sgn} y^{(n)}(t)\right)^{(n)} + q(t)\left|y(ct)\right|^{\beta} \operatorname{sgn} y(ct) = 0 \tag{5.20}$$

is equivalent to that of the ordinary differential equation (B). This observation combined with comparison Theorems 5.1 and 5.2 will lead to the following result.

Corollary 5.5. *Let either $\alpha>\beta$ or $\alpha<\beta$, and suppose that $g(t)$ in (A) satisfies*

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t}, \qquad \limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{5.21}$$

Then all solutions of (A) are oscillatory if and only if the same is true for (B).

6. Example

We present here an example which illustrates oscillation and nonoscillation theorems proved in Sections 3 and 5.

Example 6.1. Consider the equation

$$(|y^{(n)}(t)|^{\alpha}\operatorname{sgn} y^{(n)}(t))^{(n)} + t^{-\lambda}|y(t^{\gamma})|^{\beta}\operatorname{sgn} y(t^{\gamma}) = 0, \tag{6.1}$$

where α, β, γ are fixed positive constants and λ is a varying parameter.

It is easy to check that, written for (6.1),

$$(3.1) \text{ is equivalent to } \lambda > n + \alpha(n-j) + \beta\gamma j, \tag{6.2}$$

$$(3.2) \text{ is equivalent to } \lambda > 2n - j + \left(n + \frac{j-n}{\alpha}\right)\beta\gamma; \tag{6.3}$$

so that from Theorem 3.1 we see that (6.1) has a positive solution belonging to the class $P(\mathrm{I}_j)$ if and only if

$$\begin{gathered} \lambda > n + \alpha(n-j) + \beta\gamma j, \quad j \in \{0,1,\ldots,n-1\}, \\ \lambda > 2n - j + \left(n + \frac{j-n}{\alpha}\right)\beta\gamma, \quad j \in \{n, n+1,\ldots,2n-1\}. \end{gathered} \tag{6.4}$$

It follows that all solutions of (6.1) belong to $P(\mathrm{I})$ if either

$$\alpha \le \beta\gamma, \quad \lambda > 1 + \left(n + \frac{n-1}{\alpha}\right)\beta\gamma \tag{6.5}$$

or

$$\alpha > \beta\gamma, \quad \lambda > n + n\alpha. \tag{6.6}$$

It is easy to see that for (6.1) the conditions $\{(3.19), (3.20)\}$, (3.21), and (3.22) guarantee the existence of solutions of class $P(\mathrm{II}_k)$, $k \in \{1,3,\ldots,2n-1\}$ only under the condition $\alpha > \beta\gamma$. The conclusions which follow from Theorem 3.2 are

(i) (6.1) has solutions of $P(\mathrm{II}_k)$ $(1 \le k \le n)$ if

$$\alpha > \beta\gamma, \quad n + \alpha(n-k) + \beta\gamma k < \lambda \le n + \alpha(n-k) + \beta\gamma k + \alpha - \beta\gamma; \tag{6.7}$$

(ii) (6.1) has solutions of $P(\mathrm{II}_k)$ $(n < k \le 2n-1)$ if

$$\alpha > \beta\gamma, \quad 2n - k + \beta\gamma\left(n + \frac{k-n}{\alpha}\right) < \lambda \le 2n - k + \beta\gamma\left(n + \frac{k-n}{\alpha}\right) + 1 - \frac{\beta\gamma}{\alpha}. \tag{6.8}$$

We now want oscillation criteria for (6.1).

Suppose that $\alpha > \beta$. If $\gamma \le 1$, then from Theorem 5.2 we conclude that all solutions of (6.1) are oscillatory if and only if

$$\lambda \le 1 + \left(n + \frac{n-1}{\alpha}\right)\beta\gamma. \tag{6.9}$$

If $\gamma > 1$, then, applying Theorem 5.1, we see that all solutions of (6.1) are oscillatory if

$$\lambda \le 1 + \left(n + \frac{n-1}{\alpha}\right)\beta. \tag{6.10}$$

Suppose that $\alpha < \beta$. If $\gamma > 1$, then from Theorem 5.4 we conclude that all solutions of (6.1) are oscillatory if and only if

$$\lambda \leq n + n\alpha. \tag{6.11}$$

If $\gamma \leq 1$, then Theorem 5.3 applies to (6.1) and leads to the conclusion that all of its solutions are oscillatory if

$$\lambda \leq 1 + \gamma(n-1) + \alpha\gamma n. \tag{6.12}$$

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References

[1] I. T. Kiguradze, *On the oscillatory character of solutions of the equation* $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, Matematicheskii Sbornik. New Series **65 (107)** (1964), 172–187 (Russian).

[2] T. Kusano and B. S. Lalli, *On oscillation of half-linear functional-differential equations with deviating arguments*, Hiroshima Mathematical Journal **24** (1994), no. 3, 549–563.

[3] T. Kusano and M. Naito, *Comparison theorems for functional-differential equations with deviating arguments*, Journal of the Mathematical Society of Japan **33** (1981), no. 3, 509–532.

[4] T. Kusano and J. Wang, *Oscillation properties of half-linear functional-differential equations of the second order*, Hiroshima Mathematical Journal **25** (1995), no. 2, 371–385.

[5] W. E. Mahfoud, *Comparison theorems for delay differential equations*, Pacific Journal of Mathematics **83** (1979), no. 1, 187–197.

[6] T. Tanigawa, *Oscillation and nonoscillation theorems for a class of fourth order quasilinear functional differential equations*, Hiroshima Mathematical Journal **33** (2003), no. 3, 297–316.

[7] ———, *Oscillation criteria for a class of higher order nonlinear differential equations*, Memoirs on Differential Equations and Mathematical Physics **37** (2006), 137–152.

[8] T. Tanigawa and W. Fentao, *On the existence of positive solutions for a class of even order quasilinear differential equations*, Advances in Mathematical Sciences and Applications **14** (2004), no. 1, 75–85.

[9] J. Wang, *Oscillation and nonoscillation theorems for a class of second order quasilinear functional-differential equations*, Hiroshima Mathematical Journal **27** (1997), no. 3, 449–466.

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