GENERALIZED VECTOR QUASI-VARIATIONAL-LIKE INEQUALITIES

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Using maximal element theorem, we prove some existence theorems for the two types of generalized vector quasi-variational-like inequalities with non-monotonicity and non-compactness.

1. Introduction and preliminaries

Let Y be a real Hausdorff topological vector space and X be a nonempty convex subset in a real locally convex Hausdorff topological vector space E. We denote $L(E,Y)$ the space of all continuous linear operators from E into Y and by $\langle u,y\rangle$ the evaluation of $u \in L(E,Y)$ at $y \in E$. Let σ be the family of all bounded subsets of X whose union is total in E, that is, the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let β be a neighbourhood base of 0 in Y. When S runs through σ, V through β, the family

$$M(S,V) = \{l \in L(E,Y) : \cup_{x\in S}\langle l,x\rangle \subset V\} \tag{1.1}$$

is a neighbourhood base of 0 in $L(E,Y)$ at $x \in E$ (see [29, pages 79–80]). By the corollary of Schaefer [29, page 80], $L(E,Y)$ becomes a locally convex topological vector space under σ-topology, where Y is assumed a locally convex topological space.

Let $\operatorname{int} A$ and $\operatorname{Co} A$ denote the interior and convex hull of a set A, respectively. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$. Let $\eta : X \times X \to E$ and $H : X \times X \to Y$ be vector-valued mappings, $D : X \to 2^X$ and $T : X \to 2^{L(E,Y)}$ be two set-valued mappings, we introduced a new model of the generalized vector quasi-variational-like inequality, which is to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y,\bar{x})\rangle + H(\bar{x},y) \notin -\operatorname{int} C(\bar{x}). \tag{1.2}$$

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It is easy to see that $\bar{x}$ is a solution of the problem (1.2) is equivalent to $\bar{x}$ in X satisfying $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \langle T(\bar{x}), \eta(y,\bar{x})\rangle + H(\bar{x},y) \notin -\operatorname{int} C(\bar{x}), \tag{1.3}$$

where $\langle T(\bar{x}), \eta(y,\bar{x})\rangle = \cup_{v\in T(\bar{x})}\langle v, \eta(y,\bar{x})\rangle$.

The following problems are the special cases of the problem (1.2).

(i) If $H(x,y) \equiv 0$ for all $x, y \in X$, then the problem (1.2) reduces to finding $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y,\bar{x})\rangle \notin -\operatorname{int} C(\bar{x}). \tag{1.4}$$

This problem was also called generalized vector quasi-variational-like inequality and studied with certain monotonicity by Ding [13], and problem (1.4) contains as special cases the generalized vector variational-like inequality in [1, 2, 14, 15, 28] and the generalized vector quasi-variational inequality studied by Chen and Li [10] and Lee et al. [22] and those vector variational inequalities in [6–9, 11, 12, 16, 19–21, 23, 26, 30, 33–37].

(ii) If $T : X \to 2^{L(E,Y)}$ is a zero operator, then the problem (1.2) reduces to the vector quasi-equilibrium problem, which is to find $\bar{x}$ in X such that $\bar{x} \in D(\bar{x})$, and

$$H(\bar{x},y) \notin -\operatorname{int} C(\bar{x}), \quad \forall y \in D(\bar{x}). \tag{1.5}$$

Problem (1.5) includes the vector equilibrium problem researched by many authors (see [4, 5, 17, 24, 25, 27]).

In this paper, we establish existence results of solutions for both problem (1.2) and problem (1.4) with non-monotonicity and non-compactness. Our results extend and improve some main results of [15, 28].

In order to prove the main results, we need the following definitions and lemmas.

Definition 1.1 (see [15]). Let E, Y be two real topological vector spaces, X be a nonempty and convex subset of E, $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed pointed and convex cone with apex at 0 for each $x \in X$. Let $\eta : X \times X \to E$ be a single-valued mapping. $T : X \to 2^{L(E,Y)}$ is said to satisfy the generalized L-η-condition if and only if for any finite set $\{y_1, y_2, \ldots, y_n\}$ in X, $\bar{x} = \sum_{j=1}^{n} \alpha_j y_j$ with $\alpha_j \geq 0$ and $\sum_{j=1}^{n} \alpha_j = 1$, there exists $\bar{v} \in T(\bar{x})$, such that

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j \eta(y_j, \bar{x}) \right\rangle \notin -\operatorname{int} C(\bar{x}). \tag{1.6}$$

Remark 1.2. If $\eta(y,x)$ is affine in the first argument and $\forall x \in X$, $\exists v \in T(x)$, such that

$$\langle \bar{v}, \eta(x,x)\rangle \notin -\operatorname{int} C(x), \tag{1.7}$$

Then T satisfies the generalized L-η-condition.

If $\eta(y,x) = y - x$, $\forall x, y \in X$, then we have that

$$\left\langle \bar{v}, \sum_{j=1}^{n} \alpha_j (y_j - \bar{x}) \right\rangle = \langle \bar{v}, \bar{x} - \bar{x} \rangle = 0 \notin -\operatorname{int} C(\bar{x}), \quad \forall v \in T(\bar{x}), \tag{1.8}$$

and hence T satisfies the generalized L-η-condition trivially.

Definition 1.3 (see [32]). Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then

(1) T is said to be upper semicontinuous if, for any $x_0 \in X$ and for each open set U in Y containing $T(x_0)$, there is a neighbourhood V of x_0 in X such that $T(x) \subseteq U$, for all $x \in V$.

(2) T is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

(3) T is said to be closed, if the set $\{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Definition 1.4. Let $C : X \to 2^Y$ be a set-valued mapping. $H : X \times X \to Y$ is said to be 0-$C(x)$ diagonally convex with respect to the second argument if, for any finite subset $\{y_1, y_2, \dots, y_n\}$ in X, and any $x \in X$ with $x = \sum_{j=1}^{n} \alpha_j y_j$ ($\alpha_j \geq 0$, $\sum_{j=1}^{n} \alpha_j = 1$), we have

$$\sum_{j=1}^{n} \alpha_j H(\bar{x}, y_j) \in C(\bar{x}). \tag{1.9}$$

$H : X \times X \to Y$ is said to be 0-$C(x)$ diagonally concave with respect to the second argument if $-H$ is 0-$C(x)$ diagonally convex with respect to the second argument.

Remark 1.5. If $Y = R \cup \{\pm\infty\}$ and $C(x) = \{r \in R : r \geq 0\}$, then the 0-$C(x)$ diagonal concavity of H reduces to the 0-diagonal concavity of H in [38].

Lemma 1.6 (see [32]). *Let X and Y be two topological spaces. Suppose $T : X \to 2^Y$ and $K : X \to 2^Y$ are set-valued mappings having open lower sections, then* (i) *the set-valued mapping $F : X \to 2^Y$ defined by, for each $x \in X$, $F(x) = \operatorname{Co}(T(x))$ has open lower sections.* (ii) *the set-valued mapping $\theta : X \to 2^Y$ defined by, for each $x \in X$, $\theta(x) = T(x) \cap K(x)$ has open lower sections.*

Lemma 1.7 (see [3]). *Let X and Y be topological spaces. If $T : X \to 2^Y$ is an upper semicontinuous set-valued mapping with closed values, then T is closed.*

Lemma 1.8 (see [31]). *Let X and Y be topological spaces and $T : X \to 2^Y$ be an upper semicontinuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.*

Lemma 1.9 (see [18]). *Let X be a nonempty convex subset of a Hausdorff topological vector space E and $S : X \to 2^X$ be a set-valued mapping such that for each $x \in X$, $x \notin \operatorname{Co}(S(x))$ and for each $y \in X$, $S^{-1}(y)$ is open in X. Suppose further that there exist a nonempty compact* *subset N of X and a nonempty compact convex subset B of X such that* $\mathrm{Co}(S(x)) \cap B \neq \emptyset$ *for all* $x \in X \setminus N$.

Then there exists a point $\bar{x} \in X$ *such that* $S(\bar{x}) = \emptyset$.

2. Main results

In this section, we will present some existence results of solutions for the two types of generalized vector quasi-variational inequalities without monotonicity and compactness.

Theorem 2.1. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be a set-valued mapping such that $\forall x \in X$, $D(x)$ is nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y \in X$, and the set $W = \{x \in X : x \in D(x)\}$ is closed in X. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed, pointed and convex cone with* $\operatorname{int} C(x) \neq \emptyset$ *for each $x \in X$. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $M = Y\backslash(-\operatorname{int} C) : X \to 2^Y$ is upper semicontinuous on X.*
(ii) *The set-valued mapping $T : X \to 2^{L(E,Y)}$ is upper semicontinuous on X with compact values and $\eta : X \times X \to E$ is continuous with respect to the second argument, such that T satisfies the generalized L-η-condition.*
(iii) *$H : X \times X \to Y$ is continuous with respect to the first argument and* 0-$C(x)$ *diagonally convex with respect to the second argument.*
(iv) *There exist a nonempty and compact subset N of X and a nonempty, compact and convex subset B of X such that $\forall x \in X\backslash N$, $\exists \bar{y} \in B$, such that $\bar{y} \in D(x)$ and $\langle v, \eta(\bar{y}, x)\rangle + H(x, \bar{y}) \in -\operatorname{int} C(x)$, $\forall v \in T(x)$.*

Then, there exists a point $\bar{x} \in X$ such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y, \bar{x})\rangle + H(\bar{x}, y) \notin -\operatorname{int} C(\bar{x}). \tag{2.1}$$

Proof. Define a set-valued mapping $P : X \to 2^X$ by

$$\begin{aligned} P(x) &= \{y \in X : \langle T(x), \eta(y,x)\rangle + H(x,y) \subseteq -\operatorname{int} C(x)\} \\ &= \{y \in X : \langle v, \eta(y,x)\rangle + H(x,y) \in -\operatorname{int} C(x),\ \forall v \in T(x)\}, \quad \forall x \in X. \end{aligned} \tag{2.2}$$

We first prove that $x \notin \operatorname{Co} P(x)$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \mathrm{Co}(P(\bar{x}))$. Then there exists finite points $y_1, y_2, \ldots, y_n$ in X, and $\alpha_j \geq 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ and $y_j \in P(\bar{x})$ for all $j = 1, 2, \ldots, n$. That is,

$$\langle v, \eta(y_j, \bar{x})\rangle + H(\bar{x}, y_j) \in -\operatorname{int} C(\bar{x}), \quad \forall v \in T(x),\ j = 1, 2, \ldots, n. \tag{2.3}$$

Since $\operatorname{int} C(\bar{x})$ is a convex set, we obtain

$$\left\langle v, \sum_{j=1}^n \alpha_j \eta(y_j, \bar{x})\right\rangle + \sum_{j=1}^n \alpha_j H(\bar{x}, y_j) \in -\operatorname{int} C(\bar{x}), \quad \forall v \in T(x). \tag{2.4}$$

From the 0-$C(x)$ diagonal convexity with respect to the second argument of H, we have

$$\sum_{j=1}^{n} \alpha_j H(\bar{x}, y_j) \in C(\bar{x}). \tag{2.5}$$

By (2.4) and (2.5), we get, for all $v \in T(\bar{x})$,

$$\left\langle v, \sum_{j=1}^{n} \alpha_j \eta(y_j, \bar{x}) \right\rangle \in -\sum_{j=1}^{n} \alpha_j H(\bar{x}, y_j) - \operatorname{int} C(\bar{x}) \subseteq -C(\bar{x}) - \operatorname{int} C(\bar{x}) \subseteq -\operatorname{int} C(\bar{x}), \tag{2.6}$$

which contradicts the fact that T satisfies the generalized L-η-condition. Therefore $x \notin \operatorname{Co} P(x)$ for all $x \in X$. □

Now we prove that the set

$$\begin{aligned} P^{-1}(y) &= \{x \in X : \langle T(x), \eta(y,x)\rangle + H(x,y) \subseteq -\operatorname{int} C(x)\} \\ &= \{x \in X : \langle v, \eta(y,x)\rangle + H(x,y) \in -\operatorname{int} C(x),\ \forall v \in T(x)\} \end{aligned} \tag{2.7}$$

is open for each $y \in X$. That is, P has open lower sections in X. Consider the set-valued mapping $Q : X \to 2^X$ defined by

$$\begin{aligned} Q(y) &= \{x \in X : \langle T(x), \eta(y,x)\rangle + H(x,y) \not\subseteq -\operatorname{int} C(x)\} \\ &= \{x \in X : \exists v \in T(x) \text{ such that } \langle v, \eta(y,x)\rangle + H(x,y) \notin -\operatorname{int} C(x)\}. \end{aligned} \tag{2.8}$$

We only need to prove that $Q(y)$ is closed for all $y \in X$. In fact, consider a net $x_t \in Q(y)$ such that $x_t \to x \in X$. Since $x_t \in Q(y)$, there exists $s_t \in T(x_t)$ such that

$$\langle s_t, \eta(y, x_t)\rangle + H(x_t, y) \notin -\operatorname{int} C(x_t). \tag{2.9}$$

From the upper semicontinuity and compact values of T and Lemma 1.8, it suffices to find a subset $\{s_{t_j}\}$ which converges to some $s \in T(x)$. By [15, Lemma 1, page 114], we know that $\langle \cdot \rangle$ is continuous, and hence

$$\langle s_{t_j}, \eta(y, x_{t_j})\rangle + H(x_{t_j}, y) \longrightarrow \langle s, \eta(y,x)\rangle + H(x,y). \tag{2.10}$$

By Lemma 1.7 and upper semicontinuity of M, we have $\langle s, \eta(y,x)\rangle + H(x,y) \notin -\operatorname{int} C(x)$, and hence $x \in Q(y)$, $Q(y)$ is closed. Therefore, P has open lower sections in X, and by Lemma 1.6, we know that $\operatorname{Co} P : X \to 2^X$ also has open lower sections. Also define another set-valued mapping $S : X \to 2^X$ by

$$S(x) = \begin{cases} D(x) \cap \operatorname{Co} P(x) & \text{if } x \in W, \\ D(x) & \text{if } x \notin W. \end{cases} \tag{2.11}$$

Then, it is clear that $\forall x \in X$, $S(x)$ is convex, and $x \notin S(x) = \operatorname{Co} S(x)$. Since $\forall y \in X$,

$$\begin{aligned}
S^{-1}(y) &= \{x \in X : y \in S(x)\} \\
&= \{x \in W : y \in D(x) \cap \operatorname{Co} P(x)\} \cup \{x \in X \setminus W : y \in D(x)\} \\
&= (W \cap D^{-1}(y) \cap \operatorname{Co} P^{-1}(y)) \cup [(X \setminus W) \cap D^{-1}(y)] \\
&= [(W \cap D^{-1}(y) \cap \operatorname{Co} P^{-1}(y)) \cup (X \setminus W)] \cap [(W \cap D^{-1}(y) \cap \operatorname{Co} P^{-1}(y)) \cup D^{-1}(y)] \\
&= \{X \cap [(D^{-1}(y) \cap \operatorname{Co} P^{-1}(y)) \cup (X \setminus W)]\} \cap [(W \cup D^{-1}(y)) \cap (D^{-1}(y))] \\
&= [(D^{-1}(y) \cap \operatorname{Co} P^{-1}(y)) \cup (X \setminus W)] \cap D^{-1}(y) \\
&= (D^{-1}(y) \cap (\operatorname{Co} P^{-1}(y))) \cup ((X \setminus W) \cap (D^{-1}(y))),
\end{aligned} \tag{2.12}$$

and $D^{-1}(y)$, $\operatorname{Co} P^{-1}(y)$ and $X \setminus W$ are open in X, we have $S^{-1}(y)$ is open in X.

Condition (iii) implies that there exist a nonempty compact subset N of X and a nonempty compact convex subset B of X such that $S(x) \cap B = \operatorname{Co} S(x) \cap B \neq \emptyset$ for all $x \in X \setminus N$. Hence, by Lemma 1.9, $\exists \bar{x} \in X$ such that $S(\bar{x}) = \emptyset$. Since $\forall x \in X$, $D(x)$ is nonempty, we have $\bar{x} \in W$, and $D(\bar{x}) \cap \operatorname{Co} P(\bar{x}) = \emptyset$. This implies $\bar{x} \in D(\bar{x})$ and $D(\bar{x}) \cap P(\bar{x}) = \emptyset$. Consequently, $\bar{x} \in D(\bar{x})$, and $\forall y \in D(\bar{x})$, $\exists v \in T(\bar{x})$ satisfying $\langle v, \eta(y, \bar{x})\rangle + H(\bar{x}, y) \notin -\operatorname{int} C(\bar{x})$.

By Theorem 2.1 and Remark 1.2, we have the following corollary.

Corollary 2.2. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be a set-valued mapping such that $\forall x \in X$, $D(x)$ is nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y \in X$, and the set $W = \{x \in X : x \in D(x)\}$ is closed in X. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed, pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $M = Y \setminus (-\operatorname{int} C) : X \to 2^Y$ is upper semicontinuous on X.*

(ii) *The set-valued mapping $T : X \to 2^{L(E,Y)}$ is upper semicontinuous on X with compact values and $\eta : X \times X \to E$ is continuous with respect to the second argument and affine with respect to the first argument such that $\forall x \in X$, $\exists v \in T(x)$, satisfying $\langle \bar{v}, \eta(x,x)\rangle \notin -\operatorname{int} C(x)$.*

(iii) *$H : X \times X \to Y$ is continuous with respect to the first argument and 0-$C(x)$ diagonally convex with respect to the second argument.*

(iv) *There exist a nonempty and compact subset N of X and a nonempty, compact and convex subset B of X such that $\forall x \in X \setminus N$, $\exists \bar{y} \in B$, such that $\bar{y} \in D(x)$ and $\langle v, \eta(\bar{y}, x)\rangle + H(x, \bar{y}) \in -\operatorname{int} C(x)$, $\forall v \in T(x)$.*

Then, there exists a point $\bar{x} \in X$ such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y, \bar{x})\rangle + H(\bar{x}, y) \notin -\operatorname{int} C(\bar{x}). \tag{2.13}$$

If $H(x,x)=0\ \forall x\in X$, then by Theorem 2.1 and Corollary 2.2, we have the following corollary.

Corollary 2.3. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D:X\to 2^X$ be a set-valued mapping such that $\forall x\in X$, $D(x)$ is nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y\in X$, and the set $W=\{x\in X: x\in D(x)\}$ is closed in X. Let $C:X\to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed, pointed and convex cone with $\operatorname{int}C(x)\neq\emptyset$ for each $x\in X$. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $M=Y\backslash(-\operatorname{int}C):X\to 2^Y$ is upper semicontinuous on X.*

(ii) *There exist a nonempty and compact subset N of X and a nonempty, compact and convex subset B of X such that $\forall x\in X\backslash N$, $\exists\bar{y}\in B$, such that $\bar{y}\in D(x)$ and $\langle v,\eta(\bar{y},x)\rangle\in-\operatorname{int}C(x)$, $\forall v\in T(x)$.*

(iii) *The set-valued mapping $T:X\to 2^{L(E,Y)}$ is upper semicontinuous on X with compact values and $\eta:X\times X\to E$ is continuous with respect to the second argument. Moreover, one of the following conditions satisfied*

(iv) *$T:X\to 2^{L(E,Y)}$ satisfies the generalized L-η-condition.*

Or

(v) *$\eta:X\times X\to E$ is affine with respect to the first argument such that $\forall x\in X$, $\exists v\in T(x)$, satisfying $\langle\bar{v},\eta(x,x)\rangle\notin-\operatorname{int}C(x)$.*

Then, there exists a point $\bar{x}\in X$ such that $\bar{x}\in D(\bar{x})$, and

$$\forall y\in D(\bar{x}),\ \exists\hat{v}\in T(\bar{x}):\langle\hat{v},\eta(y,\bar{x})\rangle\notin-\operatorname{int}C(\bar{x}). \tag{2.14}$$

Remark 2.4. Theorem 2.1, Corollaries 2.2 and 2.3 extend and improve [15, Theorem 1 and Corollary 1] and [28, Theorem 1] without monotonicity and compactness.

If T is a zero operator, then by Theorem 2.1, we have the following corollary.

Corollary 2.5. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E. Let $D:X\to 2^X$ be a set-valued mapping such that $\forall x\in X$, $D(x)$ is nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y\in X$, and the set $W=\{x\in X:x\in D(x)\}$ is closed in X. Let $C:X\to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed, pointed and convex cone with $\operatorname{int}C(x)\neq\emptyset$ for each $x\in X$. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $M=Y\backslash(-\operatorname{int}C):X\to 2^Y$ is upper semicontinuous on X.*

(ii) *$H:X\times X\to Y$ is continuous with respect to the first argument and 0-$C(x)$ diagonally convex with respect to the second argument.*

(iii) *There exist a nonempty and compact subset N of X and a nonempty, compact and convex subset B of X such that $\forall x\in X\backslash N$, $\exists\bar{y}\in B$, such that $\bar{y}\in D(x)$ and $H(x,\bar{y})\in-\operatorname{int}C(x)$.*

Then, there exists a point $\bar{x}\in X$ such that $\bar{x}\in D(\bar{x})$, and $H(\bar{x},y)\notin-\operatorname{int}C(\bar{x}),\forall y\in D(\bar{x})$.

Theorem 2.6. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D:X\to 2^X$ be a set-valued mapping such that $\forall x\in X$, $D(x)$ is* *nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y \in X$, and the set $W = \{x \in X : x \in D(x)\}$ is closed in X. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed, pointed and convex cone with $\operatorname{int} C(x) \neq \emptyset$ for each $x \in X$. Assume that the following conditions are satisfied.*

(i) *The set-valued mapping $M = Y \backslash (-\operatorname{int} C) : X \to 2^Y$ is upper semicontinuous on X.*

(ii) *The set-valued mapping $T : X \to 2^{L(E,Y)}$ is upper semicontinuous on X with compact values and $\eta : X \times X \to E$ is continuous with respect to the second argument, and there exists a mapping $h : X \times X \to Y$, such that:*

(a) *$\forall x, y \in X$, $\exists v \in T(x)$, such that*

$$h(x,y) - \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x). \tag{2.15}$$

(b) *For any finite set $\{y_1, y_2, \dots, y_n\} \subseteq X$ and $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ with $\alpha_j \geq 0$ and $\sum_{j=1}^n \alpha_j = 1$, there is a $j \in \{1, 2, \dots, n\}$, such that $h(\bar{x}, y_j) \notin -\operatorname{int} C(\bar{x})$.*

(iii) *There exist a nonempty and compact subset N of X and a nonempty, compact and convex subset B of X such that $\forall x \in X \backslash N$, $\exists \bar{y} \in B$, such that $\bar{y} \in D(x)$ and*

$$\langle v, \eta(\bar{y}, x) \rangle \in -\operatorname{int} C(x), \quad \forall v \in T(x). \tag{2.16}$$

Then, there exists a point $\bar{x} \in X$ such that $\bar{x} \in D(\bar{x})$, and

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y, \bar{x}) \rangle \notin -\operatorname{int} C(\bar{x}). \tag{2.17}$$

Proof. Define two set-valued mappings $P : X \to 2^X$, $P_1 : X \to 2^X$ by

$$\begin{aligned} P(x) &= \{y \in X : \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x),\ \forall v \in T(x)\}, \quad \forall x \in X. \\ P_1(x) &= \{y \in X : h(x,y) \in -\operatorname{int} C(x)\}, \quad \forall x \in X. \end{aligned} \tag{2.18}$$

We first prove that $x \notin \operatorname{Co}(P_1(x))$ for all $x \in X$. To see this, suppose, by way of contradiction, that there exists some point $\bar{x} \in X$ such that $\bar{x} \in \operatorname{Co}(P_1(\bar{x}))$. Then there exists finite points $y_1, y_2, \dots, y_n$ in X, and $\alpha_j \geq 0$ with $\sum_{j=1}^n \alpha_j = 1$ such that $\bar{x} = \sum_{j=1}^n \alpha_j y_j$ and $y_j \in P_1(\bar{x})$ for all $j = 1, 2, \dots, n$. That is,

$$h(\bar{x}, y_j) \in -\operatorname{int} C(\bar{x}), \quad j = 1, 2, \dots, n. \tag{2.19}$$

This contradicts to the condition (ii)(b). Therefore $x \notin \operatorname{Co}(P_1(x))$ for all $x \in X$.

The condition (ii)(a) implies that $P_1(x) \supseteq P(x)$ for all $x \in X$. Hence, $x \notin \operatorname{Co}(P(x))$, $\forall x \in X$.

The remainder of the proof is similar to that in the proof of Theorem 2.1. □

Corollary 2.7. *Let Y be a real Hausdorff topological vector space, X be a nonempty and convex set in a real locally convex Hausdorff topological vector space E, and $L(E,Y)$ be equipped with the σ-topology. Let $D : X \to 2^X$ be a set-valued mapping such that $\forall x \in X$, $D(x)$ is nonempty and convex, $D^{-1}(y)$ is open in X, $\forall y \in X$, and the set $W = \{x \in X : x \in D(x)\}$ is closed in X. Let $C : X \to 2^Y$ be a set-valued mapping such that $C(x)$ is a closed,* *pointed and convex cone with* $\operatorname{int} C(x) \neq \emptyset$ *for each* $x \in X$. *Assume that the following conditions are satisfied.*

(i) *The set-valued mapping* $M = Y \setminus (-\operatorname{int} C) : X \to 2^Y$ *is upper semicontinuous on* X.

(ii) *The set-valued mapping* $T : X \to 2^{L(E,Y)}$ *is upper semicontinuous on* X *with compact values and* $\eta : X \times X \to E$ *is continuous with respect to the second argument, and there exists a mapping* $h : X \times X \to Y$, *such that:*

(a) $\forall x, y \in X$, $\exists v \in T(x)$, *such that*

$$h(x,y) - \langle v, \eta(y,x) \rangle \in -\operatorname{int} C(x); \tag{2.20}$$

(b) *the set* $\{y \in X : h(x,y) \in -\operatorname{int} C(x)\}$ *is convex for all* $x \in X$;

(c) $h(x,x) \notin -\operatorname{int} C(x), \forall x \in X$.

Then, there exists $\bar{x} \in X$, *such that* $\bar{x} \in D(\bar{x})$ *and* $\langle T(\bar{x}), \eta(y,\bar{x}) \rangle \notin -\operatorname{int} C(\bar{x})$, $\forall y \in D(\bar{x})$.

(iii) *There exist a nonempty and compact subset* N *of* X *and a nonempty, compact and convex subset* B *of* X *such that* $\forall x \in X \setminus N$, $\exists \bar{y} \in B$, *such that* $\bar{y} \in D(x)$ *and* $\langle v, \eta(\bar{y}, x) \rangle \in -\operatorname{int} C(x)$, $\forall v \in T(x)$.

Then, there exists a point $\bar{x} \in X$ *such that* $\bar{x} \in D(\bar{x})$, *and*

$$\forall y \in D(\bar{x}),\ \exists \hat{v} \in T(\bar{x}) : \langle \hat{v}, \eta(y,\bar{x}) \rangle \notin -\operatorname{int} C(\bar{x}). \tag{2.21}$$

Proof. Following the same argument of the proof of [15, Corollary 3], by the condition (ii)(b) and (ii)(c), we know that the condition (ii)(b) of Theorem 2.6 holds. By Theorem 2.6, we know that the conclusion is correct. □

Remark 2.8. Theorem 2.6 and Corollary 2.7, respectively, extend and improve [15, Theorem 2 and Corollary 3].

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