# GENERALIZED PARTIALLY RELAXED PSEUDOMONOTONE VARIATIONAL INEQUALITIES AND GENERAL AUXILIARY PROBLEM PRINCIPLE

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Let $T : K \to H$ be a nonlinear mapping from a nonempty closed invex subset $K$ of an infinite-dimensional Hilbert space $H$ into $H$. Let $f : K \to R$ be proper, invex, and lower semicontinuous on $K$ and let $h : K \to R$ be continuously Fréchet-differentiable on $K$ with $h'$, the gradient of $h$, $(\eta,\alpha)$-*strongly* monotone, and $(\eta,\beta)$-*Lipschitz* continuous on $K$. Suppose that there exist an $x^* \in K$, and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set $S^*$ defined by $S^* = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x,x^*)),\eta(x,x^*)\rangle\}$ is nonempty, where $K^* = \{x \in K : \|x - x^*\| \leq r\}$ and $\eta : K \times K \to H$ is $(\lambda)$-*Lipschitz* continuous with the following assumptions. (i) $\eta(x,y) + \eta(y,x) = 0$, $\eta(x,y) = \eta(x,z) + \eta(z,y)$, and $\|\eta(x,y)\| \leq r$. (ii) For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology. If, in addition, $h'$ is continuous from $H$ equipped with weak topology to $H$ equipped with strong topology, then the sequence $\{x^k\}$ generated by the general auxiliary problem principle converges to a solution $x^*$ of the variational inequality problem (VIP): $\langle T(x^*),\eta(x,x^*)\rangle + f(x) - f(x^*) \geq 0$ for all $x \in K$.



## 1. Introduction

A tremendous amount of work, applying the auxiliary problem principle in finite- as well as in infinite-dimensional Hilbert space settings, on the approximation-solvability of various classes of variational inequalities and complementarity problems, especially finite-dimensional cases, has been carried out in recent years. During the course of these investigations, there has been a significant progress in developing more generalized classes of mappings in the context of new iterative algorithms. In this paper, we intend based on a general auxiliary problem principle to present the approximation-solvability of a class of variational inequality problems (VIP) involving partially relaxed pseudomonotone mappings along with some modified results on Fréchet-differentiable functions that play a pivotal role in the development of a general framework for the auxiliary problem principle. Results thus obtained generalize/complement investigations of Argyros and

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Verma [1], El Farouq [7], Verma [20], and others. For more details on general variational inequality problems and the auxiliary problem principle, we refer to [1–23].

Let $H$ be an infinite-dimensional real Hilbert space with the inner product $\langle x,y\rangle$ and norm $\|x\|$ for all $x,y \in H$. We consider the variational inequality problem (VIP) as follows: determine an element $x^* \in K$ such that

$$\langle T(x^*),\eta(x,x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \forall x \in K, \tag{1.1}$$

where $K$ is a nonempty closed invex subset of $H$, and $\eta : K \times K \to H$ is any mapping with some additional conditions.

When $\eta(x,x^*) = x - x^*$, the VIP (1.1) reduces to the VIP: determine an element $x^* \in K$ such that

$$\langle T(x^*),x - x^*\rangle + f(x) - f(x^*) \geq 0 \quad \forall x \in K, \tag{1.2}$$

where $K$ is a nonempty closed convex subset of $H$.

When $f = 0$ in (1.2), it reduces to the following: find an element $x^* \in K$ such that

$$\langle T(x^*),x - x^*\rangle \geq 0 \quad \forall x \in K. \tag{1.3}$$

Now we recall the following auxiliary result for the approximation solvability of nonlinear variational inequality problems based on iterative procedures.

Lemma 1.1. *For elements $u,v,w \in H$,*

$$\|u\|^2 + \langle u,\eta(v,w)\rangle \geq -\frac{1}{4}\|\eta(v,w)\|^2. \tag{1.4}$$

Lemma 1.2. *For $u,v \in H$,*

$$\langle u,v\rangle = \frac{\|u+v\|^2 - \|u\|^2 - \|v\|^2}{2}. \tag{1.5}$$

*Now recall and in some cases upgrade the existing notions in the literature. Let $\eta : H \times H \to H$ be any mapping.*

*Definition 1.3.* A mapping $T : H \to H$ is called

(i) $(\eta)$-*monotone* if for each $x,y \in H$, there exists,

$$\langle T(x) - T(y),\eta(x,y)\rangle \geq 0; \tag{1.6}$$

(ii) $(\eta,r)$-*strongly* monotone if there exists a positive constant $r$ such that

$$\langle T(x) - T(y),\eta(x,y)\rangle \geq r\|x-y\|^2 \quad \forall x,y \in H; \tag{1.7}$$

(iii) $(r)$-*expansive* if

$$\|T(x) - T(y)\| \geq r\|\eta(x,y)\|; \tag{1.8}$$

(iv) expansive if $r = 1$ in (iii),

(v) $(\eta,\gamma)$-*cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), \eta(x,y) \rangle \geq r\|T(x) - T(y)\|^2 \quad \forall x, y \in H; \tag{1.9}$$

(vi) $(\eta)$-*pseudomonotone* if

$$\langle T(y), \eta(x,y) \rangle \geq 0 \Longrightarrow \langle T(x), \eta(x,y) \rangle \geq 0; \tag{1.10}$$

(vii) $(\eta,b)$-*strongly* pseudomonotone if

$$\langle T(y), \eta(x,y) \rangle \geq 0 \Longrightarrow \langle T(x), \eta(x,y) \rangle \geq b\|x - y\|^2 \quad \forall x, y \in H; \tag{1.11}$$

(viii) $(\eta,c)$-*pseudococoercive* if there exists a constant $c > 0$ such that

$$\langle T(y), \eta(x,y) \rangle \geq 0 \Longrightarrow \langle T(x), \eta(x,y) \rangle \geq c\|T(x) - T(y)\|^2 \quad \forall x, y \in H; \tag{1.12}$$

(ix) $(\eta)$-*quasimonotone* if

$$\langle T(y), \eta(x,y) \rangle > 0 \Longrightarrow \langle T(x), \eta(x,y) \rangle \geq 0 \quad \forall x, y \in H; \tag{1.13}$$

(x) $(\eta,L)$-*relaxed* (also called weakly monotone) if there is a positive constant $L$ such that

$$\langle T(x) - T(y), \eta(x,y) \rangle \geq (-L)\|x - y\|^2 \quad \forall x, y \in H; \tag{1.14}$$

(xi) $(\eta)$-*hemicontinuous* if for all $x, y, w \in H$, the function

$$t \in [0,1] \longrightarrow \langle T(y + t\eta(x,y)), w \rangle \tag{1.15}$$

is continuous;

(xii) $(\eta,\beta)$-*Lipschitz* continuous if there exists a constant $\beta \geq 0$ such that

$$\|T(x) - T(y)\| \leq \beta\|\eta(x,y)\|; \tag{1.16}$$

(xiii) $(\eta,\gamma)$-*partially* relaxed monotone if there exists a positive constant $\gamma$ such that

$$\langle T(x) - T(y), \eta(z,y) \rangle \geq (-\gamma)\|z - x\|^2 \quad \forall x, y, z \in H; \tag{1.17}$$

(xiv) $(\eta,\gamma)$-*partially* relaxed pseudomonotone if there exists a positive constant $\gamma$ such that

$$\langle T(y), \eta(z,y) \rangle \geq 0 \Longrightarrow \langle T(x), \eta(z,y) \rangle \geq (-\gamma)\|z - x\|^2 \quad \forall x, y, z \in H. \tag{1.18}$$

LEMMA 1.4. *Let $T : H \to H$ be $(\eta,\alpha)$-cocoercive and let $\eta : H \times H \to H$ be a mapping such that*

(i) $\|\eta(x,y)\| \leq \lambda\|x - y\|$;
(ii) $\eta(x,y) + \eta(y,x) = 0$;
(iii) $\eta(x,y) = \eta(x,z) + \eta(z,y)$.

*Then $T$ is $(\eta, -(\lambda^2/4\alpha))$-partially relaxed monotone.*

*Proof.* Since $T : H \to H$ is $(\eta,\alpha)$-*cocoercive*, we have

$$\begin{aligned}\langle T(x)-T(y),\eta(z,y)\rangle &= \langle T(x)-T(y),\eta(z,x)\rangle + \langle T(x)-T(y),\eta(x,y)\rangle \\ &\geq \alpha\|T(x)-T(y)\|^2 + \langle T(x)-T(y),\eta(z,x)\rangle \\ &= \alpha\Big(\|T(x)-T(y)\|^2 + \frac{1}{\alpha}\langle T(x)-T(y),\eta(z,x)\rangle\Big) \\ &\geq -\Big\{\frac{1}{4\alpha}\|\eta(z,x)\|^2\Big\} \geq -\Big\{\frac{\lambda^2}{4\alpha}\|z-x\|^2\Big\}.\end{aligned} \tag{1.19}$$

□

*Definition 1.5.* A mapping $T : H \to H$ is said to be $\mu$-*cocoercive* [2] if for each $x, y \in H$, there exists

$$\langle T(x)-T(y),x-y\rangle \geq \mu\|T(x)-T(y)\|^2, \tag{1.20}$$

where $\mu$ is a positive constant.

*Example 1.6.* Let $T : K \to H$ be nonexpansive. Then $I-T$ is 1/2-*cocoercive*, where $I$ is the identity mapping on $H$. For if $x, y \in K$, we have

$$\begin{aligned}\|(I-T)(x)-(I-T)(y)\|^2 &= \|x-y-(T(x)-T(y))\|^2 \\ &= \|x-y\|^2 - 2\langle x-y,T(x)-T(y)\rangle + \|T(x)-T(y)\|^2 \\ &\leq 2\{\|x-y\|^2 - \langle x-y,T(x)-T(y)\rangle\} \\ &= 2\langle x-y,(I-T)(x)-(I-T)(y)\rangle,\end{aligned} \tag{1.21}$$

that is,

$$\langle (I-T)(x)-(I-T)(y),x-y\rangle \geq \frac{1}{2}\|(I-T)(x)-(I-T)(y)\|^2. \tag{1.22}$$

A subset $K$ of $H$ is said to be invex if there exists a function $\eta : K\times K \to H$ such that whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$x + t\eta(y,x) \in K. \tag{1.23}$$

A function $f : K \to R$ is called invex if whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$f(x+t\eta(y,x)) \leq (1-t)f(x) + tf(y). \tag{1.24}$$

## 2. Some auxiliary results

This section deals with some auxiliary results [2] and their modified versions crucial to the approximation-solvability of VIP (1.1). Let $h : H \to R$ be a continuously Fréchet-differentiable mapping on a Hilbert space $H$. It follows that $h'(x) \in L(H,R)$—the space of all bounded linear operators from $H$ into $R$. From now on, we will denote the real number $h'(x)(y)$ by $\langle h'(x),y\rangle$ for all $x, y \in H$.

Lemma 2.1. *Let $H$ be a real Hilbert space and let $K$ be a nonempty closed invex subset of $H$. Let $h'$, the gradient of $h : K \to R$, be $(\eta,\alpha)$-strongly monotone on $K$ and let the following assumptions hold.*

(i) *There exist an $x^* \in K$ and a number $r \geq 0$ such that for all $x \in K^*$ and $t \in [0,1]$, the mapping $\eta : K \times K \to H$ satisfies*

$$||\eta(x,y)|| \leq r. \tag{2.1}$$

(ii) *The set $S^*$ defined by*

$$S^* = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\} \tag{2.2}$$

*is nonempty, where $h : K \to R$ is a continuously Fréchet-differentiable mapping, and the set $K^*$ is defined by*

$$K^* = \{x \in K : ||x - x^*|| \leq r\}. \tag{2.3}$$

*Then for all $x \in K^*$ and $(h,\eta) \in S^*$,*

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x,x^*)\rangle \geq \frac{\alpha}{2}||x - x^*||^2. \tag{2.4}$$

Lemma 2.2. *Let $H$ be a real Hilbert space and let $K$ be a nonempty closed convex subset of $H$. Let $h'$, the gradient of $h : K \to R$, be $(\alpha)$-strongly monotone on $K$ and let $h : K \to R$ be a continuously Fréchet-differentiable mapping. Then for all $x, x^* \in K$,*

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \geq \frac{\alpha}{2}||x - x^*||^2. \tag{2.5}$$

Lemma 2.3. *Let $H$ be a real Hilbert space and let $K$ be a nonempty closed invex subset of $H$. Let $h'$, the gradient of $h : K \to R$, be $(\eta,\delta)$-Lipschitz continuous on $K$ and let the following assumptions hold.*

(i) *There exist an $x^* \in K$ and a number $q \geq 0$ such that for all $x \in K_1$ and $t \in [0,1]$, the mapping $\eta : K \times K \to H$ satisfies*

$$||\eta(x,y)|| \leq q. \tag{2.6}$$

(ii) *The set $S_1$ defined by*

$$S_1 = \{(h,\eta) : h'(x^* + t(x - x^*))(x - x^*) \leq \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\} \tag{2.7}$$

*is nonempty, where $h : K \to R$ is a continuously Fréchet-differentiable mapping, and the set $K_1$ is defined by*

$$K_1 = \{x \in K : ||x - x^*|| \leq q\}. \tag{2.8}$$

*Then for all $x \in K_1$ and $(h,\eta) \in S_1$,*

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x,x^*)\rangle \leq \frac{\delta}{2}||x - x^*||^2. \tag{2.9}$$

## 3. General auxiliary problem principle

In this section, we present the approximation-solvability of the VIP (1.1) using the convergence analysis for the general auxiliary problem principle.

ALGORITHM 3.1. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate $x^{k+1}$ such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1}) \rangle + \rho(f(x) - f(x^{k+1})) \geq 0, \tag{3.1}$$

*for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $f : K \to R$ is proper, invex, and lower semicontinuous, $\rho > 0$, and $\eta : K \times K \to H$ is any mapping.*

ALGORITHM 3.2. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate $x^{k+1}$ such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle + \rho(f(x) - f(x^{k+1})) \geq 0, \tag{3.2}$$

*for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $\rho > 0$, and $K$ is a nonempty closed convex subset of $H$.*

ALGORITHM 3.3. *For arbitrarily chosen initial point $x^0 \in K$, determine an iterate $x^{k+1}$ such that*

$$\langle \rho T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1} \rangle \geq 0, \tag{3.3}$$

*for all $x \in K$, where $h : K \to R$ is continuously Fréchet-differentiable, $\rho > 0$, and $K$ is a nonempty closed convex subset of $H$.*

We now present, based on Algorithm 3.1, the approximation solvability of the VIP (1.1) in a Hilbert space setting.

THEOREM 3.4. *Let $H$ be a real infinite-dimensional Hilbert space and let $K$ be a nonempty closed invex subset of $H$. Let $T : K \to H$ be $(\eta, \gamma)$-partially relaxed pseudomonotone. Let $f : K \to R$ be proper, invex, and lower semicontinuous on $K$, let $h : K \to R$ be continuously Fréchet-differentiable on $K$ with $h'$, the gradient of $h$, $(\eta, \alpha)$-strongly monotone, and $(\eta, \beta)$-Lipschitz continuous, and let $h'$ be continuous from $H$ equipped with weak topology to $H$ equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set $S^*$ defined by*

$$S^* = \{(h, \eta) : h'(y^* + t(x - y^*))(x - y^*) \geq \langle h'(y^* + t\eta(x, y^*)), \eta(x, y^*) \rangle\} \tag{3.4}$$

*is nonempty, where*

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.5}$$

(ii) *The mapping $\eta : K \times K \to H$ is $(\lambda)$-Lipschitz continuous.*

(iii) *$\eta(u, v) + \eta(v, u) = 0$ and $\eta(u, v) = \eta(u \cdot w) + \eta(w, v)$.*

(iv) *For each fixed $y \in K$, the map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) $\|\eta(u,v)\| \leq r$.

*Then an iterate $x^{k+1}$ is a unique solution to (3.1).*

*If, in addition, $x^* \in K$ is a solution to VIP (1.1) and $\|T(x^k) - T(x^*)\| \to 0$, then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to $x^*$.*

*Proof.* First to show that $x^{k+1}$ is a unique solution to (3.1), assume that $y^{k+1}$ is another distinct solution to (3.1). Since $h'$ is $(\eta,\alpha)$-*strongly* monotone, it follows applying (3.1) that

$$-\langle h'(x^{k+1}) - h'(y^{k+1}), \eta(x^{k+1}, y^{k+1})\rangle \geq 0, \tag{3.6}$$

or

$$||x^{k+1} - y^{k+1}||^2 \leq 0, \tag{3.7}$$

a contradiction.

Since $x^* \in K$ is a solution to the VIP (1.1), we define a function $\Delta^*$ by

$$\Delta^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*,x)\rangle. \tag{3.8}$$

Then applying Lemma 2.1, we have

$$\Delta^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*,x)\rangle \geq \frac{\alpha}{2}||x^* - x||^2. \tag{3.9}$$

It follows that

$$\Delta^*(x^{k+1}) := h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), \eta(x^*, x^{k+1})\rangle. \tag{3.10}$$

Now we can write

$$\begin{aligned} \Delta^*(x^k) - \Delta^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) - \langle h'(x^k), \eta(x^{k+1}, x^k)\rangle \\ &\quad + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\ &\geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\ &\geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle \\ &\quad + \rho(f(x^{k+1}) - f(x^*)), \end{aligned} \tag{3.11}$$

for $x = x^*$ in (3.1).

Therefore, we have

$$\Delta^*(x^k) - \Delta^*(x^{k+1}) \geq \frac{\alpha}{2}||x^{k+1} - x^k||^2 + \rho\langle T(x^k), \eta(x^{k+1}, x^*)\rangle + \rho(f(x^{k+1}) - f(x^*)). \tag{3.12}$$

If we replace $x$ by $x^{k+1}$ in (1.1), we obtain

$$\langle T(x^*), \eta(x^{k+1}, x^*)\rangle + f(x^{k+1}) - f(x^*) \geq 0. \tag{3.13}$$

Since $T$ is $(\eta,\gamma)$-*partially* relaxed pseudomonotone, it implies in light of (3.13) that

$$\Delta^*(x^k)-\Delta^*(x^{k+1})\geq\frac{\alpha}{2}\|x^{k+1}-x^k\|^2-\rho\gamma\|x^{k+1}-x^k\|^2=\left(\frac{\alpha}{2}-\rho\gamma\right)\|x^{k+1}-x^k\|^2 \tag{3.14}$$

for $\rho<(\alpha/2\gamma)$.

It follows that the sequence $\{\Delta^*(x^k)\}$ is a strictly decreasing sequence except for $x^{k+1}=x^k$, and in that situation $x^k$ is a solution to (1.1). Since the difference of two consecutive terms tends to zero as $k\to\infty$, it implies that

$$\|x^{k+1}-x^k\|\longrightarrow 0\quad\text{as } k\longrightarrow\infty. \tag{3.15}$$

On the top of that, in light of Lemma 2.1, we have

$$\|x^*-x^k\|^2\leq\frac{2}{\alpha}\Delta^*(x^k), \tag{3.16}$$

and so the sequence $\{x^k\}$ is bounded. Let $x'$ be a cluster point of the sequence $\{x^k\}$, that is, there exists a subsequence $\{x^{kj}\}$ of the sequence $\{x^k\}$ such that $\{x^{kj}\}$ converges weakly to $x'$. Since $h'$ is $(\eta,\beta)$-*Lipschitz* continuous and $a<\rho$, it follows using (3.1) that for some $x\in K$, we have

$$\begin{aligned}\langle\rho T(x^k),\eta(x,x^{k+1})\rangle+\rho(f(x)-f(x^{k+1}))&\geq-\langle h'(x^{k+1})-h'(x^k),\eta(x,x^{k+1})\rangle\\&\geq-\beta\|x^{k+1}-x^k\|\|\eta(x,x^{k+1})\|,\end{aligned} \tag{3.17}$$

or

$$\langle T(x^k),\eta(x,x^{k+1})\rangle+f(x)-f(x^{k+1})\geq-\frac{\beta}{a}\|x^{k+1}-x^k\|\|\eta(x,x^{k+1})\|. \tag{3.18}$$

Since $T(x^{kj})$ converges strongly to $T(x^*)$ and $\|x^{kj+1}-x^{kj}\|\to 0$, and $f$ is invex and lower semicontinuous (and hence $f$ is weakly lower semicontinuous), it follows from (3.18) that

$$\langle T(x^*),\eta(x,x')\rangle+f(x)-f(x')\geq 0\quad\forall x\in K, \tag{3.19}$$

while

$$\begin{gathered}\langle T(x'),\eta(x^{kj},x')\rangle+f(x)-f(x^{kj})\longrightarrow 0,\\\langle T(x'),\eta(x^{kj},x')\rangle\longrightarrow 0.\end{gathered} \tag{3.20}$$

At this stage, if $T(x')=0$, then $x'$ is a solution to the VIP (1.1); and if $T(x')\neq 0$, then we express it in the form

$$\eta(y^{kj},x^{kj})=-\frac{\langle T(x'),\eta(x^{kj},x')\rangle T(x')}{\|T(x')\|^2}. \tag{3.21}$$

It follows that

$$\langle T(x'), \eta(y^{kj}, x')\rangle = 0, \tag{3.22}$$

and thus, we have

$$||\eta(y^{kj}, x^{kj})|| \longrightarrow 0. \tag{3.23}$$

It follows that

$$y^{kj} \rightharpoonup x'. \tag{3.24}$$

Applying (3.22), we have

$$0 = \langle T(x'), \eta(y^{kj}, x')\rangle = \langle T(x'), \eta(y^{kj}, x^*)\rangle + \langle T(x'), \eta(x^*, x')\rangle. \tag{3.25}$$

Since $T(x') \neq 0$, it follows that $y^{kj} \rightharpoonup x^*$ and $x^* = x'$, a solution to the VIP (1.1). □

Corollary 3.5. *Let $H$ be a real infinite-dimensional Hilbert space and let $K$ be a nonempty closed invex subset of $H$. Let $T : K \to H$ be $(\eta, \gamma)$-pseudococoercive. Let $f : K \to R$ be proper, invex, and lower semicontinuous on $K$, let $h : K \to R$ be continuously Fréchet-differentiable on $K$ with $h'$, the gradient of $h$, $(\eta, \alpha)$-strongly monotone and $(\eta, \beta)$-Lipschitz continuous, and let $h'$ be continuous from $H$ equipped with weak topology to $H$ equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set $S^*$ defined by*

$$S^* = \{(h, \eta) : h'(y^* + t(x - y^*))(x - y^*) \geq \langle h'(y^* + t\eta(x, y^*)), \eta(x, y^*)\rangle\} \tag{3.26}$$

*is nonempty, where*

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.27}$$

(ii) *The mapping $\eta : K \times K \to H$ is $(\lambda)$-Lipschitz continuous.*

(iii) *$\eta(u, v) + \eta(v, u) = 0$ and $\eta(u, v) = \eta(u \cdot w) + \eta(w, v)$.*

(iv) *For each fixed $y \in K$, the map $x \to \eta(y, x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) *$\|\eta(u, v)\| \leq r$.*

*Then an iterate $x^{k+1}$ is a unique solution to (3.1).*

*If $x^* \in K$ is a solution to VIP (1.1), then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to $x^*$.*

For $f = 0$ and $\eta(u, v) = u - v$ in Corollary 3.5, it reduces to the following corollary.

Corollary 3.6. *Let $H$ be a real infinite-dimensional Hilbert space and let $K$ be a nonempty closed convex subset of $H$. Let $T : K \to H$ be $(\gamma)$-pseudococoercive. Let $h : K \to R$ be continuously Fréchet-differentiable on $K$ with $h'$, the gradient of $h$, $(\alpha)$-strongly monotone, and $(\beta)$-Lipschitz continuous, and let $h'$ be continuous from $H$ equipped with weak topology to $H$ equipped with strong topology. Then an iterate $x^{k+1}$ is a unique solution to (3.3). If $x^* \in K$ is* *a solution to VIP (1.3), then the sequence $\{x^k\}$ generated by Algorithm 3.3 converges weakly to $x^*$.*

Note that Corollary 3.6 is proved in [7, Theorem 4.1] with an additional imposition of the uniform continuity on the mapping $T$, but we feel that the uniform continuity is not required for the convergence purposes.

THEOREM 3.7. *Let $H$ be a real infinite-dimensional Hilbert space and let $K$ be a nonempty closed invex subset of $H$. Let $T : K \to H$ be $(\eta,\gamma)$-partially relaxed pseudomonotone. Let $f : K \to R$ be proper, invex, and lower semicontinuous on $K$, let $h : K \to R$ be continuously Fréchet-differentiable on $K$ with $h'$, the gradient of $h$, $(\eta,\alpha)$-strongly monotone, and $(\eta,\beta)$-Lipschitz continuous, and let $h'$ be continuous from $H$ equipped with weak topology to $H$ equipped with strong topology. Suppose that the following assumptions hold.*

(i) *There exist a $y^* \in K$ and numbers $a > 0$, $r \geq 0$, $q \geq 0$, $\rho(a < \rho < \alpha/2\gamma)$ such that for all $t \in [0,1]$ and for all $x \in K^*$, the set $S^*$ defined by*

$$S^* = \{(h,\eta) : h'(y^* + t(x - y^*))(x - y^*) \geq \langle h'(y^* + t\eta(x,y^*)),\eta(x,y^*)\rangle\} \tag{3.28}$$

*is nonempty, where*

$$K^* = \{x \in K : ||x - y^*|| \leq r\} \subset K. \tag{3.29}$$

(ii) *The mapping $\eta : K \times K \to H$ is $(\lambda)$-Lipschitz continuous.*

(iii) *$\eta(u,v) + \eta(v,u) = 0$ and $\eta(u,v) = \eta(u \cdot w) + \eta(w,v)$.*

(iv) *For each fixed $y \in K$ the map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(v) *$\|\eta(u,v)\| \leq r$.*

*Then an iterate $x^{k+1}$ is a unique solution to (3.1). If $x^* \in K$ is a solution to VIP (1.1) and $\|T(x^k) - T(x^*)\| \to 0$, then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges weakly to $x^*$.*

*In addition, assume that*

(vi) *there exist a $y^* \in K$ such that for all $x \in K_1$, the set $S_1$ defined by*

$$S_1 = \{(h,\eta) : h'(y^* + t(x - y^*))(x - y^*) \leq \langle h'(y^* + t\eta(x,y^*)),\eta(x,y^*)\rangle\} \tag{3.30}$$

*is nonempty, where*

$$K_1 = \{x \in K : ||x - y^*|| \leq q\} \subset K, \tag{3.31}$$

*with $\|\eta(x,y^*)\| \leq q$.*

*Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to $x^*$.*

*Proof.* Since based on the proof of Theorem 3.4, $x'$ is a weak cluster point of the sequence $\{x^k\}$, we define a function $\Lambda^*$ by

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k),\eta(x',x^k)\rangle. \tag{3.32}$$

Applying Lemmas 2.1 and 2.3, we have the following:

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k), \eta(x', x^k)\rangle \geq \frac{\alpha}{2}\|x' - x^k\|^2. \tag{3.33}$$

$$\Lambda^*(x^k) = h(x') - h(x^k) - \langle h'(x^k), \eta(x', x^k)\rangle \leq \frac{\beta}{2}\|x' - x^k\|^2. \tag{3.34}$$

It follows from (3.34) that

$$\lim_{n\to\infty} \Lambda^*(x^k) = 0. \tag{3.35}$$

Applying (3.35) to (3.33), it follows that the entire sequence $\{x^k\}$ generated by Algorithm 3.1 converges to $x'$. □

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