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Research Article

Existence Theorems of Solutions for a System of Nonlinear Inclusions with an Application

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By using the iterative technique and Nadler's theorem, we construct a new iterative algorithm for solving a system of nonlinear inclusions in Banach spaces. We prove some new existence results of solutions for the system of nonlinear inclusions and discuss the convergence of the sequences generated by the algorithm. As an application, we show the existence of solution for a system of functional equations arising in dynamic programming of multistage decision processes.

1. Introduction

It is well known that the iterative technique is a very important method for dealing with many nonlinear problems (see, e.g., [1–4]). Let E be a real Banach space, let X be a nonempty subset of E, and let $A,B : X \times X \to E$ be two nonlinear mappings. Chang and Guo [5] introduced and studied the following nonlinear problem in Banach spaces:

$$A(u,u) = u, \qquad B(u,u) = u, \tag{1.1}$$

which has been used to study many kinds of differential and integral equations in Banach spaces. If $A = B$, then problem (1.1) reduces to the problem considered by Guo and Lakshmikantham [1].

On the other hand, Huang et al. [6] introduced and studied the problem of finding $u \in X$, $x \in Su$, and $y \in Tu$ such that

$$A(y,x) = u, \tag{1.2}$$

where $A : X \times X \to X$ is a nonlinear mapping and $S, T : X \to 2^X$ are two set-valued mappings. They constructed an iterative algorithm for solving this problem and gave an application to the problem of the general Bellman functional equation arising in dynamic programming.

Let $A, B : X \times X \to E$ be two nonlinear mappings, let $g : X \to E$ be a nonlinear mapping, and let $S, T : X \to 2^X$ be two set-valued mappings. Motivated by above works, in this paper, we study the following system of nonlinear inclusions problem of finding $u \in X$, $x \in Su$, and $y \in Tu$ such that

$$A(y,x) = gu, \qquad B(x,y) = gu. \tag{1.3}$$

It is easy to see that the problem (1.3) is equivalent to the following problem: find $u \in X$ such that

$$gu \in A(Tu, Su), \qquad gu \in B(Su, Tu), \tag{1.4}$$

which was considered by Huang and Fang [7] when g is an identity mapping. It is well known that problem (1.3) includes a number of variational inequalities (inclusions) and equilibrium problems as special cases (see, e.g, [8–10] and the references therein).

By using the iterative technique and Nadler's theorem [11], we construct a new algorithm for solving the system of nonlinear inclusions problem (1.3) in Banach spaces. We prove the existence of solution for the system of nonlinear inclusions problem (1.3) and the convergence of the sequences generated by the algorithm. As an application, we discuss the existence of solution for a system of functional equations arising in dynamic programming of multistage decision processes.

2. Preliminaries

Let P be a cone in E and let "$\leq$" be a partial order induced by the cone P, that is, $x \leq y$ if and only if $y - x \in P$. Recall that the cone P is said to be normal if there exists a constant $N_P > 0$ such that $\theta \leq u \leq v$ implies that $\|u\| \leq N_P \|v\|$, where θ denotes the zero element of E.

A mapping $A : E \times E \to E$ is said to be mixed monotone if for all $u_1, u_2, v_1, v_2 \in E$, $u_1 \leq u_2$ and $v_1 \leq v_2$ imply that $A(u_1, v_2) \leq A(u_2, v_1)$.

We denote by $\mathrm{CB}(X)$ the family of all nonempty closed bounded subsets of X. A set-valued mapping $F : X \to \mathrm{CB}(X)$ is said to be H-Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$H(Fx, Fy) \leq \lambda \|x - y\|, \quad \forall x, y \in X, \tag{2.1}$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $\mathrm{CB}(X)$, that is, for any $A, B \in \mathrm{CB}(X)$,

$$H(A,B) = \max\left\{ \sup_{x \in A} \inf_{y \in B} d(x,y), \sup_{y \in B} \inf_{x \in A} d(x,y) \right\}. \tag{2.2}$$

Definition 2.1. Let $S,T:E\to E$ be two single-valued mappings. A single-valued mapping $A:E\times E\to E$ is said to be (S,T)-mixed monotone if, for all $u_1,u_2,v_1,v_2\in E$,

$$u_1\le u_2,\quad v_1\le v_2\quad \text{imply that } A(Su_1,Tv_2)\le A(Su_2,Tv_1). \tag{2.3}$$

Remark 2.2. It is easy to see that, if $S=T=I$ (I is the identity mapping), then (S,T)-mixed monotonicity of A is equivalent to the mixed monotonicity of A. The following example shows that the (S,T)-mixed monotone mapping is a proper generalization of the mixed monotone mapping.

Example 2.3. Let $\mathbb{R}=(-\infty,+\infty)$, let $A:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $S,T:\mathbb{R}\to\mathbb{R}$ be defined by

$$A(x,y)=xy,\qquad S(x)=x,\qquad T(x)=-x \tag{2.4}$$

for all $x,y\in\mathbb{R}$. Then it is easy to see that A is an (S,T)-mixed monotone mapping. However, A is not a mixed monotone.

Definition 2.4. Let $S,T:E\to 2^E$ be two multivalued mappings. A single-valued mapping $A:E\times E\to E$ is said to be (S,T)-mixed monotone if, for all $u_1,u_2,v_1,v_2\in E$, $u_1\le u_2$ and $v_1\le v_2$ imply that

$$A(x_1,y_2)\le A(x_2,y_1),\quad \forall x_1\in Su_1,\ x_2\in Su_2,\ y_1\in Tv_1,\ y_2\in Tv_2. \tag{2.5}$$

Definition 2.5. If $\{x_n\}\subset E$ satisfies $x_1\le x_2\le\cdots\le x_n\le\cdots$ or $x_1\ge x_2\ge\cdots\ge x_n\ge\cdots$, then $\{x_n\}$ is said to be a monotone sequence.

Definition 2.6. Let $D\subset E$. A mapping $g:D\to E$ is said to satisfy condition (C) if, for any sequence $\{x_n\}\subset D$ satisfying $\{g(x_n)\}$ that is monotone, $g(x_n)\to g(x)$ implies that $x_n\to x$.

Remark 2.7. If g is reversible and g^{-1} is continuous, then it is easy to see that g satisfies condition (C).

3. Iterative algorithm

In this section, by using Nadler's theorem [11], we construct a new iterative algorithm for solving the system of nonlinear inclusions problem (1.3).

Let $u_0,v_0\in E$, $u_0<v_0$ (i.e., $u_0\le v_0$ and $u_0\ne v_0$) and let $D=[u_0,v_0]=\{u\in E:u_0\le u\le v_0\}$ be an order interval in E. Let $S,T:D\to \mathrm{CB}(D)$ and $g:D\to E$ such that $g(D)=E$ and $gu_0\le gv_0$. Suppose that $A:D\times D\to E$ is an (T,S)-mixed monotone mapping and $B:D\times D\to E$ is a (S,T)-mixed monotone mapping satisfying the following conditions:

(i) for any $u,v\in D$, $u\le v$ implies that

$$B(x,y)\le A(y,x),\quad \forall x\in Su,\ y\in Tv; \tag{3.1}$$

(ii) there exist two constants $a,b\in[0,1)$ such that

$$gu_0+a(gv_0-gu_0)\le B(x_0,y_0),\qquad A(y_0,x_0)\le gv_0-b(gv_0-gu_0) \tag{3.2}$$

for all $x_0\in Su_0$ and $y_0\in Tv_0$;

(iii) for $u,v \in D$, $gu \leq gv$ implies that $u \leq v$.

For u_0 and v_0, we take $x_0 \in Su_0$ and $y_0 \in Tv_0$. By virtue of $g(D) = E$, there exist $u_1, v_1 \in D$ such that

$$gu_1 = B(x_0, y_0) - a(gv_0 - gu_0), \qquad gv_1 = A(y_0, x_0) + b(gv_0 - gu_0). \tag{3.3}$$

It follows from (ii) that

$$gu_0 \leq gu_1, \qquad gv_1 \leq gv_0. \tag{3.4}$$

By condition (i), we have

$$\begin{aligned} gv_1 &= A(y_0, x_0) + b(gv_0 - gu_0) \\ &\geq B(x_0, y_0) + b(gv_0 - gu_0) \\ &= gu_1 + (a+b)(gv_0 - gu_0) \geq gu_1. \end{aligned} \tag{3.5}$$

Therefore, $gu_0 \leq gu_1 \leq gv_1 \leq gv_0$. From condition (iii), we know that $u_0 \leq u_1 \leq v_1 \leq v_0$. Now, by Nadler's theorem [11], there exist $x_1 \in Su_1$ and $y_1 \in Tv_1$ such that

$$||x_1 - x_0|| \leq (1+1)H(Su_1, Su_0), \qquad ||y_1 - y_0|| \leq (1+1)H(Tv_1, Tv_0). \tag{3.6}$$

In virtue of $g(D) = E$, there exist $u_2, v_2 \in D$ such that

$$gu_2 = B(x_1, y_1) - a(gv_1 - gu_1), \qquad gv_2 = A(y_1, x_1) + b(gv_1 - gu_1). \tag{3.7}$$

Since B is (S,T)-mixed monotone and A is (T,S)-mixed monotone,

$$\begin{gathered} gu_1 = B(x_0, y_0) - a(gv_0 - gu_0) \leq B(x_1, y_1) - a(gv_1 - gu_1) = gu_2, \\ gv_2 = A(y_1, x_1) + b(v_1 - u_1) \leq A(y_0, x_0) + b(gv_0 - gu_0) = gv_1. \end{gathered} \tag{3.8}$$

It follows from condition (i) that

$$\begin{aligned} gu_2 &= B(x_1, y_1) - a(gv_1 - gu_1) \\ &\leq A(y_1, x_1) - a(gv_1 - gu_1) \\ &= gv_2 - (a+b)(gv_1 - gu_1) \leq gv_2. \end{aligned} \tag{3.9}$$

Therefore,

$$gu_0 \leq gu_1 \leq gu_2 \leq gv_2 \leq gv_1 \leq gv_0. \tag{3.10}$$

So

$$u_0 \leq u_1 \leq u_2 \leq v_2 \leq v_1 \leq v_0. \tag{3.11}$$

By induction, we can get an iterative algorithm for solving the system of nonlinear inclusions problem (1.3) as follows.

Algorithm 3.1. Let $u_0,v_0 \in E$, $u_0 < v_0$, let $D = [u_0,v_0] = \{u \in E : u_0 \le u \le v_0\}$ be an order interval in E. Let $S,T : D \to \mathrm{CB}(D)$ and $g : D \to E$ with $g(D) = E$ and $gu_0 \le gv_0$. Suppose that $A : D \times D \to E$ is an (T,S)-mixed monotone mapping and $B : D \times D \to E$ is (S,T)-mixed monotone mapping satisfying conditions (i)–(iii). Taking $x_0 \in Su_0$ and $y_0 \in Tv_0$, we can get iterative sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, and $\{y_n\}$ as follows:

$$
\begin{gathered}
gu_{n+1} = B(x_n,y_n) - a(gv_n - gu_n),\\
gv_{n+1} = A(y_n,x_n) + b(gv_n - gu_n),\\
x_{n+1} \in Su_{n+1}, \quad ||x_{n+1} - x_n|| \le \left(1 + \frac{1}{n+1}\right) H(Su_{n+1},Su_n),\\
y_{n+1} \in Tv_{n+1}, \quad ||y_{n+1} - y_n|| \le \left(1 + \frac{1}{n+1}\right) H(Tv_{n+1},Tv_n),
\end{gathered}
\tag{3.12}
$$

$$
gu_0 \le gu_1 \le gu_2 \le \cdots \le gu_n \le \cdots \le gv_n \le \cdots \le gv_2 \le gv_1 \le gv_0, \tag{3.13}
$$

$$
u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_2 \le v_1 \le v_0 \tag{3.14}
$$

for all $n = 0,1,2,\ldots$.

Remark 3.2. From Algorithm 3.1, we can get some new algorithms for solving some special cases of problem (1.3).

4. Existence and convergence

In this section, we will prove the existence of solutions for the system of nonlinear inclusions problem (1.3) and the convergence of sequences generated by Algorithm 3.1.

Theorem 4.1. *Let E be a real Banach space, $P \subset E$ a normal cone in E, $u_0,v_0 \in E$ with $u_0 < v_0$, and $D = [u_0,v_0]$. Let $g : D \to E$ be a mapping such that $g(D) = E$, $gu_0 \le gv_0$, and g satisfies condition (C). Suppose that $S,T : D \to \mathrm{CB}(D)$ are two H-Lipschitz continuous mappings with Lipschitz constants $\alpha > 0$ and $\gamma > 0$, respectively, $A : D \times D \to E$ is a (T,S)-mixed monotone mapping and $B : D \times D \to E$ is an (S,T)-mixed monotone mapping. Assume that conditions (i)–(iii) are satisfied and*

(iv) *there exists a constant $\beta \in [0,1)$ with $a + b + \beta < 1$ such that, for any $u,v \in D$, $u \le v$ implies that*

$$
A(y,x) - B(x,y) \le \beta(gv - gu) \tag{4.1}
$$

for all $x \in Su$, $y \in Tv$.

Then there exist $u^* \in D$, $x^* \in Su^*$, and $y^* \in Tu^*$ such that

$$
\begin{gathered}
gu^* = A(y^*,x^*), \qquad gu^* = B(x^*,y^*),\\
u_n \longrightarrow u^*, \quad v_n \longrightarrow u^*, \quad x_n \longrightarrow x^*, \quad y_n \longrightarrow y^* \quad (n \longrightarrow \infty).
\end{gathered}
\tag{4.2}
$$

Proof. It follows from (3.12), (3.13), (3.14), and condition (iv) that

$$\begin{aligned}\theta \le gv_n - gu_n &= A(y_{n-1},x_{n-1}) - B(x_{n-1},y_{n-1}) + (a+b)(gv_{n-1} - gu_{n-1})\\ &\le \beta(gv_{n-1} - gu_{n-1}) + (a+b)(gv_{n-1} - gu_{n-1})\\ &= (a+b+\beta)(gv_{n-1} - gu_{n-1}) \le \cdots \le (a+b+\beta)^n(gv_0 - gu_0)\end{aligned} \tag{4.3}$$

for all $n = 1,2,\ldots$. Since the cone P is normal, we have

$$||gv_n - gu_n|| \le N_P(a+b+\beta)^n||gv_0 - gu_0||. \tag{4.4}$$

Thus, the condition $a+b+\beta \in [0,1)$ implies that

$$||gv_n - gu_n|| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.5}$$

Now we prove that $\{gu_n\}$ is a Cauchy sequence. In fact, for any $n,m \in \mathbb{N}$, if $n \le m$, then it follows from (3.14) that

$$(gv_n - gu_n) - (gu_m - gu_n) = gv_n - gu_m \in P \tag{4.6}$$

and so $gu_m - gu_n \le gv_n - gu_n$. Since P is a normal cone, we conclude that

$$||gu_m - gu_n|| \le N_P||gv_n - gu_n||. \tag{4.7}$$

Similarly, if $n > m$, we have $gu_n - gu_m \le gv_m - gu_m$ and so

$$||gu_n - gu_m|| \le N_P||gv_m - gu_m||. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$||gu_n - gu_m|| \le N_P \max\{||gv_n - gu_n||, ||gv_m - gu_m||\} \tag{4.9}$$

for all $n,m \in \mathbb{N}$. From (4.5) and (4.9), we know that $\{gu_n\}$ is a Cauchy sequence in E. Let $gu_n \to k^* \in E$ as $n \to \infty$. Since $g(D) = E$, there exists $u^* \in D$ such that $gu^* = k^*$. Now (4.5) implies that $gv_n \to gu^*$ as $n \to \infty$. Since g satisfies condition (C), we know that $u_n \to u^*$ and $v_n \to u^*$ as $n \to \infty$. Now the closedness of P implies that $gu_n \le gu^* \le gv_n$ for all $n = 1,2,\ldots$. It follows from condition (iii) that $u_n \le u^* \le v_n$ for all $n = 1,2,\ldots$. By (3.12) and the H-Lipschitz continuity of mappings S and T, we have

$$\begin{aligned}||x_{n+1} - x_n|| &\le \left(1 + \frac{1}{n+1}\right)H(Su_{n+1},Su_n) \le \left(1 + \frac{1}{n+1}\right) \cdot \alpha||u_{n+1} - u_n||,\\ ||y_{n+1} - y_n|| &\le \left(1 + \frac{1}{n+1}\right)H(Tv_{n+1},Tv_n) \le \left(1 + \frac{1}{n+1}\right) \cdot \gamma||v_{n+1} - v_n||.\end{aligned} \tag{4.10}$$

Thus, $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in D. Let

$$\lim_{n\to\infty} x_n = x^*, \qquad \lim_{n\to\infty} y_n = y^*. \tag{4.11}$$

Next, we prove that $x^* \in Su^*$ and $y^* \in Tu^*$. In fact,

$$\begin{aligned} d(x^*, Su^*) &= \inf\{||x^* - \omega|| : \omega \in Su^*\} \\ &\le ||x^* - x_n|| + d(x_n, Su^*) \le ||x^* - x_n|| + H(Su_n, Su^*) \end{aligned} \tag{4.12}$$

and so $d(x^*, Su^*) = 0$. It follows that $x^* \in Su^*$. Similarly, we have $y^* \in Tu^*$.

We now prove that $gu^* = A(y^*, x^*)$ and $gu^* = B(x^*, y^*)$. Since $u_n \le u^* \le v_n$, B is (S,T)-mixed monotone and A is (T,S)-mixed monotone, it follows from (i) that

$$\begin{aligned} gu_{n+1} &= B(x_n, y_n) - a(gv_n - gu_n) \le B(x^*, y^*) - a(gv_n - gu_n) \\ &\le A(y^*, x^*) + b(gv_n - gu_n) - (a+b)(gv_n - gu_n) \\ &\le A(y_n, x_n) + b(gv_n - gu_n) - (a+b)(gv_n - gu_n) \le gv_{n+1}. \end{aligned} \tag{4.13}$$

Therefore, $gu^* = A(y^*, x^*) = B(x^*, y^*)$. This completes the proof. □

Theorem 4.2. *Let E be a real Banach space, $P \subset E$ a normal cone in E, $u_0, v_0 \in E$ with $u_0 < v_0$, and $D = [u_0, v_0]$. Let $g : D \to E$ be a mapping such that $g(D) = E$, $gu_0 \le gv_0$, and g satisfies condition (C). Suppose that $S, T : D \to \mathrm{CB}(D)$ are two H-Lipschitz continuous mappings with Lipschitz constants $\alpha > 0$ and $\gamma > 0$, respectively, $A : D \times D \to E$ is an (T,S)-mixed monotone mapping, and $B : D \times D \to E$ is a (S,T)-mixed monotone mapping. Assume that conditions (i)–(iii) are satisfied and*

(iv)′ *for any $u, v \in D$, $u \le v$ implies that*

$$A(y, x) - B(x, y) \le L(gv - gu) \tag{4.14}$$

for all $x \in Su$, $y \in Tv$, where $L : E \to E$ is a bounded linear mapping with a spectral radius $r(L) = \beta < 1$ and $a + b + \beta < 1$.

Then there exist $u^* \in D$, $x^* \in Su^*$, and $y^* \in Tu^*$ such that

$$\begin{gathered} gu^* = A(y^*, x^*), \qquad gu^* = B(x^*, y^*), \\ u_n \longrightarrow u^*, \quad v_n \longrightarrow u^*, \quad x_n \longrightarrow x^*, \quad y_n \longrightarrow y^* \quad (n \longrightarrow \infty). \end{gathered} \tag{4.15}$$

Proof. It follows from (3.12), (3.13), (3.14), and condition (iv)′ that

$$\begin{aligned} \theta \le gv_n - gu_n &= A(y_{n-1}, x_{n-1}) - B(x_{n-1}, y_{n-1}) + (a+b)(gv_{n-1} - gu_{n-1}) \\ &\le L(gv_{n-1} - gu_{n-1}) + (a+b)(gv_{n-1} - gu_{n-1}) \\ &\le (L + (a+b)I)(gv_{n-1} - gu_{n-1}) = J(gv_{n-1} - gu_{n-1}) \end{aligned} \tag{4.16}$$

for all $n = 1, 2, \ldots$, where $J = L + (a+b)I$ and I is the identity mapping. By induction, we conclude that

$$\theta \le gv_n - gu_n \le J^n(gv_0 - gu_0) \tag{4.17}$$

for all $n = 1, 2, \ldots$. Since $r(L) = \beta < 1$, from [12, Example 10.3(b) and Theorem 10.3(b)] by Rudin, we have

$$\lim_{n\to\infty} \|J^n\|^{1/n} = r(J) \le a + b + \beta < 1. \tag{4.18}$$

This implies that there exists $n_0 \in \mathbb{N}$ such that

$$||J^n|| \leq (a+b+\beta)^n, \quad \forall n \geq n_0. \tag{4.19}$$

Since P is a normal cone and $a+b+\beta < 1$, it follows from (4.17) and (4.19) that $\|gv_n - gu_n\| \to 0$ as $n \to \infty$. The rest argument is similar to the corresponding part of the proof in Theorem 4.1 and we omit it. This completes the proof. □

If $S = T$ in Theorem 4.1, we have the following result.

Corollary 4.3. *Let E be a real Banach space, $P \subset E$ a normal cone in E, $u_0, v_0 \in E$ with $u_0 < v_0$, and $D = [u_0, v_0]$. Let $g : D \to E$ be a mapping such that $g(D) = E$, $gu_0 \leq gv_0$, and g satisfies (iii) and condition (C). Suppose that $S : D \to \mathrm{CB}(D)$ is H-Lipschitz continuous with Lipschitz constant $\alpha > 0$, and $A, B : D \times D \to E$ are both (S,S)-mixed monotone mappings such that*

(B_1) *for any $u, v \in D$, $u \leq v$ implies that*

$$B(x,y) \leq A(y,x), \quad \forall x \in Su, \ y \in Sv; \tag{4.20}$$

(B_2) *for all $u, v \in D$, $u \leq v$, there exists $\beta \in [0,1)$ such that*

$$A(y,x) - B(x,y) \leq \beta(gv - gu); \tag{4.21}$$

for all $x \in Su$, $y \in Sv$;

(B_3) *there are $a, b \in [0,1)$ with $a+b+\beta < 1$ such that*

$$gu_0 + a(gv_0 - gu_0) \leq B(u_0, v_0), \qquad A(v_0, u_0) \leq gv_0 - b(gv_0 - gu_0). \tag{4.22}$$

Then there exist $u^* \in D$ and $x^*, y^* \in Su^*$ such that

$$gu^* = B(x^*, y^*) = A(y^*, x^*), \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = u^*, \tag{4.23}$$

where

$$gu_{n+1} = B(u_n, v_n) - a(gv_n - gu_n), \qquad gv_{n+1} = A(v_n, u_n) + b(gv_n - gu_n) \tag{4.24}$$

for all $n = 1, 2, \ldots$.

If $S = I$ in Corollary 4.3, we have the following result.

Corollary 4.4. *Let E be a real Banach space, $P \subset E$ a normal cone in E, $u_0, v_0 \in E$, $u_0 < v_0$, and $D = [u_0, v_0]$. Let $g : D \to E$ be a mapping such that $g(D) = E$, $gu_0 \leq gv_0$, and g satisfies (iii) and condition (C). Suppose that $A, B : D \times D \to E$ are both mixed monotone and satisfy the following conditions:*

(C_1) *there exists $\beta \in [0,1)$ such that*

$$A(v,u) - B(u,v) \leq \beta(gv - gu) \tag{4.25}$$

for all $u, v \in D$ with $u \leq v$;

(C_2) *for all $u,v \in D$, $u \le v$ implies that*

$$B(u,v) \le A(v,u); \tag{4.26}$$

(C_3) *there are $a,b \in [0,1)$ with $a+b+\beta<1$ such that*

$$gu_0 + a(gv_0 - gu_0) \le B(u_0,v_0), \qquad A(v_0,u_0) \le gv_0 - b(gv_0 - gu_0). \tag{4.27}$$

Then there exists $u^* \in D$ such that

$$gu^* = A(u^*,u^*) = B(u^*,u^*), \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = u^*, \tag{4.28}$$

where

$$gu_{n+1} = B(u_n,v_n) - a(gv_n - gu_n), \qquad gv_{n+1} = A(v_n,u_n) + b(gv_n - gu_n) \tag{4.29}$$

for all $n = 1,2,\ldots$.

5. An application

Dynamic programming, because of its wide applicability, has evoked much interest among people of various discipline. See, for example, [13–17] and the references therein.

Let Y and Z be two Banach spaces, $G \subset Y$ a state space, $\Delta \subset Z$ a decision space, and $\mathbb{R} = (-\infty,+\infty)$. We denote by $B(G)$ the set of all bounded real-valued functional defined on G. Define $\|f\| = \sup_{x\in G}|f(x)|$. Then $(B(G),\|\cdot\|)$ is a Banach space. Let

$$P = \{f \in B(G) : f(x) \ge 0,\ \forall x \in G\}. \tag{5.1}$$

Obviously, P is a normal cone. In this section, we consider a system of functional equations as follows.

Find a bounded functional $f : G \to \mathbb{R}$ such that

$$\begin{gathered} f_1 \in Sf(x), \quad f_2 \in Tf(x), \\ gf(x) = \sup_{y\in\Delta}\left[\varphi(x,y) + F_1(x,y,f_1(W(x,y)),f_2(W(x,y)))\right], \\ gf(x) = \sup_{y\in\Delta}\left[\varphi(x,y) + F_2(x,y,f_2(W(x,y)),f_1(W(x,y)))\right] \end{gathered} \tag{5.2}$$

for all $x \in G$, where $W : G\times\Delta \to G$, $\varphi : G\times\Delta \to \mathbb{R}$, $F_1,F_2 : G\times\Delta\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $S,T : B(G) \to 2^{B(G)}$, and $g : B(G) \to B(G)$.

As an application of Theorem 4.1, we have the following result concerned with the existence of solution for the system of functional equations problem (5.2).

Theorem 5.1. *Suppose that*

(1) *φ, F_1, and F_2 are bounded;*

(2) *there exist two bounded functionals $u_0,v_0 : G \to \mathbb{R}$ with $u_0 \ne v_0$, $u_0(x) \le v_0(x)$ for all $x \in G$, and suppose that $S,T : D = [u_0,v_0] \to \mathrm{CB}(D)$ are H-Lipschitz continuous with Lipschitz constants $\alpha > 0$ and $\gamma > 0$, respectively;*

(3) $g : D \to B(G)$ *satisfies* $g(D) = B(G)$, $gu_0 \le gv_0$, *and*

(a) *for any* $\{u_n\} \subset D$ *with* $\{gu_n\}$ *being monotone,* $u \in D$, *if* $gu_n \to gu$, *then* $u_n \to u$;

(b) *for any* $u, v \in D$, *if* $u(x) \le v(x)$, *for all* $x \in G$, *then* $gu(x) \le gv(x)$, *for all* $x \in G$;

(4) *there exists a constant* $\beta \in [0,1)$ *such that, for any* $u, v \in D$, *if* $u(x) \le v(x)$ *for all* $x \in G$, *then*

$$\begin{aligned} &F_1(x,y,\omega(W(x,y)),z(W(x,y))) - F_2(x,y,z(W(x,y)),\omega(W(x,y))) \\ &\quad \le \beta\big(gv(x) - gu0(x)\big) \end{aligned} \tag{5.3}$$

for all $z \in Su$, $\omega \in Tv$, $x \in G$, *and* $y \in \Delta$;

(5) *for any* $u, v \in D$ *with* $u(x) \le v(x)$ *for all* $x \in G$,

$$F_2(x,y,z(W(x,y)),\omega(W(x,y))) \le F_1(x,y,\omega(W(x,y)),z(W(x,y))) \tag{5.4}$$

for all $z \in Su$, $\omega \in Tv$, $x \in G$, *and* $y \in \Delta$;

(6) *for any* $z \in Su_0$, $\omega \in Tv_0$, $x \in G$, *and* $y \in \Delta$,

$$\begin{aligned} gu_0(x) + a\big(gv_0(x) - gu_0(x)\big) &\le F_2(x,y,z(W(x,y)),\omega(W(x,y))), \\ F_1(x,y,\omega(W(x,y)),z(W(x,y))) &\le gv_0(x) - b\big(gv_0(x) - gu_0(x)\big), \end{aligned} \tag{5.5}$$

where $a, b \in [0,1)$ *with* $a + b + \beta < 1$;

(7) *for any* $u_1, u_2, v_1, v_2 \in D$, *if* $u_1(x) \le u_2(x)$ *and* $v_1 \le v_2(x)$ *for all* $x \in G$, *then*

$$\begin{aligned} F_2(x,y,y_1(W(x,y)),x_2(W(x,y))) &\le F_2(x,y,y_2(W(x,y)),x_1(W(x,y))), \\ F_1(x,y,x_1(W(x,y)),y_2(W(x,y))) &\le F_1(x,y,x_2(W(x,y)),y_1(W(x,y))) \end{aligned} \tag{5.6}$$

for all $x_1 \in Su$, $x_2 \in Su_2$, $y_1 \in Tv_1$, $y_2 \in Tv_2$, $x \in G$, *and* $y \in \Delta$.

Then there exist $u^* \in D$, $z^* \in Su^*$, *and* $\omega^* \in Tu^*$ *such that*

$$\begin{aligned} gu^* &= \sup_{y\in\Delta}\{\varphi(x,y) + F_1(x,y,\omega^*(W(x,y)),z^*W(x,y))\}, \\ gu^* &= \sup_{y\in\Delta}\{\varphi(x,y) + F_2(x,y,z^*(W(x,y)),\omega^*W(x,y))\} \end{aligned} \tag{5.7}$$

for all $x \in G$.

Proof. For any $u, v \in D$, we define the mappings A, B as follows:

$$\begin{aligned} A(u,v)(x) &= \sup_{y\in\Delta}\big[\omega(x,y) + F_1(x,y,u(W(x,y)),v(W(x,y)))\big], \\ B(u,v)(x) &= \sup_{y\in\Delta}\big[\omega(x,y) + F_2(x,y,u(W(x,y)),v(W(x,y)))\big] \end{aligned} \tag{5.8}$$

for all $x \in G$. From (1.1) and (4.7), we know that $A, B : D \times D \to B(G)$ are (T, S)-mixed monotone and (S, T)-mixed monotone, respectively. By assumptions (1.3)–(4.5), it is easy to check that A, B and S, T satisfy all the conditions of Theorem 4.1. Thus, Theorem 4.1 implies that there exist $u^* \in D$, $z^* \in Su^*$, and $\omega^* \in Tu^*$ such that $gu^* = A(\omega^*, z^*) = B(z^*, \omega^*)$, that is,

$$\begin{aligned} gu^* &= \sup_{y\in\Delta}\{\varphi(x,y) + F_1(x,y,\omega^*(W(x,y)), z^* W(x,y))\}, \\ gu^* &= \sup_{y\in\Delta}\{\varphi(x,y) + F_2(x,y,z^*(W(x,y)), \omega^* W(x,y))\} \end{aligned} \tag{5.9}$$

for all $x \in G$. This completes the proof. □

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