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Research Article

Generalized Augmented Lagrangian Problem and Approximate Optimal Solutions in Nonlinear Programming

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We introduce some approximate optimal solutions and a generalized augmented Lagrangian in nonlinear programming, establish dual function and dual problem based on the generalized augmented Lagrangian, obtain approximate KKT necessary optimality condition of the generalized augmented Lagrangian dual problem, prove that the approximate stationary points of generalized augmented Lagrangian problem converge to that of the original problem. Our results improve and generalize some known results.

1. Introduction

It is well known that dual method and penalty function method are popular methods in solving nonlinear optimization problems. Many constrained optimization problems can be formulated as an unconstrained optimization problem by dual method or penalty function method. Recently, a general class of nonconvex constrained optimization problem has been reformulated as unconstrained optimization problem via augmented Lagrangian [1].

In [1], Rockafellar and Wets introduced an augmented Lagrangian for minimizing an extended real-valued function. Based on the augmented Lagrangian, a strong duality result without any convexity requirement in the primal problem was obtained under mild conditions. A necessary and sufficient condition for the exact penalization based on the augment Lagrangian function was given [1]. Chen et al. [2] and Huang and Yang [3] used augmented Lagrangian functions to construct the set-valued dual functions and corresponding dual problems and obtained weak and strong duality results of multiobjective optimization problem. More recently a generalized augmented Lagrangian was introduced in [4] by Huang and Yang. They relaxed the convexity on the augmented function, and many papers in the literature are devoted to investigate augmented Lagrangian problems. Necessary and sufficient optimality conditions, duality theory, saddle point theory as well as exact penalization results between the original constrained optimization problems and its unconstrained augmented Lagrangian problems have been established under mild conditions (see, e.g., [5–9]). It is worth noting that most of these results are established on the basis of assumption that the set of optimal solutions of the primal constrained optimization problems is not empty.

However, many mathematical programming problems do not have an optimal solution, moreover sometimes we do not need to find an exact optimal solution due to the fact that it is often very hard to find an exact optimal solution even if it does exist. As a mater of fact, many numerical methods only yield approximate optimal solutions, thus we have to resort to approximate solution of nonlinear programming (see [10–14]). In [10] Liu used exact penalty function to transform a multiobjective programming problem with inequality constraints into an unconstrained problem and derived the Kuhn-Tucker conditions for ϵ-Pareto optimality of primal problem. In [14] Huang and Yang investigated relationship between approximate optimal values of nonlinear Lagrangian problem and that of primal problem. As we known, Ekeland's variational principle and penalty function methods are effective tools to study approximate solutions of constrained optimization problems and the augmented Lagrangian functions have some similar properties of penalty functions. Thus it is possible to apply them in the study of approximate solutions of constrained optimization problems.

In this paper, based on the results in [4, 10, 14], we investigate the possibility of obtaining the various versions of approximate solutions to a constrained optimization problem by solving an unconstrained programming problem formulated by using a generalized augmented Lagrangian function. As an application, an approximate KKT optimality condition is obtained for a kind of approximate solutions to the generalized augmented Lagrangian problem. We prove that the approximate stationary points of the generalized augmented Lagrangian problem converge to that of the original problems. Our results generalized Huang and Yang's corresponding results in [4, 6, 9] into approximate case which is more practical from computational viewpoint.

The paper is organized as follows. In Section 2, we present some concepts, basic assumptions, and preliminary results. In Section 3, we obtain an approximate KKT optimality condition of generalized augmented Lagrangian problem and prove that the approximate stationary points of the generalized augmented Lagrangian problem converge to that of the original problem.

2. Preliminaries

In this section, we present some definitions and Ekeland's variational principle. Consider the following constrained optimization problem:

$$\begin{aligned} \inf f(x) \quad & \text{s.t. } x \in X, \\ g_j(x) = 0, \quad & j = 1,\dots,m, \end{aligned} \tag{P}$$

where $X \subset \mathbb{R}^n$ is a nonempty and closed set, $f : X \to \mathbb{R}$, $g_j : X \to \mathbb{R}$, f and g_j are continuously differentiable functions. Let $S = \{x \in X,\ g_j(x) = 0,\ j = 1,\dots,m\}$, it is clear that S is the set of feasible solutions. For any $\epsilon > 0$, we denote by S_ϵ the set of ϵ feasible solution, that is,

$$S_\epsilon = \{x \in X : g_j(x) = \epsilon,\ j = 1,\dots,m\}, \tag{2.1}$$

and by M_P the optimal value of problem (P).

Let $u \in \mathbb{R}$, we define a function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$:

$$F(x,u) = \begin{cases} f(x), & \text{if } g_j(x) \le u, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

So we have a perturbed problem

$$\inf F(x,u) \quad \text{s.t. } x \in \mathbb{R}^n. \tag{P*'}$$

Define the optimal value function by $p(u) = \inf_{x\in\mathbb{R}^n} F(x,u)$, obviously $p(0)$ is the optimal value of problem (P).

Definition 2.1 [1]. (i) A function $g : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty,+\infty\}$ is called *level-bounded* if, for any $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n; g(x) \le \alpha\}$ is bounded. (ii) A function $h : \mathbb{R}^n \times \mathbb{R}^m \to R \cup \{-\infty,+\infty\}$ with values $h(x,u)$ is called level-bounded in x locally uniformly in u if, for each $\overline{u} \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, there exists a neighborhood $V_{\overline{u}}$ of $\overline{u}$ along with a bounded set $D \subset \mathbb{R}^n$ such that $\{x \in \mathbb{R}^n : h(x,v) \le \alpha\} \subset D$ for all $v \in V_{\overline{u}}$.

Definition 2.2 [4]. A function $\sigma : \mathbb{R}^m \to \mathbb{R}_+ \cup \{+\infty\}$ is called a generalized augmented function if it is proper, lower semicontinuous (lsc), level-bounded on $\mathbb{R}^m$, $\operatorname{argmin}_y \sigma(y) = \{0\}$, and $\sigma(0) = 0$.

Define the dualizing parameterization function:

$$\overline{f}_p(x,u) = f(x) + \delta_{R^m_-}(G(x)+u) + \delta_X(x), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{2.3}$$

where $G(x) = \{g_1(x),\dots,g_m(x)\}$, δ_D is the indicator function of the set D, that is,

$$\delta_D(z) = \begin{cases} 0, & \text{if } z \in D, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.4}$$

So a class of generalized augmented Lagrangians of (P) with dualizing parameterization function $\overline{f}_p(x,u)$ defined by (2.3) can be expressed as

$$l_p(x,y,r) = \inf\{\overline{f}_p(x,u) - \langle y,u\rangle + r\sigma(u) :\ u \in \mathbb{R}^m\}, \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,\ r \ge 0. \tag{2.5}$$

When $\sigma(u) = [\sum_{j=1}^m |u_j|]^\alpha$ $(\alpha > 0)$, the above abstract-generalized augmented Lagrangian can be formulated as the following generalized augmented Lagrangian:

$$l_p(x,y,r) = f(x) + \sum_{j=1}^m y_j g_j(x) + \left[\sum_{j=1}^m |g_j(x)|\right]^\alpha. \tag{2.6}$$

In this paper, we will focus on the problems about the above generalized augmented Lagrangian.

The generalized augmented Lagrangian problem (Q) corresponding to l_p is defined as

$$\psi_p(y,r) = \inf\{l_p(x,y,r);\ x \in \mathbb{R}^n\} \quad y \in \mathbb{R}^m,\ r \geq 0. \tag{2.7}$$

The following various definitions of approximate solutions are taken from Loridan [11].

Definition 2.3. Let $\epsilon > 0$, the point $x^* \in S$ is said to be an ϵ-solution of (P) if

$$f(x^*) \leq f(x) + \epsilon \quad \forall x \in S. \tag{2.8}$$

Definition 2.4. Let $\epsilon > 0$, the point $x^* \in S$ is said to be an ϵ-quasi solution of (P) if

$$f(x^*) \leq f(x) + \epsilon\|x - x^*\| \quad \forall x \in S. \tag{2.9}$$

Definition 2.5. Let $\epsilon > 0$, the point $x^* \in S$ is said to be a regular ϵ-solution of (P) if it is both an ϵ solution and an ϵ-quasi solution of (P).

Definition 2.6. Let $\epsilon > 0$, the point $x^* \in S_\epsilon$ is said to be an almost ϵ-solution of (P) if

$$f(x^*) \leq f(x) + \epsilon \quad \forall x \in S. \tag{2.10}$$

Definition 2.7. The point $x^* \in S$ is said to be an almost regular ϵ-solution of (P) if it is both an almost ϵ-solution and a regular ϵ-solution of (P).

Proposition 2.8 (Ekeland's variational principle) [13]. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be proper lower semicontinous function which is bounded below. Then for any $\epsilon > 0$, there exists $x^* \in S$ such that*

$$\begin{gathered} f(x^*) \leq f(x) + \epsilon, \quad \forall x \in S, \\ f(x^*) < f(x) + \epsilon\|x - x^*\|, \quad \forall x \in S \backslash \{x^*\}. \end{gathered} \tag{2.11}$$

3. Main results

In this section, we will discuss some approximate optimality conditions of constrained optimization problem, obtain necessary condition for an approximate solution of generalized augmented Lagrangian problem (Q), and prove that the approximate stationary points of (Q) converges to that of the primal problem (P). We say that the linear independence constrained qualification (LICQ in short) for (P) holds at $\bar{x}$ if $\{\nabla g_j(\bar{x}) : j \in J_1(\bar{x})\}$ is linearly independent. Suppose that $\bar{x} \in \mathbb{R}^n$ is a local optimal solution to (P) and the (LICQ) for (P) holds at $\bar{x}$. Then the first-order necessary optimality condition is that there exists $\mu_j \geq 0$, $j = 1,\ldots,m$, such that

$$\nabla f(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) = 0. \tag{3.1}$$

Proposition 3.1. *Suppose $\overline{x}_\epsilon \in R^n$ is a ϵ-quasi solution for (P) and the (LICQ) for (P) holds at $\overline{x}_\epsilon \in \mathbb{R}^n$. Then first-order approximate necessary conditions hold that there exists real numbers $\mu_j(\epsilon) \geq 0$, $j = 1,\dots,m$, such that*

$$\left| \nabla f(\overline{x}_\epsilon) + \sum_{j=1}^{m} \mu_j(\epsilon) \nabla g_j(\overline{x}_\epsilon) \right| \leq \epsilon. \tag{3.2}$$

Proof. From the definition of ϵ-quasi solution, we have that there exists $\overline{x}_\epsilon \in S$ such that

$$f(\overline{x}_\epsilon) \leq f(x) + \epsilon \|x - \overline{x}_\epsilon\| \quad \forall x \in S. \tag{3.3}$$

We conclude that $\overline{x}_\epsilon$ is a local optimal solution of the following constrained optimization problem (P*):

$$\inf \{ f(x) + \epsilon \|x - \overline{x}_\epsilon\| \} \quad \text{s.t. } x \in S. \tag{P*}$$

For the objective function, $\{ f(x) + \epsilon \|x - \overline{x}_\epsilon\| \}$ is only locally Lipschitz. Thus we apply Proposition 2.8 and obtain the KKT necessary condition of (P*):

$$\nabla f(\overline{x}_\epsilon) + \xi\epsilon + \sum_{j=1}^{m} \mu_j(\epsilon) \nabla g_j(\overline{x}_\epsilon) = 0 \quad \xi \in [-1,1]. \tag{3.4}$$

It follows that

$$\left| \nabla f(\overline{x}_\epsilon) + \sum_{j=1}^{m} \mu_j(\epsilon) \nabla g_j(\overline{x}_\epsilon) \right| \leq \epsilon. \tag{3.5}$$

□

It is easy to see that the generalized augmented Lagrangian function is a nonsmooth function, moreover it is not locally Lipschitz when $0 < \alpha < 1$. Thus it is necessary that we divide the generalized augmented Lagrangian problems into the following two parts:

$$\alpha > 1, \quad 0 < \alpha < 1. \tag{3.6}$$

First let us consider the case (1), the generalized augmented Lagrangian function is a nonsmooth function, thus we have the following conclusion.

Proposition 3.2. *For any $\epsilon > 0$, suppose $x_\epsilon \in \mathbb{R}^n$ is a ϵ-quasi solution of generalized augmented Lagrangian problem (Q), then*

$$\left|\nabla f(x_\epsilon) + \sum_{j=1}^{m} \nabla g_j(x_\epsilon)\left\{y_j + \theta r\alpha\left[\sum_{j=1}^{m} |g_j(x_\epsilon)|\right]^{\alpha-1}\right\}\right| \le \epsilon, \tag{3.7}$$

where $\theta \in [-1,1]$.

Proof. Since $x_\epsilon \in \mathbb{R}^n$ is a ϵ-quasi solution of generalized augmented Lagrangian problem (Q), we can see that

$$f(x_\epsilon)+\sum_{j=1}^{m} y_j g_j(x_\epsilon)+\left[\sum_{j=1}^{m} |g_j(x_\epsilon)|\right]^{\alpha} \le f(x)+\sum_{j=1}^{m} y_j g_j(x)+\left[\sum_{j=1}^{m} |g_j(x)|\right]^{\alpha} +\epsilon\|x-x_\epsilon\|, \tag{3.8}$$

thus we have that x_ϵ is a local optimal solution of the following optimization problem (P^{**}):

$$\inf\left\{f(x)+\sum_{j=1}^{m} y_j g_j(x)+\left[\sum_{j=1}^{m} |g_j(x)|\right]^{\alpha} +\epsilon\|x-x_\epsilon\|,\ x\in\mathbb{R}^n\right\}. \tag{P^{**}}$$

Since the objective function of (P^{**}) is only locally Lipschitz. Thus we apply the corollary of Proposition 2.4.3 in [15] and Example 2.1.2 in [15] and obtain the approximate KKT necessary condition of (P^{**}):

$$\left|\nabla f(x_\epsilon) + \sum_{j=1}^{m} \nabla g_j(x_\epsilon)\left\{y_j + \theta r\alpha\left[\sum_{j=1}^{m} |g_j(x_\epsilon)|\right]^{\alpha-1}\right\}\right| \le \epsilon \tag{3.9}$$

□

Theorem 3.3 (convergence analysis). *Suppose $\{y_k\} \in \mathbb{R}^m$ is bounded, $0 < r_k \to +\infty$ as $k \to +\infty$, $x_\epsilon^k \in \mathbb{R}^n$ is generated by some methods for solving the following problem (Q_k):*

$$\inf\{l_p(x,y^k,r_k);\ x\in\mathbb{R}^n\} \quad y^k\in\mathbb{R}^m,\ r_k\ge 0. \tag{3.10}$$

Assume that there exist n, $N \in \mathbb{R}$ such that $f(x_\epsilon^k) \ge n$, $l_p(x_\epsilon^k, y^k, r_k) \le N$ for any k. Then every limit point x_ϵ^* of $\{x_\epsilon^k\}$ is feasible to the primal problem (P). Further assume that each x_ϵ^k satisfies the approximate first-order necessary optimality condition stated in Proposition 3.2 and the (LICP) of (P) holds at x_ϵ^*. Then x_ϵ^* satisfies the approximate first-order necessary optimality condition of (P).

Proof. Without loss of generality, we suppose that $x_\epsilon^k \to x_\epsilon^*$. Noting that $l_p(x_\epsilon^k, y^k, r_k) \le N$ for any k, so we can see

$$f(x_\epsilon^k) + \sum_{j=1}^{m} y_j^k g_j(x_\epsilon^k) + r_k\left[\sum_{j=1}^{m} |g_j(x_\epsilon^k)|\right]^{\alpha} \le N. \tag{3.11}$$

Moreover, since $f(x_\epsilon^k) \geq n$ and $y^k \in \mathbb{R}^m$ is bounded, thus there exist $N_1 \in R$ such that

$$r_k\left[\sum_{j=1}^{m} |g_j(x_\epsilon^k)|\right]^\alpha \leq N_1,$$
$$\left[\sum_{j=1}^{m} |g_j(x_\epsilon^k)|\right]^\alpha \leq \frac{N_1}{r_k}. \tag{3.12}$$

It is clear that $g_j(x_\epsilon^*) = 0$ as $r_k \to +\infty$. Therefore, x_ϵ^* is a feasible solution to (P). □

Letting $\nu_j^k = \{y_j^k + \theta r\alpha[\sum_{j=1}^{m} |g_j(x_\epsilon^k)|]^{\alpha-1}\}$, $j = 1,\ldots,m$, where $\theta \in [-1,1]$, the inequality (3.7) can be formulated as

$$\left|\nabla f(x_\epsilon^k) + \sum_{j=1}^{m} \nu_j^k \nabla g_j(x_\epsilon^k)\right| \leq \epsilon. \tag{3.13}$$

Now we prove by contradiction that the sequence $\sum_{j=1}^{m} |\nu_j^k|$ is bounded as $k \to +\infty$. Otherwise without loss of generality, we assume that $\sum_{j=1}^{m} |\nu_j^k| \to +\infty$, then we can see that

$$\lim_{k\to+\infty} \frac{\nu_j^k}{\sum_{j=1}^{m} |\nu_j^k|} = \nu_j^*, \quad j = 1,\ldots,m. \tag{3.14}$$

Dividing (3.13) by $\sum_{j=1}^{m} |\nu_j^k|$ and letting k to the limit, we can derive that

$$\left|\sum_{j=1}^{m} \nu_j^* \nabla g_j(x_\epsilon^*)\right| = 0. \tag{3.15}$$

This contradicts with the (LICQ) of (P) which holds at x_ϵ^*. Hence $\sum_{j=1}^{m} |\nu_j^k|$ is bounded and without loss of generality, we can assume that

$$\nu_j^k \longrightarrow \nu_j, \quad j = 1,\ldots,m. \tag{3.16}$$

Thus taking limit in (3.14) and applying (3.16), we can obtain the approximate first-order necessary condition of (P).

Next let's consider the case $0 < \alpha < 1$. It is clear that the generalized augmented Lagrangian function $l_p(x,y,r)$ is a nonlocal Lipschitz nonsmooth function when $0 < \alpha < 1$. However, we have not founded one that is suitable for our purpose of convergence analysis of the second case. Fortunately, we may smooth $l_p(x,y,r)$ by approximation.

Definition 3.4. For any $0 < \epsilon_k \to 0$ as $k \to +\infty$, the following function is called an approximate generalized augmented Lagrangian:

$$l_p(x,y,r,\epsilon_k) = f(x) + \sum_{j=1}^{m} y_j g_j(x) + r\left[\sum_{j=1}^{m} \sqrt{g_j(x)^2 + \epsilon_k^2}\right]^\alpha. \tag{3.17}$$

It is clear that the approximate generalized augmented Lagrangian is a smooth function.

So we have the corresponding approximate generalized augmented Lagrangian problem (Q_ϵ) can be expressed as follows:

$$\inf\{l_p(x,y,r,\epsilon_k),\ x\in\mathbb{R}^n\}\quad y\in\mathbb{R}^m,\ r\geq 0. \tag{3.18}$$

For this approximate generalized augmented Lagrangian function, it is necessary to consider error estimation between generalized augmented Lagrangian function and the approximate generalized augmented Lagrangian function. The following Lemma is about the error estimation

Lemma 3.5. *For generalized augmented Lagrangian function and approximate generalized augmented Lagrangian function, the following statement holds:*

$$l_p(x,y,r,\epsilon_k)-l_p(x,y,r)\leq rm\epsilon_k, \tag{3.19}$$

where $\epsilon_k\to 0$ *as* $k\to+\infty$.

Proof. From (2.6) and (3.17), we can see that

$$\begin{aligned}&\left\{f(x)+\sum_{j=1}^m y_jg_j(x)+r\left[\sum_{j=1}^m\sqrt{g_j(x)^2+\epsilon_k^2}\right]^\alpha\right\}-\left\{f(x)+\sum_{j=1}^m y_jg_j(x)+r\left[\sum_{j=1}^m\sqrt{g_j(x)^2}\right]^\alpha\right\}\\&\quad=r\left\{\left[\sum_{j=1}^m\sqrt{g_j(x)^2+\epsilon_k^2}\right]^\alpha-\left[\sum_{j=1}^m\sqrt{g_j(x)^2}\right]^\alpha\right\}.\end{aligned} \tag{3.20}$$

For $\sqrt{g_j(x)^2+\epsilon_k^2}-\sqrt{g_j(x)^2}\leq\epsilon_k$, thus we have that

$$\left[\sum_{j=1}^m\sqrt{g_j(x)^2+\epsilon_k^2}\right]-\left[\sum_{j=1}^m\sqrt{g_j(x)^2}\right]\leq m\epsilon_k, \tag{3.21}$$

letting $M=\sum_{j=1}^m\sqrt{g_j(x)^2}$, then we can derive that

$$\left[\sum_{j=1}^m\sqrt{g_j(x)^2+\epsilon_k^2}\right]^\alpha-\left[\sum_{j=1}^m\sqrt{g_j(x)^2}\right]^\alpha\leq(M+m\epsilon_k)^\alpha-M^\alpha. \tag{3.22}$$

Since $0<\alpha<1$, when $M+m\epsilon_k\geq 1$, we can see that

$$(M+m\epsilon_k)^\alpha-M^\alpha\leq M+m\epsilon_k-M=m\epsilon_k, \tag{3.23}$$

when $M+m\epsilon_k<1$, we have that

$$(M+m\epsilon_k)^\alpha-M^\alpha\leq\xi_k,\quad \xi_k\in(0,1). \tag{3.24}$$

However, we can see $\epsilon_k\to 0$ as $k\to+\infty$, thus we have that $\xi_k\to 0$. Without lose of generality, we can derive that $m\epsilon_k=\xi_k$ when k is sufficiently large. Thus we have the following statement:

$$l_p(x,y,r,\epsilon_k) - l_p(x,y,r) \le rm\epsilon_k. \tag{3.25}$$

□

Next we will discuss approximate optimality of approximate generalized augmented Lagrangian problem (Q_ϵ).

Proposition 3.6 (approximate optimality condition). *Assume that $\overline{x}_\epsilon \in \mathbb{R}^n$ is a ϵ-quasi solution of (Q_ϵ), then*

$$\left|\nabla f(\overline{x}_\epsilon)+\sum_{j=1}^{m}\left\{y_j+r\alpha\left[\sum_{j=1}^{m}\sqrt{g_j(\overline{x}_\epsilon)^2+\epsilon_k^2}\right]^{\alpha-1}\sum_{j=1}^{m}\left[g_j(\overline{x}_\epsilon)^2+\epsilon_k^2\right]^{-1/2}g_j(\overline{x}_\epsilon)\right\}\nabla g_j(\overline{x}_\epsilon)\right| \le \epsilon, \tag{3.26}$$

where $\epsilon_k \to 0$, as $k \to +\infty$.

Proof. From the definition of ϵ-quasi solution, we have that

$$l_p(\overline{x}_\epsilon,y,r,\epsilon_k) \le l_p(x,y,r,\epsilon_k) + \epsilon\|x-\overline{x}_\epsilon\|. \tag{3.27}$$

From (3.17), we can see that

$$\begin{aligned} &f(\overline{x}_\epsilon)+\sum_{j=1}^{m}y_jg_j(\overline{x}_\epsilon)+r\left[\sum_{j=1}^{m}\sqrt{g_j(\overline{x}_\epsilon)^2+\epsilon_k^2}\right]^{\alpha} \\ &\quad \le f(x)+\sum_{j=1}^{m}y_jg_j(x)+r\left[\sum_{j=1}^{m}\sqrt{g_j(x)^2+\epsilon_k^2}\right]^{\alpha}+\epsilon\|x-\overline{x}_\epsilon\|; \end{aligned} \tag{3.28}$$

it is clear that $\overline{x}_\epsilon$ is a local optimal solution of the following optimization problem:

$$\inf_{x\in R^n}\left\{f(x)+\sum_{j=1}^{m}y_jg_j(x)+r\left[\sum_{j=1}^{m}\sqrt{g_j(x)^2+\epsilon_k^2}\right]^{\alpha}+\epsilon\|x-\overline{x}_\epsilon\|\right\}. \tag{3.29}$$

Since the objective function of the above problem is local Lipschitz Thus we apply the corollary of Proposition 2.4.3 in [15] and Example 2.1.3 in [15], and obtain the KKT necessary condition:

$$\nabla f(\overline{x}_\epsilon)+\sum_{j=1}^{m}\left\{y_j+r\alpha\left[\sum_{j=1}^{m}\sqrt{g_j(\overline{x}_\epsilon)^2+\epsilon_k^2}\right]^{\alpha-1}\sum_{j=1}^{m}\left[g_j(\overline{x}_\epsilon)^2+\epsilon_k^2\right]^{-1/2}g_j(\overline{x}_\epsilon)\right\}\nabla g_j(\overline{x}_\epsilon)+\xi\epsilon=0, \tag{3.30}$$

where $\xi \in [-1,1]$, thus we have that

$$\left|\nabla f(\overline{x}_\epsilon)+\sum_{j=1}^{m}\left\{y_j+r\alpha\left[\sum_{j=1}^{m}\sqrt{g_j(\overline{x}_\epsilon)^2+\epsilon_k^2}\right]^{\alpha-1}\sum_{j=1}^{m}\left[g_j(\overline{x}_\epsilon)^2+\epsilon_k^2\right]^{-1/2}g_j(\overline{x}_\epsilon)\right\}\nabla g_j(\overline{x}_\epsilon)\right| \le \epsilon. \tag{3.31}$$

□

Theorem 3.7 (convergence analysis). *Assume that $y^k \in \mathbb{R}^m$ is bounded, $0 < r_k \to +\infty$ as $k \to +\infty$, $\overline{x}_\epsilon^k \in \mathbb{R}^n$ is generated by some methods for solving the following problem (Q_ϵ^k):*

$$\inf\{l_p(x, y^k, r_k); x \in \mathbb{R}^n\} \quad y^k \in \mathbb{R}^m,\ r_k \geq 0. \tag{3.32}$$

Suppose that there exist n, $N \in \mathbb{R}$ such that for any k, $f(\overline{x}_\epsilon^k) \geq n$, $l_p(\overline{x}_\epsilon^k, y^k, r_k, \epsilon_k) \leq N$. Then every limit point $\overline{x}_\epsilon$ of $\{\overline{x}_\epsilon^k\}$ is feasible to the primal problem (P). Further assume that each $\overline{x}_\epsilon^k$ satisfies the approximate first-order necessary optimality condition stated in Proposition 3.6 and the (LICP) of (P) holds at $\overline{x}_\epsilon$. Then $\overline{x}_\epsilon$ satisfies the approximate first-order necessary optimality condition of (P).

Proof. Without loss of generality, we assume that $\overline{x}_\epsilon^k \to \overline{x}_\epsilon$. From $l_p(\overline{x}_\epsilon^k, y_k, r_k) \leq N$, we have that

$$f(\overline{x}_\epsilon^k) + \sum_{j=1}^{m} y_j^k g_j(\overline{x}_\epsilon^k) + r_k \left[\sum_{j=1}^{m} \sqrt{g_j(\overline{x}_\epsilon^k)^2 + \epsilon_k^2}\right]^\alpha \leq N. \tag{3.33}$$

Since $f(\overline{x}_\epsilon^k) \geq n$ and $\{y^k\} \in \mathbb{R}^m$ be bounded, so there exist $N_1 \in R^n$ such that

$$r_k \left[\sum_{j=1}^{m} \sqrt{g_j(\overline{x}_\epsilon^k)^2 + \epsilon_k^2}\right]^\alpha \leq N_1 \tag{3.34}$$

when $k \to +\infty$, we have that $g_j(\overline{x}_\epsilon) = 0$ and $\overline{x}_\epsilon$ is a feasible solution to (P). □

Since $\overline{x}_\epsilon^k$ satisfies approximate optimality condition stated in Proposition 3.6. Let

$$\mu_j^k = \left\{ y_j^k + r_k \alpha \left[\sum_{j=1}^{m} \sqrt{g_j(x_\epsilon^k)^2 + \epsilon_k^2}\right]^{\alpha-1} \sum_{j=1}^{m} \left[g_j(arx_\epsilon^k)^2 + \epsilon_k^2\right]^{-1/2} g_j(x_\epsilon^{k\alpha}) \right\}. \tag{3.35}$$

From (3.26) we have that

$$\left| \nabla f(x_\epsilon^k) + \sum_{j=1}^{m} \mu_j^k \nabla g_j(x_\epsilon^k) \right| \leq \epsilon. \tag{3.36}$$

Now we prove by contradiction that the sequence $\sum_{j=1}^m |\mu_j^k|$ is bounded as $k \to +\infty$. Otherwise without loss of generality, we assume that $\sum_{j=1}^m |\mu_j^k| \to +\infty$, then we can see that

$$\lim_{k \to +\infty} \frac{\mu_j^k}{\sum_{j=1}^m |\mu_j^k|} = \mu_j^* \quad j = 1, \ldots, m. \tag{3.37}$$

We divide (3.26) by $\sum_{j=1}^m |\mu_j^*|$ and take $k \to +\infty$, we have that

$$\left| \sum_{j=1}^{m} \mu_j^* \nabla g_j(x^*) \right| = 0. \tag{3.38}$$

This contradicts the (LICQ) of (P) which holds at $\bar{x}_\epsilon$. So we have that $\sum_{j=1}^{m} |\mu_j^k|$ is bounded. So without loss of generality, we assume that

$$\mu_j^k \longrightarrow \mu_j^*, \quad j = 1,\ldots,m. \tag{3.39}$$

Taking $k \to +\infty$ in (3.26) and applying (3.39), then we can derive the approximate first-order necessary condition of (P).

4. Conclusion

As we know, Lagrangian method is a powerful tool to transform the constrained optimization problem into an unstrained optimization problem. However, it will cause dual gap between primal problem and dual one without some convexity requirements. In [4, 6, 9], Huang and Yang introduced a generalized augmented Lagrangian and studied various properties of generalized augmented Lagrangian problem based on an assumption that the set of exact optimal solutions of the primal constrained optimization problem is not empty. But many mathematical programming problems do not have an optimal solution, moreover sometimes we do not need to find an exact optimal solution due to the fact that it is often very hard to find an exact optimal solution even if it does exist. As a matter of fact, many numerical methods only yield approximate optimal solutions. So in this paper, we consider the ϵ-quasi optimal solution and the generalized augmented Lagrangian in nonlinear programming without the requirement that the set of optimal solutions of the primal constrained optimization problems is not empty, establish dual function and dual problem based on the generalized augmented Lagrangian, obtain approximate KKT necessary optimality condition of the generalized augmented Lagrangian dual problem, and prove that the approximate stationary points of generalized augmented Lagrangian problem converge to that of the original problem. Our results generalized Huang and Yang's corresponding results in [4, 6, 9] into approximate case which is more suitable for numerical test.

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