

Research Article

Perturbed Iterative Algorithms for Generalized Nonlinear Set-Valued Quasivariational Inclusions Involving Generalized m -Accretive Mappings

Mao-Ming Jin

Received 24 August 2006; Revised 10 January 2007; Accepted 14 January 2007

Recommended by H. Bevan Thompson

A new class of generalized nonlinear set-valued quasivariational inclusions involving generalized m -accretive mappings in Banach spaces are studied, which included many variational inclusions studied by others in recent years. By using the properties of the resolvent operator associated with generalized m -accretive mappings, we established the equivalence between the generalized nonlinear set-valued quasi-variational inclusions and the fixed point problems, and some new perturbed iterative algorithms, proved that its proximate solution converges strongly to its exact solution in real Banach spaces.

Copyright © 2007 Mao-Ming Jin. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In 1994, Hassouni and Moudafi [1] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. Since then, Adly [2], Ding [3], Ding and Luo [4], Huang [5, 6], Huang et al. [7], Ahmad and Ansari [8] have obtained some important extensions of the results in various different assumptions. For more details, we refer to [1–29] and the references therein.

In 2001, Huang and Fang [16] were the first to introduce the generalized m -accretive mapping and give the definition of the resolvent operator for the generalized m -accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for the generalized m -accretive mappings in Banach spaces. For further works, we refer to Huang [15], Huang et al. [19] and Huang et al. [20].

Recently, Huang and Fang [17] introduced a new class of maximal η -monotone mapping in Hilbert spaces, which is a generalization of the classical maximal monotone mapping, and studied the properties of the resolvent operator associated with the maximal

η -monotone mapping. They also introduced and studied a new class of nonlinear variational inclusions involving maximal η -monotone mapping in Hilbert spaces.

Motivated and inspired by the research work going on in this field, we introduce and study a new class of generalized nonlinear set-valued quasivariational inclusions involving generalized m -accretive mappings in Banach spaces, which include many variational inclusions studied by others in recent years. By using the properties of the resolvent operator associated with generalized m -accretive mappings, we establish the equivalence between the generalized nonlinear set-valued quasivariational inclusions and the fixed point problems, and some new perturbed iterative algorithms, prove that its proximate solution converges to its exact solution in real Banach spaces. The results presented in this paper extend and improve the corresponding results in the literature.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space equipped with norm $\| \cdot \|$, X^* is the topological dual space of X , $CB(X)$ is the family of all nonempty closed and bounded subset of X , 2^X is a power set of X , $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$ defined by

$$D(A, B) = \max \left\{ \sup_{u \in A} d(u, B), \sup_{v \in B} d(A, v) \right\} \quad \forall A, B \in CB(X), \tag{2.1}$$

where $d(u, B) = \inf_{v \in B} d(u, v)$ and $d(A, v) = \inf_{u \in A} d(u, v)$.

Suppose that $\langle \cdot, \cdot \rangle$ is the dual pair between X and X^* , $J : X \rightarrow 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) = \{ f \in X^* : \langle x, f \rangle = \|x\|^2, \|x\| = \|f\| \}, \quad x \in X, \tag{2.2}$$

and j is a selection of normalized duality mapping J .

Definition 2.1. A single-valued mapping $g : X \rightarrow X$ is said to be k -strongly accretive if there exists $k > 0$ such that for any $x, y \in X$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle g(x) - g(y), j(x - y) \rangle \geq k \|x - y\|^2. \tag{2.3}$$

Definition 2.2. A single-valued mapping $N : X \times X \rightarrow X$ is said to be γ -Lipschitz continuous with respect to the first argument if there exists a constant $\gamma > 0$ such that

$$\|N(x, \cdot) - N(y, \cdot)\| \leq \gamma \|x - y\| \quad \forall x, y \in X. \tag{2.4}$$

In a similar way, we can define Lipschitz continuity of $N(\cdot, \cdot)$ with respect to the second argument.

Definition 2.3. A set-valued mapping $S : X \rightarrow 2^X$ is said to be ξ - D -Lipschitz continuous if there exists $\xi > 0$ such that

$$D(S(x), S(y)) \leq \xi \|x - y\| \quad \forall x, y \in X. \tag{2.5}$$

Definition 2.4. A mapping $\eta : X \times X \rightarrow X^*$ is said to be

(i) accretive if for any $x, y \in X$,

$$\langle x - y, \eta(x, y) \rangle \geq 0; \quad (2.6)$$

(ii) strictly accretive if for any $x, y \in X$,

$$\langle x - y, \eta(x, y) \rangle \geq 0, \quad (2.7)$$

and equality holds if and only if $x = y$;

(iii) α -strongly accretive if there exists a constant $\alpha > 0$ such that

$$\langle x - y, \eta(x, y) \rangle \geq \alpha \|x - y\|^2 \quad \forall x, y \in X; \quad (2.8)$$

(iv) β -Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|\eta(x, y)\| \leq \beta \|x - y\| \quad \forall x, y \in X. \quad (2.9)$$

Definition 2.5 [16]. Let $\eta : X \times X \rightarrow X^*$ be a single-valued mapping. A set-valued mapping $M : X \rightarrow 2^X$ is said to be

(i) η -accretive if for any $x, y \in X$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad u \in M(x), v \in M(y); \quad (2.10)$$

(ii) strictly η -accretive if for any $x, y \in X$,

$$\langle u - v, \eta(x, y) \rangle \geq 0, \quad u \in M(x), v \in M(y), \quad (2.11)$$

and equality holds if and only if $x = y$;

(iii) μ -strongly η -accretive if there exists a constant $\mu > 0$ such that

$$\langle u - v, \eta(x, y) \rangle \geq \mu \|x - y\|^2 \quad \forall x, y \in X, u \in M(x), v \in M(y); \quad (2.12)$$

(iv) generalized m -accretive if M is η -accretive and $(I + \rho M)(X) = X$ for any $\rho > 0$, where I is the identity mapping.

Remark 2.6. If X is a smooth Banach space, $\eta(x, y) = J(x - y)$ for all x, y in X , then Definition 2.5 reduces to the usual definitions of accretiveness of the set-valued mapping M in smooth Banach spaces.

LEMMA 2.7 [30]. *Let X be a real Banach space and let $J : X \rightarrow 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall j(x + y) \in J(x + y). \quad (2.13)$$

LEMMA 2.8 [16]. Let $\eta : X \times X \rightarrow X$ be a strictly accretive mapping and let $M : X \rightarrow 2^X$ be a generalized m -accretive mapping. Then the following conclusions hold:

- (1) $\langle x - y, \eta(u, v) \rangle \geq 0 \quad \forall (y, v) \in \text{graph}(M)$ implies $(x, u) \in \text{graph}(M)$, where $\text{graph}(M) = \{(x, u) \in X \times X : x \in M(u)\}$;
- (2) the mapping $(I + \rho M)^{-1}$ is single-valued for any $\rho > 0$.

Based on Lemma 2.8, we can define the resolvent operator for a generalized m -accretive mapping M as follows:

$$J_\rho^M(z) = (I + \rho M)^{-1}(z) \quad \forall z \in X, \tag{2.14}$$

where $\rho > 0$ is a constant and $\eta : X \times X \rightarrow X^*$ is a strictly accretive mapping.

LEMMA 2.9 [16]. Let $\eta : X \times X \rightarrow X^*$ be a δ -strongly accretive and τ -Lipschitz continuous mapping. Let $M : X \rightarrow 2^X$ be a generalized m -accretive mapping. Then the resolvent operator J_ρ^M for M is τ/δ -Lipschitz continuous, that is,

$$\|J_\rho^M(u) - J_\rho^M(v)\| \leq \frac{\tau}{\delta} \|u - v\| \quad \forall u, v \in X. \tag{2.15}$$

3. Variational inclusions

In this section, by using the resolvent operator for the generalized m -accretive mapping and the results obtained in Section 2, we introduce and study a new class of generalized nonlinear set-valued quasivariational inclusion problem involving generalized m -accretive mappings, and prove that its proximate solution converges strongly to its exact solution in real Banach spaces.

Let $S, T, G : X \rightarrow CB(X)$ and $M(\cdot, \cdot) : X \times X \rightarrow 2^X$ be set-valued mappings such that for any given $t \in X, M(t, \cdot) : X \rightarrow 2^X$ is a generalized m -accretive mapping. Let $g : X \rightarrow X$ and $N(\cdot, \cdot) : X \times X \rightarrow X$ be nonlinear mappings. For any $f \in X$, we consider the following problem.

Find $x \in X, w \in S(x), y \in T(x), z \in G(x)$ such that

$$f \in N(w, y) + M(z, g(x)), \tag{3.1}$$

which is called the generalized nonlinear set-valued quasivariational inclusion problem involving generalized m -accretive mappings.

Some special cases of problem (3.1) are as follows.

- (I) If $S, T, G : X \rightarrow X$ is a single-valued mapping, then problem (3.1) reduced to finding $x \in X$ such that

$$f \in N(S(x), T(x)) + M(G(x), g(x)), \tag{3.2}$$

which is called the nonlinear quasivariational inclusion problem.

- (II) If $X = H$ is a Hilbert space and $\eta(u, v) = u - v$, then problem (3.1) becomes the usual nonlinear quasivariational inclusion with a maximal monotone mapping M .

Remark 3.1. For a suitable choice of S, T, G, N, M, g, f , and the space X , a number of known classes of variational inequalities (inclusion) and quasivariational inequalities

(inclusion) can be obtained as special cases of generalized nonlinear set-valued quasivariational inclusion (3.1).

LEMMA 3.2. *Problem (3.1) has a solution (x, w, y, z) , where $x \in X, w \in S(x), y \in T(x), z \in G(x)$ if and only if (p, x, w, y, z) , where $p \in X$, is a solution of implicit resolvent equation*

$$p = g(x) - \rho(N(w, y) - f), \quad g(x) = J_\rho^{M(z, \cdot)}(p), \quad (3.3)$$

where $J_\rho^{M(z, \cdot)} = (I + \rho M(z, \cdot))^{-1}$ and $\rho > 0$ is a constant.

Proof. This directly follows from the definition of $J_\rho^{M(z, \cdot)}$. □

Now Lemma 3.2 and Nadler's theorem [31] allow us to suggest the following iterative algorithm.

ALGORITHM 3.3. *Assume that $S, T, G : X \rightarrow CB(X)$, and $M(\cdot, \cdot) : X \times X \rightarrow 2^X$ are set-valued mappings such that for any given $t \in X$, $M(t, \cdot) : X \rightarrow 2^X$ is a generalized m -accretive mapping and $g : X \rightarrow X$ is a strongly accretive and Lipschitz continuous mapping. Let $N(\cdot, \cdot) : X \times X \rightarrow X$ be a nonlinear mapping. For any $f \in X$ and for given $p_0 \in X, x_0 \in X$ and $w_0 \in S(x_0), y_0 \in T(x_0), z_0 \in G(x_0)$, compute the sequences $\{p_n\}, \{x_n\}, \{w_n\}, \{y_n\}$, and $\{z_n\}$ defined by the iterative schemes*

$$\begin{aligned} g(x_n) &= J_\rho^{M(z_n, \cdot)} p_n, \\ w_n \in S(x_n), \quad \|w_n - w_{n+1}\| &\leq (1 + (1+n)^{-1})D(S(x_n), S(x_{n+1})), \\ y_n \in T(x_n), \quad \|y_n - y_{n+1}\| &\leq (1 + (1+n)^{-1})D(T(x_n), T(x_{n+1})), \quad n = 0, 1, 2, \dots, \\ z_n \in G(x_n), \quad \|z_n - z_{n+1}\| &\leq (1 + (1+n)^{-1})D(G(x_n), G(x_{n+1})), \\ p_{n+1} &= (1 - \lambda)p_n + \lambda(g(x_n) - \rho N(w_n, y_n) + \rho f) + \lambda e_n, \end{aligned} \quad (3.4)$$

where $0 < \lambda \leq 1$ is a constant and $e_n \in X$ is the errors while considering the approximation in computation.

If $S, T, G : X \rightarrow X$ are single-valued mappings, then Algorithm 3.3 can be degenerated to the following algorithm for problem (3.2).

ALGORITHM 3.4. *For any $f \in X$ and for given $p_0 \in X, x_0 \in X$, we can obtain sequences $\{p_n\}, \{x_n\}$ satisfying*

$$\begin{aligned} g(x_n) &= J_\rho^{M(G(x_n), \cdot)} p_n, \\ p_{n+1} &= (1 - \lambda)p_n + \lambda(g(x_n) - \rho N(S(x_n), T(x_n)) + \rho f) + \lambda e_n, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (3.5)$$

where $0 < \lambda \leq 1$ is a constant and $e_n \in X$ is the errors while considering the approximation in computation.

Remark 3.5. If we choose suitable S, T, G, N, M, g , and the space X , then Algorithm 3.3 can be degenerated to a number of algorithm for solving variational inequalities (inclusions).

THEOREM 3.6. *Let X be a real Banach space. Let $\eta : X \times X \rightarrow X^*$ be δ -strongly accretive and τ -Lipschitz continuous, let $S, T, G : X \rightarrow CB(X)$ be α, β and γ -D-Lipschitz continuous, respectively, let $g : X \rightarrow X$ be k -strongly accretive and ξ -Lipschitz continuous. Let $N(\cdot, \cdot) : X \times X \rightarrow X$ be r, t -Lipschitz continuous with respect to the first and second arguments, respectively. Let $M : X \times X \rightarrow 2^X$ be such that for each fixed $t \in X$, $M(t, \cdot)$ is a generalized m -accretive mapping. Suppose that there exist constants $\rho > 0$ and $\mu > 0$ such that for each $x, y, z \in X$,*

$$\|J_\rho^{M(x, \cdot)}(z) - J_\rho^{M(y, \cdot)}(z)\| \leq \mu \|x - y\|, \tag{3.6}$$

$$\rho < \frac{\sqrt{k + 1.5 - \mu^2 \gamma^2} - b\xi}{b(r\alpha + t\beta)}, \quad b\xi < \sqrt{k + 1.5 - \mu^2 \gamma^2}, \quad b = \frac{\tau}{\delta}, \tag{3.7}$$

$$\lim_{n \rightarrow \infty} \|e_n\| = 0, \quad \sum_{n=0}^{\infty} \|e_{n+1} - e_n\| < \infty. \tag{3.8}$$

Then there exist $p, x \in X, w \in S(x), y \in T(x), z \in G(x)$ satisfy the implicit resolvent equation (3.3) and the iterative sequences $\{p_n\}, \{x_n\}, \{w_n\}, \{y_n\}$, and $\{z_n\}$ generated by Algorithm 3.3 converge strongly to p, x, w, y , and z in X , respectively.

Proof. From condition (3.6), Lemma 2.9, and γ -Lipschitz continuity of G , we have

$$\begin{aligned} & \|J_\rho^{M(z_{n+1}, \cdot)} p_{n+1} - J_\rho^{M(z_n, \cdot)} p_n\| \\ & \leq \|J_\rho^{M(z_{n+1}, \cdot)} p_{n+1} - J_\rho^{M(z_n, \cdot)} p_{n+1}\| + \|J_\rho^{M(z_n, \cdot)} p_{n+1} - J_\rho^{M(z_n, \cdot)} p_n\| \\ & \leq \mu \|z_{n+1} - z_n\| + \frac{\tau}{\delta} \|p_{n+1} - p_n\| \\ & \leq \mu \gamma \left(1 + \frac{1}{n}\right) \|x_{n+1} - x_n\| + \frac{\tau}{\delta} \|p_{n+1} - p_n\|. \end{aligned} \tag{3.9}$$

Since g is k -strongly accretive mapping, from Algorithm 3.3, Lemma 2.7, and (3.9), for any $j(x_{n+1} - x_n) \in J(x_{n+1} - x_n)$, we have

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_n + (g(x_{n+1}) - g(x_n)) - (J_\rho^{M(z_{n+1}, \cdot)} p_{n+1} - J_\rho^{M(z_n, \cdot)} p_n)\|^2 \\ &\leq \|J_\rho^{M(z_{n+1}, \cdot)} p_{n+1} - J_\rho^{M(z_n, \cdot)} p_n\|^2 - 2 \langle g(x_{n+1}) - g(x_n) + x_{n+1} - x_n, j(x_{n+1} - x_n) \rangle \\ &\leq \left(\mu \gamma \left(1 + \frac{1}{n}\right) \|x_{n+1} - x_n\| + \frac{\tau}{\delta} \|p_{n+1} - p_n\| \right)^2 \\ &\quad - 2 \langle g(x_{n+1}) - g(x_n), j(x_{n+1} - x_n) \rangle - 2 \langle x_{n+1} - x_n, j(x_{n+1} - x_n) \rangle \\ &\leq \left(2\mu^2 \gamma^2 \left(1 + \frac{1}{n}\right)^2 - 2k - 2 \right) \|x_{n+1} - x_n\|^2 + 2 \frac{\tau^2}{\delta^2} \|p_{n+1} - p_n\|^2, \end{aligned} \tag{3.10}$$

which implies

$$\|x_{n+1} - x_n\| \leq \frac{b}{\sqrt{k + 1.5 - \mu^2 \gamma^2 (1 + (1/n))^2}} \|p_{n+1} - p_n\|, \quad (3.11)$$

where $b = \tau/\delta$.

Since N is r, t -Lipschitz continuous with respect to the first, second arguments, respectively, S, T are α, β -Lipschitz continuous, respectively, and g is ξ -Lipschitz continuous, by (3.4), we obtain

$$\begin{aligned} \|p_{n+2} - p_{n+1}\| &= \|(1 - \lambda)p_{n+1} + \lambda[g(x_{n+1}) - \rho N(w_{n+1}, y_{n+1}) + \rho f] \\ &\quad + \lambda e_{n+1} - \{(1 - \lambda)p_n + \lambda[g(x_n) - \rho N(w_n, y_n) + \rho f] + \lambda e_n\}| \\ &\leq (1 - \lambda)\|p_{n+1} - p_n\| + \lambda\|g(x_{n+1}) - g(x_n)\| \\ &\quad + \lambda\rho(\|N(w_{n+1}, y_{n+1}) - N(w_n, y_{n+1})\| + \|N(w_n, y_{n+1}) - N(w_n, y_n)\|) \\ &\quad + \lambda\|e_{n+1} - e_n\| \leq (1 - \lambda)\|p_{n+1} - p_n\| \\ &\quad + \lambda\left[\xi + \rho\left(1 + \frac{1}{n}\right)(r\alpha + t\beta)\right]\|x_{n+1} - x_n\| + \lambda\|e_{n+1} - e_n\|. \end{aligned} \quad (3.12)$$

It follows from (3.11) and (3.12) that

$$\begin{aligned} \|p_{n+2} - p_{n+1}\| &\leq \left(1 - \lambda + \frac{\lambda b[\xi + \rho(1 + (1/n))(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2 \gamma^2 (1 + (1/n))^2}}\right)\|p_{n+1} - p_n\| + \lambda\|e_{n+1} - e_n\| \\ &= [1 - \lambda(1 - h_n)]\|p_{n+1} - p_n\| + \lambda\|e_{n+1} - e_n\| \\ &= \theta_n\|p_{n+1} - p_n\| + \lambda\|e_{n+1} - e_n\|, \end{aligned} \quad (3.13)$$

where

$$\theta_n = 1 - \lambda(1 - h_n), \quad h_n = \frac{b[\xi + \rho(1 + (1/n))(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2 \gamma^2 (1 + (1/n))^2}}. \quad (3.14)$$

Letting

$$\theta = 1 - \lambda(1 - h), \quad h = \frac{b[\xi + \rho(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2 \gamma^2}}, \quad (3.15)$$

we know that $h_n \rightarrow h$ and $\theta_n \rightarrow \theta$ as $n \rightarrow \infty$. It follows from (3.7) and $0 < \lambda \leq 1$ that $0 < h < 1$ and $0 < \theta < 1$, and so there exists a positive number $\theta_* \in (0, 1)$ such that $\theta_n < \theta_*$ for

all $n \geq N$. Therefore, for all $n \geq N$, by (3.13), we now know that

$$\begin{aligned} \|p_{n+2} - p_{n+1}\| &\leq \theta_* \|p_{n+1} - p_n\| + \lambda \|e_{n+1} - e_n\| \\ &\leq \theta_* (\theta_* \|p_n - p_{n-1}\| + \lambda \|e_n - e_{n-1}\|) + \lambda \|e_{n+1} - e_n\| \\ &= \theta_*^2 \|p_n - p_{n-1}\| + \lambda \theta_* \|e_n - e_{n-1}\| + \lambda \|e_{n+1} - e_n\| \\ &\leq \dots \leq \theta_*^{n+1-N} \|p_{N+1} - p_N\| + \sum_{i=1}^{n+1-N} \theta_*^{i-1} \lambda \|e_{n+1-(i-1)} - e_{n+1-i}\|, \end{aligned} \tag{3.16}$$

which implies that for any $m > n > N$, we have

$$\begin{aligned} \|p_m - p_n\| &\leq \sum_{j=n}^{m-1} \|p_{j+1} - p_j\| \\ &\leq \sum_{j=n}^{m-1} \theta_*^{j+1-N} \|p_{N+1} - p_N\| + \sum_{j=n}^{m-1} \sum_{i=1}^{j+1-N} \theta_*^{i-1} \lambda \|e_{n+1-(i-1)} - e_{n+1-i}\|. \end{aligned} \tag{3.17}$$

Since $0 < \lambda \leq 1$ and $\theta_* \in (0, 1)$, it follows from (3.8) and (3.17) that $\lim_{m,n \rightarrow \infty} \|p_m - p_n\| = 0$, and hence $\{p_n\}$ is a Cauchy sequence in X . Let $p_n \rightarrow p$ as $n \rightarrow \infty$. From (3.11), we know that sequence $\{x_n\}$ is also a Cauchy sequence in X . Let $x_n \rightarrow x$ as $n \rightarrow \infty$.

On the other hand, from the Lipschitzian continuity of S, T, G , and Algorithm 3.3, we have

$$\begin{aligned} \|w_n - w_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) D(S(x_n), S(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \alpha \|x_n - x_{n+1}\|, \\ \|y_n - y_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \beta \|x_n - x_{n+1}\|, \\ \|z_n - z_{n+1}\| &\leq \left(1 + \frac{1}{n+1}\right) D(G(x_n), G(x_{n+1})) \leq \left(1 + \frac{1}{n+1}\right) \gamma \|x_n - x_{n+1}\|. \end{aligned} \tag{3.18}$$

Since $\{x_n\}$ is a Cauchy sequence, from (3.18), we know that $\{w_n\}, \{y_n\}$, and $\{z_n\}$ are also Cauchy sequences. Let $w_n \rightarrow w, y_n \rightarrow y$, and $z_n \rightarrow z$ as $n \rightarrow \infty$. From Algorithm 3.3,

$$p_{n+1} = (1 - \lambda)p_n + \lambda(g(x_n) - \rho N(w_n, y_n) + \rho f) + \lambda e_n. \tag{3.19}$$

By the assumptions and $\lim_{n \rightarrow \infty} \|e_n\| = 0$, we have

$$\begin{aligned} p &= g(x) - \rho(N(w, y) - f), \\ g(x_n) &= J_\rho^{M(z_n, \cdot)} p_n \implies g(x) = J_\rho^{M(z, \cdot)} p. \end{aligned} \tag{3.20}$$

From (3.20), we have p, x, w, y, z satisfy the implicit resolvent equation (3.3).

Now we will prove that $w \in S(x)$, $y \in T(x)$, and $z \in G(x)$. In fact, since $w_n \in S(x_n)$ and

$$\begin{aligned} d(w_n, S(x)) &\leq \max \left\{ d(w_n, S(x)), \sup_{v \in S(x)} d(S(x_n), v) \right\} \\ &\leq \max \left\{ \sup_{u \in S(x_n)} (u, S(x)), \sup_{v \in S(x)} d(S(x_n), v) \right\} \\ &= D(S(x_n), S(x)), \end{aligned} \quad (3.21)$$

we have

$$\begin{aligned} d(w, S(x)) &\leq \|w - w_n\| + d(w_n, S(x)) \leq \|w - w_n\| + D(S(x_n), S(x)) \\ &\leq \|w - w_n\| + \gamma \|x_n - x\| \rightarrow 0. \end{aligned} \quad (3.22)$$

This implies that $w \in S(x)$. Similarly, we know that $y \in T(x)$ and $z \in G(x)$. This completes the proof. \square

If $S, T, G: X \rightarrow X$ are single-valued mappings, then Theorem 3.6 can be degenerated to the following theorem.

THEOREM 3.7. *Let $X, g, \eta, N(\cdot, \cdot), M(\cdot, \cdot)$ be the same as in Theorem 3.6, and let $S, T, G: X \rightarrow X$ be α, β, γ -Lipschitz continuous single-valued mappings, respectively. If conditions (3.6)–(3.8) hold, then the sequences $\{x_n\}$ generated by Algorithm 3.4 converges strongly to the unique solution x of problem (3.2).*

Proof. By Theorem 3.6, problem (3.2) has a solution $x \in X$ and $x_n \rightarrow x$ as $n \rightarrow \infty$. Now we prove that x is a unique solution of problem (3.2). Let $x^* \in X$ be another solution of problem (3.2). Then

$$g(x^*) = J_\rho^{M(G(x^*), \cdot)} m(x^*), \quad m(x^*) = g(x^*) - \rho(N(S(x^*), T(x^*)) - f). \quad (3.23)$$

We have

$$\begin{aligned} \|x - x^*\|^2 &= \|x - x^* + (g(x) - g(x^*)) - (J_\rho^{M(G(x), \cdot)} m(x) - J_\rho^{M(G(x^*), \cdot)} m(x^*))\|^2 \\ &\leq \|J_\rho^{M(G(x), \cdot)} m(x) - J_\rho^{M(G(x^*), \cdot)} m(x^*)\|^2 - 2\langle g(x) - g(x^*) + x - x^*, j(x - x^*) \rangle \\ &\leq (\|J_\rho^{M(G(x), \cdot)} m(x) - J_\rho^{M(G(x^*), \cdot)} m(x)\| + \|J_\rho^{M(G(x^*), \cdot)} m(x) - J_\rho^{M(G(x^*), \cdot)} m(x^*)\|)^2 \\ &\quad - 2\langle g(x) - g(x^*), j(x - x^*) \rangle - 2\langle x - x^*, j(x - x^*) \rangle \\ &\leq \left(\mu \|G(x) - G(x^*)\| + \frac{\tau}{\delta} \|m(x) - m(x^*)\| \right)^2 - 2(k+1)\|x - x^*\|^2 \\ &\leq 2(\mu^2 \gamma^2 - k - 1)\|x - x^*\|^2 + 2\frac{\tau^2}{\delta^2} \|m(x) - m(x^*)\|^2. \end{aligned} \quad (3.24)$$

This implies that

$$\|x - x^*\| \leq \frac{b}{\sqrt{k + 1.5 - \mu^2 \gamma^2}} \|m(x) - m(x^*)\|, \quad (3.25)$$

where $b = \tau/\delta$. Furthermore,

$$\begin{aligned} \|m(x) - m(x^*)\| &= \|g(x) - g(x^*) - \rho(N(S(x), T(x)) - N(S(x^*), T(x^*)))\| \\ &\leq \|g(x) - g(x^*)\| + \rho(\|N(S(x), T(x)) - N(S(x^*), T(x))\| \\ &\quad + \|N(S(x^*), T(x)) - N(S(x^*), T(x^*))\|) \\ &\leq [\xi + \rho(r\alpha + t\beta)] \|x - x^*\|. \end{aligned} \quad (3.26)$$

Combining (3.25) and (3.26), we have

$$\|x - x^*\| \leq \frac{b[\xi + \rho(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2 \gamma^2}} \|x - x^*\| = h \|x - x^*\|, \quad (3.27)$$

where

$$h = \frac{b[\xi + \rho(r\alpha + t\beta)]}{\sqrt{k + 1.5 - \mu^2 \gamma^2}}. \quad (3.28)$$

It follows from (3.7) that $0 < h < 1$ and so $x = x^*$. This completes the proof. \square

Acknowledgments

The author would like to thank the referees for their valuable comments and suggestions leading to the improvements of this paper. This work was supported by the National Natural Science Foundation of China (10471151) and the Educational Science Foundation of Chongqing, Chongqing, China.

References

- [1] A. Hassouni and A. Moudafi, "A perturbed algorithm for variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 185, no. 3, pp. 706–712, 1994.
- [2] S. Adly, "Perturbed algorithms and sensitivity analysis for a general class of variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 2, pp. 609–630, 1996.
- [3] X. P. Ding, "Perturbed proximal point algorithms for generalized quasivariational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 1, pp. 88–101, 1997.
- [4] X. P. Ding and C. L. Luo, "Perturbed proximal point algorithms for general quasi-variational-like inclusions," *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 153–165, 2000.
- [5] N.-J. Huang, "On the generalized implicit quasivariational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 197–210, 1997.
- [6] N.-J. Huang, "A new completely general class of variational inclusions with noncompact valued mappings," *Computers & Mathematics with Applications*, vol. 35, no. 10, pp. 9–14, 1998.

- [7] N.-J. Huang, M.-R. Bai, Y. J. Cho, and S. M. Kang, "Generalized nonlinear mixed quasi-variational inequalities," *Computers & Mathematics with Applications*, vol. 40, no. 2-3, pp. 205–215, 2000.
- [8] R. Ahmad and Q. H. Ansari, "An iterative algorithm for generalized nonlinear variational inclusions," *Applied Mathematics Letters*, vol. 13, no. 5, pp. 23–26, 2000.
- [9] R. Ahmad, A. H. Siddiqi, and Z. Khan, "Proximal point algorithm for generalized multivalued nonlinear quasi-variational-like inclusions in Banach spaces," *Applied Mathematics and Computation*, vol. 163, no. 1, pp. 295–308, 2005.
- [10] S. S. Chang, Y. J. Cho, B. S. Lee, and I. H. Jung, "Generalized set-valued variational inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 409–422, 2000.
- [11] C. E. Chidume, K. R. Kazmi, and H. Zegeye, "Iterative approximation of a solution of a general variational-like inclusion in Banach spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2004, no. 22, pp. 1159–1168, 2004.
- [12] Y.-P. Fang and N.-J. Huang, " H -accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 647–653, 2004.
- [13] F. Giannessi and A. Maugeri, Eds., *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, NY, USA, 1995.
- [14] X. He, "On ϕ -strongly accretive mappings and some set-valued variational problems," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 504–511, 2003.
- [15] N.-J. Huang, "Nonlinear implicit quasi-variational inclusions involving generalized m -accretive mappings," *Archives of Inequalities and Applications*, vol. 2, no. 4, pp. 413–425, 2004.
- [16] N.-J. Huang and Y.-P. Fang, "Generalized m -accretive mappings in Banach spaces," *Journal of Sichuan University*, vol. 38, no. 4, pp. 591–592, 2001.
- [17] N.-J. Huang and Y.-P. Fang, "A new class of general variational inclusions involving maximal η -monotone mappings," *Publicationes Mathematicae Debrecen*, vol. 62, no. 1-2, pp. 83–98, 2003.
- [18] N.-J. Huang and Y.-P. Fang, "Iterative processes with errors for nonlinear set-valued variational inclusions involving accretive type mappings," *Computers & Mathematics with Applications*, vol. 47, no. 4-5, pp. 727–738, 2004.
- [19] N.-J. Huang, Y.-P. Fang, and Y. J. Cho, "Generalized m -accretive mappings and variational inclusions in Banach spaces," *Journal of Concrete and Applicable Mathematics*, vol. 3, no. 1, pp. 31–40, 2005.
- [20] N.-J. Huang, Y.-P. Fang, and C. X. Deng, "A new class of generalized nonlinear variational inclusions in Banach spaces," in *Proceedings of International Conference on Mathematical Programming*, M. Y. Yue, J. Y. Han, L. S. Zhang, and S. Z. Zhang, Eds., pp. 207–214, Shanghai University Press, Shanghai, China, 2004.
- [21] M.-M. Jin, "A new proximal point algorithm with errors for nonlinear variational inclusions involving generalized m -accretive mappings," *Nonlinear Analysis Forum*, vol. 9, no. 1, pp. 87–96, 2004.
- [22] M.-M. Jin and Q.-K. Liu, "Nonlinear quasi-variational inclusions involving generalized m -accretive mappings," *Nonlinear Functional Analysis and Applications*, vol. 9, no. 3, pp. 485–494, 2004.
- [23] M.-M. Jin, "Generalized nonlinear implicit quasivariational inclusions with relaxed monotone mappings," *Advances in Nonlinear Variational Inequalities*, vol. 7, no. 2, pp. 173–181, 2004.
- [24] K. R. Kazmi, "Mann and Ishikawa type perturbed iterative algorithms for generalized quasi-variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 209, no. 2, pp. 572–584, 1997.
- [25] L.-W. Liu and Y.-Q. Li, "On generalized set-valued variational inclusions," *Journal of Mathematical Analysis and Applications*, vol. 261, no. 1, pp. 231–240, 2001.

- [26] R. U. Verma, "Generalized variational inequalities involving multivalued relaxed monotone operators," *Applied Mathematics Letters*, vol. 10, no. 4, pp. 107–109, 1997.
- [27] R. U. Verma, "A-monotonicity and applications to nonlinear variational inclusion problems," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2004, no. 2, pp. 193–195, 2004.
- [28] R. U. Verma, "Approximation-solvability of a class of A-monotone variational inclusion problems," *Journal of the Korean Society for Industrial and Applied Mathematics*, vol. 8, no. 1, pp. 55–66, 2004.
- [29] X. Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, NY, USA, 1999.
- [30] W. V. Petryshyn, "A characterization of strict convexity of Banach spaces and other uses of duality mappings," *Journal of Functional Analysis*, vol. 6, pp. 282–291, 1970.
- [31] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

Mao-Ming Jin: Department of Mathematics, Yangtze Normal University, Chongqing 408003, Fuling, China
Email address: mmj1898@163.com