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Research Article

Superlinear Equations Involving Nonlinearities Limited by Asymptotically Homogeneous Functions

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We obtain a solution of the quasilinear equation $-\Delta_p u = f(u)$ in Ω, $u = 0$, on $\partial\Omega$. Here the nonlinearity f is superlinear at zero, and it is located near infinity between two functions that belong to a class of functions where the Ambrosetti-Rabinowitz condition is satisfied. More precisely, we consider the class of functions that are asymptotically homogeneous of index q.

1. Introduction

Consider the problem

$$\begin{aligned} -\Delta_p u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$, with $N \geq 3$ and $1 < p < N$. We assume that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz function satisfying the condition

(f_1) $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$.

It is well known that problems involving the p-Laplacian operator appear in many contexts. Some of these problems come from different areas of applied mathematics and physics. For example, they may be found in the study of non-Newtonian fluids, nonlinear elasticity, and reaction diffusions. For discussions about problems modelled by these boundary value problems, see, for example, [1].

One of the most widely used results for solving problem (1.1) is the mountain pass theorem. In order to apply this theorem, it is necessary that the Euler-Lagrange functional associated to the problem has the Palais-Smale property. One way to ensure this is to assume that f satisfies some Ambrosetti-Rabinowitz-type condition (see, e.g., [2] or [3]). Another technique used for obtaining solutions of problem (1.1) is the blowup method due to Gidas and Spruck [4]. In order to use any of the techniques above, it is necessary that the nonlinearity f has subcritical growth.

The object of this paper is to study problem (1.1) for nonlinearities f which do not necessarily satisfy the classical Ambrosetti-Rabinowitz condition, but are limited by functions that do satisfy that condition. We mention recent work on existence of solutions of problem (1.1) where a combination of blowup arguments and nonexistence results for $\mathbb{R}^N$ is used. Azizieh and Clément [5] studied the case $1 < p \leq 2$. It is assumed that the domain Ω is strictly convex and that there exist positive constants C_1, C_2, and q, where $p < q \leq N(p-1)/(N-p)$, such that for all $s > 0$, the function f satisfies the condition

$$C_1 s^q \leq f(s) \leq C_2 s^q. \tag{1.2}$$

Topological techniques and blowup methods are used in [5].

Figueiredo and Yang [6] studied the case $p = 2$. The nonlinearity f is assumed to be a differentiable subcritical function satisfying condition (1.2) for s large. Variational methods, Morse's index, and blowup methods are used.

Recently, a more general nonlinearity f, which may depend on the gradient, is studied in [7] where convex assumptions are not imposed on the domain. The nonlinearity must be located, however, in a region defined by an inequality like the one which appears in (1.2). Therefore, in [7] there is a stronger restriction on the growth of the nonlinearity than the one we are imposing.

In this paper, we assume that the nonlinearity f satisfies condition (f_1) and that it is bounded from below and from above by functions which are asymptotically homogeneous of index q. Following ideas of [5–7], we obtain the existence of a solution of problem (1.1). (See Theorem 4.1. By definition, a function h is asymptotically homogeneous of index q if and only if $h : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\lim_{t\to\infty}(h(ts))/(h(t)) = s^q$, for all $s \in (0,\infty)$.)

Observe that our method works if f is a locally Lipschitz function satisfying both condition (f_1) and inequality (1.2) for s large. Thus our result is an improvement because we do not impose either the regularity condition on the function f (as in [6]) or condition (1.2) for all $s \geq 0$ (as in [5, 7]). Also, we note that we do not assume any convex assumption on Ω.

The paper is organized as follows. Section 2 contains some properties of asymptotically homogeneous functions of index q as well as a result of existence. In Section 3, we state some known estimates and Harnack inequalities. In Section 4, we formulate and prove our main result, Theorem 4.1.

2. Asymptotically homogeneous nonlinearities

Asymptotically homogeneous nonlinearities are considered in the study of existence of radial solutions of superlinear equations, as well as in probabilities (see [8], as well as [9, 10]). An example is the function given by $h(s) = s^q/\ln(e+s)$, which motivates in part our study. Note that the function h satisfies the next two limits:

$$\lim_{s\to\infty}\frac{h(s)}{s^r}=0 \quad \text{if } q\le r, \qquad \lim_{s\to\infty}\frac{h(s)}{s^r}=\infty \quad \text{if } r<q. \tag{2.1}$$

Thus h is not asymptotic to any power at infinity. It does, however, satisfy the following property.

(P) For all $\varepsilon>0$, there exist positive constants C_1, C_2, and s_0 such that

$$C_1 s^{q-\varepsilon}\le h(s)\le C_2 s^{q+\varepsilon}, \quad \forall s>s_0. \tag{2.2}$$

In general, we have the following.

Proposition 2.1. *If h is a continuous function that is asymptotically homogeneous of index q, then it satisfies property (P). Moreover, one has*

$$\lim_{s\to\infty}\frac{H(s)}{sh(s)}=\frac{1}{q+1}, \tag{2.3}$$

where H is the primitive of h.

Proof. For the proof of property (P), we refer the reader to [8, page 4, inequality (10)]. Limit (2.3) follows from Karamata's theorem (see [9]). □

We thus have that near infinity, asymptotically homogeneous nonlinearities lie between two different powers. Further, by equality (2.3), they satisfy the classical Ambrosetti-Rabinowitz condition. The following follows from the mountain pass theorem.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N\ge 3$. Let f be an asymptotically homogeneous nonlinearity of index q such that $p-1<q<(N(p-1)+p)/(N-p)$. Suppose that f satisfies condition (f_1). Then there exists at least one positive solution of problem (1.1).*

3. Some previous estimates

Here we first state some lemmas which will be useful to prove our principal result. We note that here and throughout all the paper, C, C_1, C_2, and M stand for positive constants which may vary from one expression to another, but are always independent of u.

We will use the following weak Harnack inequality due to Trudinger (see [11]).

Lemma 3.1. *Let u be a nonnegative weak solution of $-\Delta_p u\ge 0$ in Ω. Take $\gamma\in(0,N(p-1)/(N-p))$ and let B_R be a ball of radius R such that B_{2R} is included in Ω. Then there exists $C=C(N,p,\gamma)$ such that*

$$\inf_{B_R} u\ge CR^{-N/\gamma}\|u\|_{L^\gamma(B_{2R})}. \tag{3.1}$$

A slight modification of the proof of [7, Lemma 2.1] allows us to show the following lemma (see also [12] and the references therein).

Lemma 3.2. *Let u be a nonnegative weak solution of the inequality*

$$-\Delta_p u \geq u^q - Mu^{p-1}, \tag{3.2}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $q > p-1$. Take $\gamma \in (0,q)$ and let B_{R_0} be a ball of radius R such that B_{2R_0} is included in Ω.

Then, there exists a positive constant $C = (N,m,p,\gamma,R_0)$ such that

$$\int_{B_R} u^\gamma \leq CR^{(N-p\gamma)/(q+1-p)}, \tag{3.3}$$

for all $R \in (0,R_0)$.

4. An existence result

In this section, we consider two fixed continuous functions h_0, $h_1 : \mathbb{R}^+ \to \mathbb{R}^+$ which are asymptotically homogeneous of index q, where $p-1 < q < N(p-1)/(N-p)$.

It follows from Proposition 2.1 that h_1 and h_2 are superlinear at infinity, that is,

$$\lim_{s\to\infty} \frac{h_i(s)}{s^{p-1}} = \infty \quad \text{for } i = 0,1. \tag{4.1}$$

Our existence result is the following.

Theorem 4.1. *Let Ω be a bounded C^2-domain in $\mathbb{R}^N$. Let f be a locally Lipschitz function satisfying condition (f_1). Further, assume that there exist positive constants C_1, C_2, and s_0 such that f satisfies the condition*

$$C_1 h_0(s) \leq f(s) \leq C_2 h_1(s), \quad \forall s > s_0. \tag{4.2}$$

Then problem (1.1) has at least one positive solution.

Proof. By (4.2), there exist positive constants K_1 and K_2 such that

$$C_1 h_0(s) - K_1 \leq f(s) \leq C_2 h_1(s) + K_2, \quad \text{for } s > 0. \tag{4.3}$$

By Proposition 2.1, we have that f satisfies property (P).

For each $n \in \mathbb{N}$, we next define the function

$$f_n(s) = \begin{cases} f(s) & \text{if } 0 \leq s < n, \\ f(s_0)\left(h_1(s_0)\right)^{-1} h_1(s) & \text{if } s \geq n. \end{cases} \tag{4.4}$$

It is not difficult to verify that the function f_n satisfies condition (f_1). Observe that the function f_n also satisfies inequality (4.3) and property (P), where the constants are taken as independent of n.

Now consider the equation

$$-\Delta_p u = f_n(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

Since the function f_n is asymptotically homogeneous of index q, we conclude that a solution u_n of this equation exists by Theorem 2.2. To complete the proof of Theorem 4.1, we need to show that there exists an n such that $\|u_n\|_\infty \le n$.

Suppose to the contrary that $\|u_n\|_\infty > n$, for all n. Take $M_n = \|u_n\|_\infty$. Let $x_n \in \Omega$ be such that $u_n(x_n) = M_n$. Denote

$$\delta_n = d(x_n,\partial\Omega), \qquad \widetilde{\delta}_n = \sup\left\{\delta;\ x \in B_\delta(x_n) \Longrightarrow u_n(x) > \frac{M_n}{2}\right\}. \tag{4.6}$$

It is simple to prove that $\widetilde{\delta}_n$ is well defined. Moreover, we have $0 < \widetilde{\delta}_n < \delta_n$.

Claim 1. There exists $\widetilde{x}_n \in \Omega$ such that $d(x_n,\widetilde{x}_n) = \widetilde{\delta}_n$ and $u_n(\widetilde{x}_n) = M_n/2$.

Assume that $u_n(x) > M_n/2$ for all x such that $d(x_n,x) = \widetilde{\delta}_n$, then by continuity, the existence of $\varepsilon > 0$ can be proved such that $u_n(x) > M_n/2$ for all x in $B_{\widetilde{\delta}_n+\varepsilon}(x_n)$ which is a contradiction with the definition of $\widetilde{\delta}_n$.

Claim 2. Define $\widetilde{h}_1(s) = \max_{t\in[0,s]} h_1(t)$. Then, there exists c such that $0 < c < \widetilde{\delta}_n(\widetilde{h}_1(M_n)/M_n^{p-1})^{1/p}$ for n large.

We first note that the function $\widetilde{h}_1$ is not decreasing and satisfies

$$\lim_{s\to+\infty} \widetilde{h}_1(s) = +\infty. \tag{4.7}$$

Moreover, we have that for all $\varepsilon > 0$, there exist positive constants C_1, C_2, and s_1 such that

$$C_1 s^{q-\varepsilon} \le \widetilde{h}_1(s) \le C_2 s^{q+\varepsilon}, \quad \forall s > s_1. \tag{4.8}$$

We may suppose, passing to a subsequence, that $\widetilde{\delta}_n(\widetilde{h}_1(M_n)/M_n^{p-1})^{1/p} < 1$ for all n; since in other cases, there is nothing to prove. Define Ω_n by

$$\left\{z \in \mathbb{R}^N : \left(x_n + \left(\frac{\widetilde{h}_1(M_n)}{M_n^{p-1}}\right)^{-1/p} z\right) \in \Omega\right\}. \tag{4.9}$$

For $z \in \Omega_n$, define the normalized sequence

$$v_n(z) = M_n^{-1} u_n\left(x_n + \left(\frac{\widetilde{h}_1(M_n)}{M_n^{p-1}}\right)^{-1/p} z\right). \tag{4.10}$$

We have

$$\begin{gathered} -\Delta_p v_n = g_n(v_n) \quad \text{in } \Omega_n, \\ v_n(0) = 1, \quad 0 \le v_n \le 1, \end{gathered} \tag{4.11}$$

where

$$g_n(s) = \frac{f_n(M_n s)}{\widetilde{h}_1(M_n)}, \quad 0 \le s \le 1. \tag{4.12}$$

By the definition of $\widetilde{h}_1$, it follows, according to (4.3), that for all $n \in \mathbb{N}$,

$$g_n(v_n) \le \frac{C_2 h_1(M_n v_n) + K_2}{\widetilde{h}_1(M_n)} \le C_2 + \frac{K_2}{\widetilde{h}_1(M_n)}. \tag{4.13}$$

By using $C^{1,\tau}$ regularity result up to the boundary (see [13]), we conclude that

$$\sup_{|x| \le \widetilde{\delta}_n(\widetilde{h}_1(M_n)/M_n^{p-1})^{1/p}} \|\nabla v_n\| < C, \tag{4.14}$$

for certain $C > 0$.

The mean value theorem implies that

$$\begin{aligned}\frac{1}{2} &= v_n(0) - v_n\left(\left(\frac{\widetilde{h}_1(M_n)}{M_n^p}\right)^{1/p}(\widetilde{x}_n - x_n)\right)\\ &\le \sup_{|x| \le \widetilde{\delta}_n(\widetilde{h}_1(M_n)/M_n^{p-1})^{1/p}} \|\nabla v_n\| \widetilde{\delta}_n \left(\frac{\widetilde{h}_1(M_n)}{M_n^{p-1}}\right)^{1/p}\\ &\le C\widetilde{\delta}_n \left(\frac{\widetilde{h}_1(M_n)}{M_n^{p-1}}\right)^{1/p},\end{aligned} \tag{4.15}$$

which proves the claim.

Claim 3. There exist $\tau_n > 0$ and $y_n \in \Omega$ such that $B_{2\tau_n}(y_n) \subset \Omega$; $0 < \lim \tau_n < \infty$, and passing to a subsequence, we have

$$\inf_{x \in B_{\tau_n}(y_n)} u_n(x) \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{4.16}$$

Passing to a subsequence, we only need to consider two cases.

Case 1. If $\lim \delta_n = 0$, let $z_n \in \partial\Omega$ be the point such that $\delta_n = d(x_n, z_n)$. Denote by ν_n the unit exterior normal of $\partial\Omega$ at z_n. For τ sufficiently small but fixed, take $y_n = z_n - 2\tau\nu_n$ (we use the regularity of Ω). Let $x \in B_{\widetilde{\delta}_n}(x_n)$, then we have for n large that

$$d(x, y_n) \le d(x, x_n) + d(x_n, y_n) < \delta_n + d(x_n, y_n) = 2\tau, \tag{4.17}$$

which implies that $B_{\widetilde{\delta}_n}(x_n) \subset B_{2\tau}(y_n)$.

Fix ε positive such that

$$\frac{N(q+\varepsilon+1-p)}{p} < \frac{N(p-1)}{(N-p)}, \tag{4.18}$$

and take γ such that

$$\frac{N(q+\varepsilon+1-p)}{p} < \gamma < \frac{N(p-1)}{(N-p)}. \tag{4.19}$$

Using Lemma 3.1 and Claim 2, we get

$$\begin{aligned}\inf_{B_\tau(y_n)} u_n &\geq C\tau^{-N/\gamma}\|u_n\|_{L^\gamma(B_{2\tau}(y_n))} \geq C\tau^{-N/\gamma}\left(\int_{B_{B_{\widetilde{\delta}_n}(x_n)}} u_n^\gamma\right)^{1/\gamma} \\ &\geq C_1\tau^{-N/\gamma}\left(\widetilde{\delta}_n^N M_n^\gamma\right)^{1/\gamma} \geq C_2\tau^{-N/\gamma}\left(\left(\frac{M_n^{p-1}}{\widetilde{h}_1(M_n)}\right)^{N/p} M_n^\gamma\right)^{1/\gamma}.\end{aligned} \tag{4.20}$$

Now, take $\tau_n = \tau$ and use inequality (4.8) to obtain

$$\inf_{B_{\tau_n}(y_n)} u_n \geq C\tau^{-N/\gamma}\left(M_n^{-N(q+\varepsilon+1-p)/p+\gamma}\right)^{1/\gamma} \longrightarrow \infty, \tag{4.21}$$

as n goes to ∞.

Case 2. If $\lim \delta_n > 0$, taking $y_n = x_n$, and choosing $\tau_n = \delta_n/2$, we obtain a similar conclusion and Claim 3 is proved.

To conclude the proof of Theorem 4.1, observe that by property (P) for h_0 and estimate (4.3), the function u_n verifies

$$-\Delta_p u_n \geq C_1 u_n^{q-\varepsilon} - M u_n^{p-1} \quad \text{in } \Omega. \tag{4.22}$$

Now, choose γ so that $0 < \gamma < q - \varepsilon$. By Lemma 3.2, we have

$$\int_{B_{\tau_n}(y_n)} u_n^\gamma \leq C\tau_n^{(N-p\gamma)/(q+1-p)}. \tag{4.23}$$

This is a contradiction with Claim 3. □

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