# SUPPLEMENTS TO KNOWN MONOTONICITY RESULTS AND INEQUALITIES FOR THE GAMMA AND INCOMPLETE GAMMA FUNCTIONS

A. LAFORGIA AND P. NATALINI

*Received 29 June 2005; Accepted 3 July 2005*

We denote by $\Gamma(a)$ and $\Gamma(a;z)$ the gamma and the incomplete gamma functions, respectively. In this paper we prove some monotonicity results for the gamma function and extend, to $x > 0$, a lower bound established by Elbert and Laforgia (2000) for the function $\int_0^x e^{-t^p}dt = [\Gamma(1/p) - \Gamma(1/p;x^p)]/p$, with $p > 1$, only for $0 < x < (9(3p+1)/4(2p+1))^{1/p}$.



## 1. Introduction and background

In a paper of 1984, Kershaw and Laforgia [4] investigated, for real $\alpha$ and positive $x$, some monotonicity properties of the function $x^\alpha[\Gamma(1+1/x)]^x$ where, as usual, $\Gamma$ denotes the gamma function defined by

$$\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}dt, \quad a > 0. \tag{1.1}$$

In particular they proved that for $x > 0$ and $\alpha = 0$ the function $[\Gamma(1+1/x)]^x$ decreases with $x$, while when $\alpha = 1$ the function $x[\Gamma(1+1/x)]^x$ increases. Moreover they also showed that the values $\alpha = 0$ and $\alpha = 1$, in the properties mentioned above, cannot be improved if $x \in (0,+\infty)$. In this paper we continue the investigation on the monotonicity properties for the gamma function proving, in Section 2, the following theorem.

Theorem 1.1. *The functions* $f(x) = \Gamma(x+1/x)$, $g(x) = [\Gamma(x+1/x)]^x$ *and* $h(x) = \Gamma'(x+1/x)$ *decrease for* $0 < x < 1$, *while increase for* $x > 1$.

In Section 3, we extend a result previously established by Elbert and Laforgia [2] related to a lower bound for the integral function $\int_0^x e^{-t^p}dt$ with $p > 1$. This function can be expressed by the gamma function (1.1) and incomplete gamma function defined by

$$\Gamma(a;z) = \int_z^\infty e^{-t}t^{a-1}dt, \quad a > 0,\ z > 0. \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 48727, Pages 1–8
DOI 10.1155/JIA/2006/48727

In fact we have

$$\int_0^x e^{-t^p}\,dt = \frac{\Gamma(1/p) - \Gamma(1/p;x^p)}{p}. \tag{1.3}$$

If $p = 2$ it reduces, by means of a multiplicative constant, to the well-known error function $\operatorname{erf}(x)$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \tag{1.4}$$

or to the complementary error function $\operatorname{erf}c(x)$

$$\operatorname{erf}c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \tag{1.5}$$

Many authors established inequalities for the function $\int_0^x e^{-t^p}\,dt$.

Gautschi [3] proved the following lower and upper bounds

$$\frac{1}{2}\left[\left(x^p+2\right)^{1/p} - x\right] < e^{x^p}\int_x^\infty e^{-t^p}\,dt \le a_p\left[\sqrt{x^2+\frac{1}{a_p}} - x\right], \tag{1.6}$$

where $p > 1$, $x \ge 0$ and

$$a_p = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{p/(p-1)}. \tag{1.7}$$

The integral in (1.6) can be expressed in the following way

$$\int_x^\infty e^{-t^p}\,dt = \frac{1}{p}\Gamma\left(\frac{1}{p};x^p\right) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) - \int_0^x e^{-t^p}\,dt. \tag{1.8}$$

Alzer [1] found the following inequalities

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-x^p}\right)^{1/p} < \int_0^x e^{-t^p}\,dt < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-\alpha x^p}\right)^{1/p}, \tag{1.9}$$

where $p > 1$, $x > 0$ and

$$\alpha = \left[\Gamma\left(1+\frac{1}{p}\right)\right]^{-p}. \tag{1.10}$$

Feng Qi and Sen-lin Guo [5] establisched, among others, the following lower bounds for $p > 1$

$$\frac{1}{2}x\left(1+e^{-x^p}\right) \le \int_0^x e^{-t^p}\,dt, \tag{1.11}$$

if $0 < x < (1-1/p)^{1/p}$, while

$$\frac{1}{2}\left(1-\frac{1}{p}\right)^{1/p}\left(1+e^{1/p-1}\right)+\left(x-\left(1-\frac{1}{p}\right)^{1/p}\right)e^{-((x+(1-1/p)^{1/p})/2)^p} \leq \int_0^x e^{-t^p}\,dt, \tag{1.12}$$

if $x > (1-1/p)^{1/p}$.

Elbert and Laforgia established in [2] the following estimations for the functions $\int_0^x e^{t^p}dt$ and $\int_0^x e^{-t^p}\,dt$

$$1+\frac{u(x^p)}{p+1} < \frac{1}{x}\int_0^x e^{t^p}\,dt < 1+\frac{u(x^p)}{p}, \quad \text{for } x>0,\ p>1, \tag{1.13}$$

$$1-\frac{v(x^p)}{p+1} < \frac{1}{x}\int_0^x e^{-t^p}\,dt, \quad \text{for } 0<x<\left(\frac{9(3p+1)}{4(2p+1)}\right)^{1/p},\ p>1, \tag{1.14}$$

where

$$u(x)=\int_0^x \frac{e^t-1}{t}\,dt, \qquad v(x)=\int_0^x \frac{1-e^{-t}}{t}\,dt. \tag{1.15}$$

In Section 3 we prove the following extension of the lower bound (1.14).

Theorem 1.2. *For $p>1$, the inequality (1.14) holds for $x>0$.*

We conclude this paper, Section 4, showing some numerical results related to this last theorem.

## 2. Proof of Theorem 1.1

*Proof.* It is easy to note that $\min_{x>0}(x+1/x)=2$, consequently $\Gamma'(x+1/x)>0$ for every $x>0$. We have

$$f'(x)=\left(1-\frac{1}{x^2}\right)\Gamma'\left(x+\frac{1}{x}\right). \tag{2.1}$$

Since $f'(x)<0$ for $x\in(0,1)$ and $f'(x)>0$ for $x>1$ it follows that $f(x)$ decreases for $0<x<1$, while increases for $x>1$.

Now consider $G(x)=\log[g(x)]$. We have $G(x)=x\log[\Gamma(x+1/x)]$. Then

$$\begin{aligned} G'(x) &= \log\left[\Gamma\left(x+\frac{1}{x}\right)\right]+\left(x-\frac{1}{x}\right)\psi\left(x+\frac{1}{x}\right),\\ G''(x) &= 2\psi\left(x+\frac{1}{x}\right)+\left(x-\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)\psi'\left(x+\frac{1}{x}\right). \end{aligned} \tag{2.2}$$

Since $G'(1)=0$ and $G''(x)>0$ for $x>0$ it follows that $G'(x)<0$ for $x\in(0,1)$ and $G'(x)>0$ for $x\in(1,+\infty)$. Therefore $G(x)$, and consequently $g(x)$, decrease for $0<x<1$, while increase for $x>1$.

Finally

$$h'(x)=\left(1-\frac{1}{x^2}\right)\Gamma''\left(x+\frac{1}{x}\right). \tag{2.3}$$

Since $\Gamma''(x+1/x)>0$, hence $h'(x)<0$ for $x\in(0,1)$ and $h'(x)>0$ for $x>1$. It follows that $h(x)$ decreases on $0<x<1$, while increases for $x>1$. □

## 3. Proof of Theorem 1.2

By means the series expansion of the exponential function $e^{-t^p}$, we have

$$\int_0^x e^{-t^p}dt=\sum_{n=0}^{\infty}(-1)^n\frac{x^{np+1}}{(np+1)n!},\qquad v(x^p)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{np}}{nn!},\tag{3.1}$$

consequently the inequality (1.14) is equivalent to the following

$$1-\frac{1}{p+1}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{np}}{nn!}<\frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{np+1}}{(np+1)n!},\tag{3.2}$$

that is,

$$1-\frac{x^p}{p+1}+\frac{x^{2p}}{(p+1)2\cdot 2!}-\frac{x^{3p}}{(p+1)3\cdot 3!}+\cdots<1-\frac{x^p}{p+1}+\frac{x^{2p}}{(2p+1)2!}-\frac{x^{3p}}{(3p+1)3!}+\cdots.\tag{3.3}$$

Since for every integer $n$

$$\frac{1}{(np+1)n!}-\frac{1}{n(p+1)n!}=-\frac{n-1}{(p+1)n\cdot n!(np+1)},\tag{3.4}$$

by putting $z=x^p$ the inequality (1.14) is equivalent to

$$s(z)=\frac{1}{p+1}\sum_{n=2}^{\infty}(-1)^n\frac{n-1}{(np+1)n\cdot n!}z^n>0;\tag{3.5}$$

it is clear that the series to the right-hand side of (3.5) is convergent for any $z\in\mathbb{R}$. We can observe that, for $p>1$,

$$(p+1)s_3(z)=\sum_{n=2}^{3}(-1)^n\frac{n-1}{(np+1)n\cdot n!}z^n=z^2\left(\frac{1}{4(2p+1)}-\frac{z}{9(3p+1)}\right)>0\tag{3.6}$$

when $0<z<9(3p+1)/4(2p+1)$. As a consequence of a well known property of Leibniz type series we have $0<s_3(z)<s(z)$ for $0<z<9(3p+1)/4(2p+1)$ just like was proved by Elbert and Laforgia in [2].

It is easy to observe that $z=0$ represents a relative minimum point for the function $s(z)$ defined in (3.5). In fact we have $s(z)>0$ for $z<0$ and $0<z<9(3p+1)/4(2p+1)$.

Now we can prove Theorem 1.2 by using the following lemma.

Lemma 3.1. *The function $s(z)$, defined in (3.5), have not any relative maximum point in the interval* $(0,+\infty)$.

*Proof.* For any $n \geq 1$ consider the partial sum of series (3.5)

$$(p+1)s_{2n}(z) = \sum_{k=2}^{2n}(-1)^k \frac{k-1}{(kp+1)k \cdot k!} z^k \tag{3.7}$$

and multiply this expression by $pz^{1/p}$; we have

$$pz^{1/p}(p+1)s_{2n}(z) = \sum_{k=2}^{2n}(-1)^k \frac{k-1}{k \cdot k!((kp+1)/p)} z^{(kp+1)/p}. \tag{3.8}$$

Deriving and dividing by $z^{1/p-1}$ we obtain

$$(p+1)\left(s_{2n}(z) + pzs'_{2n}(z)\right) = \sum_{k=2}^{2n}(-1)^k \frac{k-1}{k \cdot k!} z^k. \tag{3.9}$$

A new derivation give us the following expression

$$(p+1)\left((p+1)s'_{2n}(z) + pzs''_{2n}(z)\right) = \sum_{k=2}^{2n}(-1)^k \frac{k-1}{k!} z^{k-1}. \tag{3.10}$$

Dividing by $z$ and re-writing, in equivalent way, the indexes into the sum to the right-hand side, the last expression yields

$$(p+1)\left((p+1)\frac{s'_{2n}(z)}{z} + ps''_{2n}(z)\right) = \sum_{k=0}^{2n-2}(-1)^k \frac{k+1}{(k+2)!} z^k. \tag{3.11}$$

Now consider the following series

$$\sum_{k=0}^{\infty}(-1)^k \frac{k+1}{(k+2)!} z^k; \tag{3.12}$$

we have for every $z \in \mathbb{R}$

$$\begin{aligned}
\sum_{k=0}^{\infty}(-1)^k \frac{k+1}{(k+2)!} z^k &= \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(k+1)!} - \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(k+2)!} \\
&= \left(1 - \frac{z}{2} + \frac{z^2}{3!} - \frac{z^3}{4!} + \cdots\right) - \left(\frac{1}{2} - \frac{z}{3!} + \frac{z^2}{4!} - \frac{z^3}{5!} + \cdots\right) \\
&= \frac{1}{z}\left(z - \frac{z^2}{2} + \frac{z^3}{3!} - \frac{z^4}{4!} + \cdots\right) - \frac{1}{z^2}\left(\frac{z^2}{2} - \frac{z^3}{3!} + \frac{z^4}{4!} - \frac{z^5}{5!} + \cdots\right) \\
&= \frac{1 - e^{-z}}{z} - \frac{e^{-z} - 1 + z}{z^2} = \frac{f(z)}{z^2},
\end{aligned} \tag{3.13}$$

where $f(z) = 1 - (z+1)e^{-z}$.

Since $f(0) = 0$ and $f'(z) = ze^{-z} > 0$ for $z > 0$, it follows that $f(z) > 0$ $\forall z \in (0, +\infty)$. From (3.11), by $n \to +\infty$, we obtain

$$(p+1)\left((p+1)\frac{s'(z)}{z} + ps''(z)\right) = \frac{f(z)}{z^2}, \tag{3.14}$$

for every $z \in \mathbb{R}$. If we assume that $\bar{z} > 0$ is a relative maximum point of $s(z)$ then $s'(\bar{z}) = 0$ and $s''(\bar{z}) < 0$, but this produces an evident contradiction when we substitute $z = \bar{z}$ in (3.14). □

*Proof of Theorem 1.2.* Since $s(z) > 0$ $\forall z \in (0, 9(3p+1)/4(2p+1))$, if we assume the existence of a point $\bar{z} > 9(3p+1)/4(2p+1)$ such that $s(\bar{z}) < 0$ then there exists at least a point $\zeta \in (9(3p+1)/4(2p+1), \bar{z})$ such that $s(\zeta) = 0$. Let $\zeta$, eventually, be the smallest positive zero of $s(z)$, hence we have $s(0) = s(\zeta) = 0$ and $s(z) > 0$ $\forall z \in (0, \zeta)$. It follows therefore, that there exists a relative maximum point $z_0 \in (0, \zeta)$ for the function $s(z)$, but this is in contradiction whit Lemma 3.1. □

## 4. Concluding remark on Theorem 1.2

In this concluding section we report some numerical results, obtained by means the computer algebra system Mathematica ©, which justify the importance of the result obtained by means of Theorem 1.2. We briefly put

$$I(x) = \int_0^x e^{-t^p} dt, \tag{4.1}$$

while denote with

$$A(x) = \Gamma\left(1 + \frac{1}{p}\right)\left(1 - e^{-x^p}\right)^{1/p} \tag{4.2}$$

the lower bound established by Alzer [1], with

$$G(x) = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) - e^{-x^p} a_p \left[\sqrt{x^2 + \frac{1}{a_p}} - x\right] \tag{4.3}$$

that one established by Gautschi [3], with

$$Q(x) = \frac{1}{2}\left(1 - \frac{1}{p}\right)^{1/p}\left(1 + e^{1/p-1}\right) + \left(x - \left(1 - \frac{1}{p}\right)^{1/p}\right)e^{-((x+(1-1/p)^{1/p})/2)^p} \tag{4.4}$$

that one established by Qi-Guo [5] when $x > (1 - 1/p)^{1/p}$, and finally with

$$E(x) = 1 - \frac{\nu(x^p)}{p+1} \tag{4.5}$$

that one established by Elbert-Laforgia [2].

Therefore the following numerical results are obtained:

(i) for $p = 50$ and $x = 1.026 > (9(3p+1)/4(2p+1))^{1/p} = 1.023456$, we have

$$\begin{aligned} I(x) - E(x) &= 0.000272222, \\ I(x) - A(x) &= 0.000417332, \\ I(x) - G(x) &= -0.0108717, \\ I(x) - Q(x) &= 0.301341; \end{aligned} \tag{4.6}$$

(ii) for $p = 100$ and $x = 1.013 > (9(3p+1)/4(2p+1))^{1/p} = 1.01222$,

$$\begin{aligned} I(x) - E(x) &= 0.0000690398, \\ I(x) - A(x) &= 0.000205222, \\ I(x) - G(x) &= -0.0107205, \\ I(x) - Q(x) &= 0.308547; \end{aligned} \tag{4.7}$$

(iii) for $p = 200$ and $x = 1.0065 > (9(3p+1)/4(2p+1))^{1/p} = 1.0061$,

$$\begin{aligned} I(x) - E(x) &= 0.0000173853, \\ I(x) - A(x) &= 0.000101731, \\ I(x) - G(x) &= -0.106414, \\ I(x) - Q(x) &= 0.312265. \end{aligned} \tag{4.8}$$

In these three numerical examples we can note that there exist values of $x > (9(3p+1)/4(2p+1))^{1/p}$ such that $E(x)$ represents the best lower bound of $I(x)$ with respect to $A(x)$, $Q(x)$, and $G(x)$. Moreover we state that this is always true in general, more preciously we state the following conjecture: for any $p > 1$, there exists a right neighbourhood of $(9(3p+1)/4(2p+1))^{1/p}$ such that $E(x)$ represents the best lower bound of $I(x)$ with respect to $A(x)$, $Q(x)$, and $G(x)$.

## References


[1] H. Alzer, *On some inequalities for the incomplete gamma function*, Mathematics of Computation **66** (1997), no. 218, 771–778.

[2] Á. Elbert and A. Laforgia, *An inequality for the product of two integrals relating to the incomplete gamma function*, Journal of Inequalities and Applications **5** (2000), no. 1, 39–51.

[3] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, Journal of Mathematics and Physics **38** (1959), 77–81.

[4] D. Kershaw and A. Laforgia, *Monotonicity results for the gamma function*, Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali **119** (1985), no. 3-4, 127–133 (1986).

[5] F. Qi and S.-L. Guo, *Inequalities for the incomplete gamma and related functions*, Mathematical Inequalities & Applications **2** (1999), no. 1, 47–53.

A. Laforgia: Department of Mathematics, Roma Tre University, Largo San Leonardo Murialdo 1, 00146 Rome, Italy
*E-mail address*: laforgia@mat.uniroma3.it

P. Natalini: Department of Mathematics, Roma Tre University, Largo San Leonardo Murialdo 1, 00146 Rome, Italy
*E-mail address*: natalini@mat.uniroma3.it