Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 24716, 8 pages
doi:10.1155/2007/24716

Research Article

A Multidimensional Functional Equation Having Quadratic Forms as Solutions

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Received 7 July 2007; Accepted 3 September 2007

Recommended by Vijay Gupta

We obtain the general solution and the stability of the m-variable quadratic functional equation $f(x_1+y_1,\dots,x_m+y_m)+f(x_1-y_1,\dots,x_m-y_m)=2f(x_1,\dots,x_m)+2f(y_1,\dots,y_m)$. The quadratic form $f(x_1,\dots,x_m)=\sum_{1\le i\le j\le m}a_{ij}x_ix_j$ is a solution of the given functional equation.

1. Introduction

In this paper, let X and Y be real vector spaces. A mapping f is called a *quadratic form* if there exist $a_{ij}\in\mathbb{R}$ $(1\le i\le j\le m)$ such that

$$f(x_1,\dots,x_m)=\sum_{1\le i\le j\le m}a_{ij}x_ix_j \tag{1.1}$$

for all $x_1,\dots,x_m\in X$.

For a mapping $f:X^m\to Y$, consider the m-variable quadratic functional equation

$$f(x_1+y_1,\dots,x_m+y_m)+f(x_1-y_1,\dots,x_m-y_m)=2f(x_1,\dots,x_m)+2f(y_1,\dots,y_m). \tag{1.2}$$

When $X=Y=\mathbb{R}$, the quadratic form $f:\mathbb{R}^m\to\mathbb{R}$ given by

$$f(x_1,\dots,x_m)=\sum_{1\le i\le j\le m}a_{ij}x_ix_j \tag{1.3}$$

is a solution of (1.2).

For a mapping $g : X \to Y$, consider the quadratic functional equation

$$g(x+y)+g(x-y) = 2g(x)+2g(y). \tag{1.4}$$

In 1989, Aczél [1] proposed the solution of (1.4). Later, many different quadratic functional equations were solved by numerous authors [2–6].

In this paper, we investigate the relation between (1.2) and (1.4). And we find out the general solution and the generalized Hyers-Ulam stability of (1.2).

2. Results

The m-variable quadratic functional equation (1.2) induces the quadratic functional equation (1.4) as follows.

Theorem 2.1. *Let $f : X^m \to Y$ be a mapping satisfying (1.2) and let $g : X \to Y$ be the mapping given by*

$$g(x) := f(x,\dots,x) \tag{2.1}$$

for all $x \in X$, then g satisfies (1.4).

Proof. By (1.2) and (2.1),

$$\begin{aligned} g(x+y)+g(x-y) &= f(x+y,\dots,x+y)+f(x-y,\dots,x-y) \\ &= 2f(x,\dots,x)+2f(y,\dots,y) = 2g(x)+2g(y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. □

The quadratic functional equation (1.4) induces the m-variable quadratic functional equation (1.2) with an additional condition.

Theorem 2.2. *Let $a_{ij} \in \mathbb{R}$ $(1 \le i \le j \le m)$ and $g : X \to Y$ be a mapping satisfying (1.4). If $f : X^m \to Y$ is the mapping given by*

$$f(x_1,\dots,x_m) := \sum_{i=1}^{m} a_{ii}g(x_i) + \frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i+x_j)-g(x_i-x_j)\right] \tag{2.3}$$

for all $x_1,\dots,x_m \in X$, then f satisfies (1.2). Furthermore, (2.1) holds if

$$\sum_{1\le i\le j\le m} a_{ij} = 1. \tag{2.4}$$

Proof. By (1.4) and (2.3),

$$
\begin{aligned}
& f(x_1+y_1,\ldots,x_m+y_m)+f(x_1-y_1,\ldots,x_m-y_m) \\
&\quad =\sum_{i=1}^{m} a_{ii}\left[g(x_i+y_i)+g(x_i-y_i)\right] \\
&\qquad +\frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i+y_i+x_j+y_j)-g(x_i+y_i-x_j-y_j)\right] \\
&\qquad +\frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i-y_i+x_j-y_j)-g(x_i-y_i-x_j+y_j)\right] \\
&\quad =2\sum_{i=1}^{m} a_{ii}\left[g(x_i)+g(y_i)\right] \\
&\qquad +\frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i+y_i+x_j+y_j)+g(x_i-y_i+x_j-y_j)\right] \\
&\qquad -\frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i+y_i-x_j-y_j)+g(x_i-y_i-x_j+y_j)\right] \\
&\quad =2\sum_{i=1}^{m} a_{ii}\left[g(x_i)+g(y_i)\right] \\
&\qquad +\frac{1}{2}\sum_{1\le i<j\le m} a_{ij}\left[g(x_i+x_j)+g(y_i+y_j)-g(x_i-x_j)-g(y_i-y_j)\right] \\
&\quad =2f(x_1,\ldots,x_m)+2f(y_1,\ldots,y_m)
\end{aligned}
\tag{2.5}
$$

for all $x_1,\ldots,x_m,y_1,\ldots,y_m\in X$.

Letting $x=y=0$ and $y=x$ in (1.4), respectively,

$$
g(0)=0,\qquad g(2x)=4g(x) \tag{2.6}
$$

for all $x\in X$. By (2.3) and the above two equalities,

$$
f(x,\ldots,x)=\sum_{i=1}^{m} a_{ii}g(x)+\frac{1}{4}\sum_{1\le i<j\le m} a_{ij}\left[g(2x)-g(0)\right]=\sum_{1\le i\le j\le m} a_{ij}g(x)=g(x) \tag{2.7}
$$

for all $x\in X$. □

Example 2.3. The function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=x^2$ satisfies (1.4). By Theorem 2.2, the quadratic form $f:\mathbb{R}^m\to\mathbb{R}$ given by

$$
f(x_1,\ldots,x_m)=\sum_{1\le i\le j\le m} a_{ij}x_iy_j \tag{2.8}
$$

satisfies (1.2).

Example 2.4. Let $g:\mathbb{C}\to\mathbb{C}$ be the function given by $g(z)=z\bar{z}$. Then, it satisfies the quadratic functional equation (1.4). If $f:\mathbb{C}^m\to\mathbb{C}$ is the mapping given by (2.3), that is,

$$f(z_1,\dots,z_m)=\sum_{i=1}^{m} z_i\overline{z_i}\left(\sum_{j=1}^{i} a_{ji}+\frac{1}{2}\sum_{j=i+1}^{m} a_{ij}\right), \tag{2.9}$$

then f satisfies the m-variable quadratic functional equation (1.2).

Example 2.5. Let $M_2(\mathbb{R})$ be the real vector space of all 2×2 real matrices and $g:M_2(\mathbb{R})\to\mathbb{R}$ the determinant function given by

$$g(A)=\det(A) \tag{2.10}$$

for all $A\in M_2(\mathbb{R})$. Then, it satisfies (1.4). Using (2.3), $f:M_2(\mathbb{R})\times M_2(\mathbb{R})\to\mathbb{R}$ is given by $f(A,B)=(a_{11}+(1/2)a_{12})\det(A)+(a_{22}+(1/2)a_{12})\det(B)(a_{11},a_{12},a_{21},a_{22}\in\mathbb{R})$. Also, f satisfies (1.2).

In the following theorem, we find out the general solution of the m-variable quadratic functional equation (1.2).

Theorem 2.6. *A mapping $f:X^m\to Y$ satisfies (1.2) if and only if there exist symmetric biadditive mappings $S_1,\dots,S_m:X^2\to Y$ and biadditive mappings $M_{ij}:X^2\to Y\,(1\le i<j\le m)$ such that*

$$f(x_1,\dots,x_m)=\sum_{i=1}^{m} S_i(x_i,x_i)+\sum_{1\le i<j\le m} M_{ij}(x_i,x_j) \tag{2.11}$$

for all $x_1,\dots,x_m\in X$.

Proof. We first assume that f is a solution of (1.2). Define $f_1,\dots,f_m:X\to Y$ by $f_1(x):=f(x,0,\dots,0),\dots,f_m(x):=f(0,\dots,0,x)$ for all $x\in X$. One can easily verify that $f_1,\dots,f_m$ are quadratic. By [1], there exist symmetric biadditive mappings $S_1,\dots,S_m:X^2\to Y$ such that $f_1(x)=S_1(x,x),\dots,f_m(x)=S_m(x,x)$ for all $x\in X$. Define $M_{ij}:X^2\to Y$ by

$$\begin{aligned}M_{ij}(x,y):=&\,f(0,\dots,0,x,0,\dots,0,y,0,\dots,0)-f(0,\dots,0,x,0,\dots,0,0,0,\dots,0)\\&-f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\end{aligned} \tag{2.12}$$

for all i,j with $1\le i<j\le m$ and all $x,y\in X$. On the right-hand side of (2.12), x and y are the ith and the jth components, respectively. Then, M_{ij} are biadditive for all i,j with

$1 \le i < j \le m$. Indeed, by (1.2) and (2.12), we obtain

$$\begin{aligned}
&M_{ij}(x_1+x_2,y)\\
&\quad= f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) - f(0,\dots,0,x_1+x_2,0,\dots,0,0,0,\dots,0)\\
&\qquad - f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0)\\
&\qquad - \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) + f(0,\dots,0,x_1+x_2,0,\dots,0,-y,0,\dots,0)\right]\\
&\quad= \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) - f(0,\dots,0,x_1+x_2,0,\dots,0,-y,0,\dots,0)\right]\\
&\quad= f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0)\\
&\qquad - \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) + f(0,\dots,0,x_1+x_2,0,\dots,0,-y,0,\dots,0)\right]\\
&\quad= f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) - f(0,\dots,0,x_1+x_2,0,\dots,0,0,0,\dots,0)\\
&\qquad - f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= \frac{1}{2}\left[2f(0,\dots,0,x_1+x_2,0,\dots,0,y,0,\dots,0) + 2f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\right.\\
&\qquad\quad \left.- 2f(0,\dots,0,x_1+x_2,0,\dots,0,0,0,\dots,0)\right] - 2f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,2y,0,\dots,0) - f(0,\dots,0,x_1+x_2,0,\dots,0,0,0,\dots,0)\right]\\
&\qquad - 2f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,2y,0,\dots,0) + f(0,\dots,0,x_1-x_2,0,\dots,0,0,0,\dots,0)\right]\\
&\qquad - \frac{1}{2}\left[f(0,\dots,0,x_1+x_2,0,\dots,0,0,0,\dots,0) + f(0,\dots,0,x_1-x_2,0,\dots,0,0,0,\dots,0)\right]\\
&\qquad - 2f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= f(0,\dots,0,x_1,0,\dots,0,y,0,\dots,0) + f(0,\dots,0,x_2,0,\dots,0,y,0,\dots,0)\\
&\qquad - f(0,\dots,0,x_1,0,\dots,0,0,0,\dots,0) - f(0,\dots,0,x_2,0,\dots,0,0,0,\dots,0)\\
&\qquad - 2f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\\
&\quad= f(0,\dots,0,x_1,0,\dots,0,y,0,\dots,0)\\
&\qquad - \left[f(0,\dots,0,x_1,0,\dots,0,0,0,\dots,0) + f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\right]\\
&\qquad + f(0,\dots,0,x_2,0,\dots,0,y,0,\dots,0)\\
&\qquad - \left[f(0,\dots,0,x_2,0,\dots,0,0,0,\dots,0) + f(0,\dots,0,0,0,\dots,0,y,0,\dots,0)\right]\\
&\quad= M_{ij}(0,\dots,0,x_1,0,\dots,0,y,0,\dots,0) + M_{ij}(0,\dots,0,x_2,0,\dots,0,y,0,\dots,0)
\end{aligned} \tag{2.13}$$

for all $x_1, x_2, y \in X$. Similarly,

$$\begin{aligned} &M_{ij}(0,\dots,0,x,0,\dots,0,y_1+y_2,0,\dots,0) \\ &\quad = M_{ij}(0,\dots,0,x,0,\dots,0,y_1,0,\dots,0) + M_{ij}(0,\dots,0,x,0,\dots,0,y_2,0,\dots,0) \end{aligned} \tag{2.14}$$

for all $x, y_1, y_2 \in X$.

Conversely, we assume that there exist symmetric biadditive mappings $S_1,\dots,S_m : X^2 \to Y$ and biadditive mappings $M_{ij} : X^2 \to Y$ $(1 \le i < j \le m)$ such that

$$f(x_1,\dots,x_m) = \sum_{i=1}^{m} S_i(x_i,x_i) + \sum_{1\le i<j\le m} M_{ij}(x_i,x_j) \tag{2.15}$$

for all $x_1,\dots,x_m \in X$. Since M_{ij} $(1 \le i < j \le m)$ are biadditive and $S_1,\dots,S_m$ are symmetric biadditive,

$$\begin{aligned} &f(x_1+y_1,\dots,x_m+y_m) + f(x_1-y_1,\dots,x_m-y_m) \\ &\quad = \sum_{i=1}^{m} S_i(x_i+y_i,x_i+y_i) + \sum_{1\le i<j\le m} M_{ij}(x_i+y_i,x_j+y_j) \\ &\qquad + \sum_{i=1}^{m} S_i(x_i-y_i,x_i-y_i) + \sum_{1\le i<j\le m} M_{ij}(x_i-y_i,x_j-y_j) \\ &\quad = \sum_{i=1}^{m} \left[S_i(x_i,x_i) + 2S_i(x_i,y_i) + S_i(y_i,y_i)\right] \\ &\qquad + \sum_{1\le i<j\le m} \left[M_{ij}(x_i,x_j) + M_{ij}(x_i,y_j) + M_{ij}(y_i,x_j) + M_{ij}(y_i,y_j)\right] \\ &\qquad + \sum_{i=1}^{m} \left[S_i(x_i,x_i) - 2S_i(x_i,y_i) + S_i(y_i,y_i)\right] \\ &\qquad + \sum_{1\le i<j\le m} \left[M_{ij}(x_i,x_j) - M_{ij}(x_i,y_j) - M_{ij}(y_i,x_j) + M_{ij}(y_i,y_j)\right] \\ &\quad = 2\left[\sum_{i=1}^{m} S_i(x_i,x_i) + \sum_{1\le i<j\le m} M_{ij}(x_i,x_j)\right] \\ &\qquad + 2\left[\sum_{i=1}^{m} S_i(y_i,y_i) + \sum_{1\le i<j\le m} M_{ij}(y_i,y_j)\right] \\ &\quad = 2f(x_1,\dots,x_m) + 2f(y_1,\dots,y_m) \end{aligned} \tag{2.16}$$

for all $x_1,\dots,x_m,y_1,\dots,y_m \in X$. □

Let Y be complete and let $\varphi : X^{2m} \to [0,\infty)$ be a function satisfying

$$\widetilde{\varphi}(x_1,\dots,x_m,y_1,\dots,y_m) := \sum_{j=0}^{\infty} \frac{1}{4^{j+1}} \varphi(2^j x_1,\dots,2^j x_m, 2^j y_1,\dots,2^j y_m) < \infty \tag{2.17}$$

for all $x_1,\dots,x_m,y_1,\dots,y_m \in X$.

Theorem 2.7. *Let $f : X^m \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x_1+y_1,\dots,x_m+y_m)+f(x_1-y_1,\dots,x_m-y_m) \\ &\quad -2f(x_1,\dots,x_m)-2f(y_1,\dots,y_m)\| \le \varphi(x_1,\dots,x_m,y_1,\dots,y_m) \end{aligned} \tag{2.18}$$

for all $x_1,\dots,x_m,y_1,\dots,y_m \in X$. Then, there exists a unique m-variable quadratic mapping $F : X^m \to Y$ such that

$$\|f(x_1,\dots,x_m)-F(x_1,\dots,x_m)\| \le \tilde{\varphi}(x_1,\dots,x_m,x_1,\dots,x_m) \tag{2.19}$$

for all $x_1,\dots,x_m \in X$. The mapping F is given by

$$F(x_1,\dots,x_m) := \lim_{j\to\infty}\frac{1}{4^j}f(2^jx_1,\dots,2^jx_m) \tag{2.20}$$

for all $x_1,\dots,x_m \in X$.

Proof. Letting $y_1 = x_1,\dots,y_m = x_m$ in (2.18), we have

$$\left\|f(x_1,\dots,x_m)-\frac{1}{4}[f(0,\dots,0)+f(2x_1,\dots,2x_m)]\right\| \le \frac{1}{4}\varphi(x_1,\dots,x_m,x_1,\dots,x_m) \tag{2.21}$$

for all $x_1,\dots,x_m \in X$. Thus, we obtain

$$\begin{aligned} &\left\|\frac{1}{4^j}f(2^jx_1,\dots,2^jx_m)-\frac{1}{4^{j+1}}[f(0,\dots,0)+f(2^{j+1}x_1,\dots,2^{j+1}x_m)]\right\| \\ &\quad \le \frac{1}{4^{j+1}}\varphi(2^jx_1,\dots,2^jx_m,2^jx_1,\dots,2^jx_m) \end{aligned} \tag{2.22}$$

for all $x_1,\dots,x_m \in X$ and all j. For given integers l, n $(0 \le l < n)$, we get

$$\begin{aligned} &\left\|\frac{1}{4^l}f(2^lx_1,\dots,2^lx_m)-\frac{1}{4^n}[f(0,\dots,0)+f(2^nx_1,\dots,2^nx_m)]\right\| \\ &\qquad \sum_{j=l}^{n-1}\frac{1}{4^{j+1}}\varphi(2^jx_1,\dots,2^jx_m,2^jx_1,\dots,2^jx_m) \end{aligned} \tag{2.23}$$

for all $x_1,\dots,x_m \in X$. By (2.23), the sequence $\{(1/4^j)f(2^jx_1,\dots,2^jx_m)\}$ is a Cauchy sequence for all $x_1,\dots,x_m \in X$. Since Y is complete, the sequence $\{(1/4^j)f(2^jx_1,\dots,2^jx_m)\}$ converges for all $x_1,\dots,x_m \in X$. Define $F : X^m \to Y$ by

$$F(x_1,\dots,x_m) := \lim_{j\to\infty}\frac{1}{4^j}f(2^jx_1,\dots,2^jx_m) \tag{2.24}$$

for all $x_1,\ldots,x_m \in X$. By (2.18), we have

$$\left\|\frac{1}{4^j} f\left(2^j(x_1+y_1),\ldots,2^j(x_m+y_m)\right) + \frac{1}{4^j} f\left(2^j(x_1-y_1),\ldots,2^j(x_m-y_m)\right) - \frac{2}{4^j} f\left(2^j x_1,\ldots,2^j x_m\right) - \frac{2}{4^j} f\left(2^j y_1,\ldots,2^j y_m\right)\right\| \le \frac{1}{4^j}\varphi\left(2^j x_1,\ldots,2^j x_m, 2^j y_1,\ldots,2^j y_m\right) \tag{2.25}$$

for all $x_1,\ldots,x_m,y_1,\ldots,y_m \in X$ and all j. Letting $j \to \infty$ and using (2.17), we see that F satisfies (1.2). Setting $l=0$ and taking $n \to \infty$ in (2.23), one can obtain the inequality (2.19). If $G: X^m \to Y$ is another m-variable quadratic mapping satisfying (2.19), we obtain

$$\begin{aligned} &\|F(x_1,\ldots,x_m) - G(x_1,\ldots,x_m)\| \\ &\quad = \frac{1}{4^n}\|F(2^n x_1,\ldots,2^n x_m) - G(2^n x_1,\ldots,2^n x_m)\| \\ &\quad \le \frac{1}{4^n}\|F(2^n x_1,\ldots,2^n x_m) - f(2^n x_1,\ldots,2^n x_m)\| \\ &\qquad + \frac{1}{4^n}\|f(2^n x_1,\ldots,2^n x_m) - G(2^n x_1,\ldots,2^n x_m)\| \\ &\quad \le \frac{2}{4^n}\tilde{\varphi}(2^n x_1,\ldots,2^n x_m, 2^n x_1,\ldots,2^n x_m) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \end{aligned} \tag{2.26}$$

for all $x_1,\ldots,x_m \in X$. Hence, the mapping F is the unique m-variable quadratic mapping, as desired. □

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