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Research Article

A Note on the q-Genocchi Numbers and Polynomials

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We discuss new concept of the q-extension of Genocchi numbers and give some relations between q-Genocchi polynomials and q-Euler numbers.

1. Introduction

The Genocchi numbers G_n, $n = 0,1,2,\ldots$, which can be defined by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi, \tag{1.1}$$

have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas, [1–13]. It is easy to find the values $G_1 = 1$, $G_3 = G_5 = G_7 = \cdots = 0$, and even coefficients are given by $G_{2m} = 2(1-2^{2n})B_{2n} = 2nE_{2n-1}(0)$, where B_n is a Bernoulli number and $E_n(x)$ is an Euler polynomial. The first few Genocchi numbers for $n = 2,4,\ldots$ are $-1,-3,17,-155,2073,\ldots$. The Euler polynomials are well known as

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \text{ (see [1, 3, 7–9])}. \tag{1.2}$$

By (1.1) and (1.2) we easily see that

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}, \quad \text{where } \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \text{ (cf. [4–6])}. \tag{1.3}$$

For $m, n \geq 1$ and, m odd, we have

$$(n^m - n)G_m = \sum_{k=1}^{m-1} \binom{m}{k} n^k G_k Z_{m-k}(n-1), \tag{1.4}$$

where $Z_m(n) = 1^m - 2^m + 3^m - \cdots + (-1)^{n-1}n^m$, see [3, 13]. From (1.15) we derive

$$2t = \sum_{n=0}^{\infty} \left((G+1)^n + G_n\right)\frac{t^n}{n!}, \tag{1.5}$$

where we use the technique method notation by replacing G^m by $G_m (m \geq 0)$, symbolically. By comparing the coefficients on both sides in (1.5), we see that

$$G_0 = 0, \qquad (G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1.6}$$

Let p be a fixed odd prime, and let $\mathbb{C}_p$ denote the p-adic completion of the algebraic closure of $\mathbb{Q}_p(= p$-adic number field). For d is a fixed positive integer with $(p,d) = 1$, let

$$\begin{gathered} X = X_d = \varprojlim_{N} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \\ X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{d}p^N\}, \end{gathered} \tag{1.7}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

Ordinary q-calculus is now very well understood from many different points of view. Let us consider a complex number $q \in \mathbb{C}$ with $|q| < 1$ (or $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$) as an indeterminate. The q-basic numbers are defined by

$$[x]_q = \frac{q^x - 1}{q - 1}, \quad [x]_{-q} = \frac{-(-q)^x + 1}{q+1}, \quad \text{for } x \in \mathbb{R}. \tag{1.8}$$

We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $f \in \mathrm{UD}(\mathbb{Z}_p)$, if the difference quotients

$$F_f(x,y) = \frac{f(x) - f(y)}{x - y} \tag{1.9}$$

have a limit $l = f'(a)$ as $(x,y) \to (a,a)$.

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, let us start with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N\mathbb{Z}_p) \tag{1.10}$$

representing a q-analogue of Riemann sums for f, (cf. [5]). The integral of f on $\mathbb{Z}_p$ will be defined as limit ($n \to \infty$) of those sums, when it exists. The p-adic q-integral of the function $f \in \mathrm{UD}(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{0\le x<p^N} f(x)q^x, \text{ (see [5, 10–12]).} \tag{1.11}$$

In the previous paper [4, 9], the author constructed the q-extension of Euler polynomials by using p-adic q-fermionic integral on $\mathbb{Z}_p$ as follows:

$$E_{n,q}(x) = \int_{\mathbb{Z}_p} [t+x]_q^n d\mu_{-q}(t), \quad \text{where } \mu_{-q}(x+p^N\mathbb{Z}_p) = \frac{(-q)^x}{[p^N]_{-q}}. \tag{1.12}$$

From (1.12), we note that

$$E_{n,q}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}} q^{lx}, \text{ see [4].} \tag{1.13}$$

The q-extension of Genocchi numbers is defined as

$$g_q^*(t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,q}^* \frac{t^n}{n!}, \text{ see [4].} \tag{1.14}$$

The following formula is well known in [4, 7]:

$$E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}^*}{k+1}. \tag{1.15}$$

The modified q-Euler numbers are defined as

$$\xi_{0,q} = \frac{[2]_q}{2}, \qquad (q\xi+1)^k + \xi_{k,q} = \begin{cases} [2]_q & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases} \tag{1.16}$$

with the usual convention of replacing ξ^i by $\xi_{i,q}$, see [10]. Thus, we derive the generating function of $\xi_{n,q}$ as follows:

$$F_q(t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} \xi_{n,q} \frac{t^n}{n!}. \tag{1.17}$$

Now we also consider the q-Euler polynomials $\xi_{n,q}(x)$ as

$$F_q(t,x) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} \xi_{n,q}(x) \frac{t^n}{n!}. \tag{1.18}$$

From (1.18) we note that

$$\xi_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \xi_{l,q} q^{lx} [x]_q^{n-l}, \text{ see [10].} \tag{1.19}$$

In the recent, several authors studied the q-extension of Genocchi numbers and polynomials (see [1, 2, 5–7, 12]). In this paper we discuss the new concept of the q-extension of Genocchi numbers and give the same relations between q-Genocchi numbers and q-Euler numbers.

2. q-extension of Genocchi numbers

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Now we consider the q-extension of Genocchi numbers as follows:

$$g_q(t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k]_q t} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}. \tag{2.1}$$

In (2.1), it is easy to show that $\lim_{q\to 1} g_q(t) = 2t/(e^t+1) = \sum_{n=0}^{\infty} G_n(t^n/n!)$. From (2.1) we derive

$$\begin{aligned} g_q(t) &= [2]_q t \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} [k]_q^m \frac{t^m}{m!} = [2]_q \sum_{k=0}^{\infty} (-1)^k \sum_{m=1}^{\infty} m[k]_q^{m-1} \frac{t^m}{m!} \\ &= [2]_q \sum_{k=0}^{\infty} (-1)^k \sum_{m=0}^{\infty} m[k]_q^{m-1} \frac{t^m}{m!}. \end{aligned} \tag{2.2}$$

By (2.2), we easily see that

$$g_q(t) = [2]_q \sum_{m=0}^{\infty} \left(m \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} (-1)^l \frac{1}{1+q^l} \right) \frac{t^m}{m!}. \tag{2.3}$$

From (2.1) and (2.3) we note that

$$\sum_{m=0}^{\infty} G_{m,q} \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left(m[2]_q \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l}{1+q^l} \right) \frac{t^m}{m!}. \tag{2.4}$$

By comparing the coefficients on both sides in (2.4), we have the following theorem.

Theorem 2.1. *For $m \geq 0$,*

$$G_{m,q} = m[2]_q \left(\frac{1}{1-q} \right)^{m-1} \sum_{l=0}^{m-1} \binom{m-1}{l} \frac{(-1)^l}{1+q^l}. \tag{2.5}$$

From Theorem 2.1, we easily derive the following corollary.

Corollary 2.2. *For $k \in \mathbb{N}$,*

$$G_{0,q} = 0, \qquad (qG+1)^k + G_{k,q} = \begin{cases} \dfrac{[2]_q^2}{2} & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \tag{2.6}$$

with the usual convention of replacing G^i by $G_{i,q}$.

Remark 2.3. We note that Corollary 2.2 is the q-extension of (1.6). By (1.15)–(1.19) and Corollary 2.2, we obtain the following theorem.

Theorem 2.4. *For* $n \in \mathbb{N}$

$$\xi_{n,q} = \frac{G_{n+1,q}}{n+1}. \tag{2.7}$$

From (1.18) we derive

$$\begin{aligned} F_q(x,t) &= [2]_q \sum_{n=0}^{\infty} (-1)^n e^{[n+x]_q t} = q^x t \frac{[2]_q}{q^x t} e^{[x]_q t} \sum_{n=0}^{\infty} (-1)^n e^{q^x [n]_q t} \\ &= e^{[x]_q t} \sum_{n=0}^{\infty} q^{nx} \frac{G_{n+1,q}}{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}}{k+1} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

By (2.8), we easily see that

$$\xi_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \frac{G_{k+1,q}}{k+1}. \tag{2.9}$$

This formula can be considered as the q-extension of (1.3). Let us consider the q-analogue of Genocchi polynomials as follows:

$$g_q(x,t) = [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.10}$$

Thus, we note that $\lim_{q \to 1} g_q(x,t) = (2t/(e^t+1)) e^{xt} = \sum_{n=0}^{\infty} G_n(x)(t^n/n!)$. From (2.10), we easily derive

$$G_{n,q}(x) = [2]_q n \left(\frac{1}{1-q} \right)^{n-1} \sum_{l=0}^{n-1} \frac{(-1)^l}{1+q^l} q^{lx} \binom{n-1}{l}. \tag{2.11}$$

By (2.10) we also see that

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} &= [2]_q t \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t} = [2]_q t \sum_{a=0}^{m-1} (-1)^a \sum_{k=0}^{\infty} (-1)^k e^{[k+(a+x)/m]_{q^m} [m]_q t} \\ &= \frac{[2]_q}{[m]_q [2]_{q^m}} \sum_{a=0}^{m-1} (-1)^a \left([m]_q t [2]_{q^m} \sum_{k=0}^{\infty} (-1)^k e^{[m]_q t [k+(a+x)/m]_{q^m}} \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{[2]_q}{[m]_q [2]_{q^m}} \sum_{a=0}^{m-1} (-1)^a [m]_q^n G_{n,q^m}\left(\frac{x+a}{m} \right) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{[2]_q}{[2]_{q^m}} [m]_q^{n-1} \sum_{a=0}^{m-1} (-1)^a G_{n,q^m}\left(\frac{x+a}{m} \right) \right) \frac{t^n}{n!}, \quad \text{where } m \in \mathbb{N} \text{ odd}. \end{aligned} \tag{2.12}$$

Therefore, we obtain the following theorem.

Theorem 2.5. *Let $m(=\text{odd}) \in \mathbb{N}$. Then the distribution of the q-Genocchi polynomials will be as follows:*

$$G_{n,q}(x) = \frac{[2]_q}{[2]_{q^m}}[m]_q^{n-1}\sum_{a=0}^{m-1}(-1)^a G_{n,q^m}\left(\frac{x+a}{m}\right), \tag{2.13}$$

where n is positive integer.

Theorem 2.5 will be used to construct the p-adic q-Genocchi measures which will be treated in the next section. Let χ be a primitive Dirichlet character with a conductor $d(=\text{odd}) \in \mathbb{N}$. Then the generalized q-Genocchi numbers attached to χ are defined as

$$g_{\chi,q}(t) = [2]_q t\sum_{a=0}^{d-1}\chi(n)(-1)^n e^{[n]_q t} = \sum_{n=0}^{\infty} G_{n,\chi,q}\frac{t^n}{n!}. \tag{2.14}$$

From (2.14), we derive

$$G_{n,\chi,q} = \frac{[2]_q}{[2]_{q^d}}[d]_q^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a)G_{n,q^d}\left(\frac{a}{d}\right). \tag{2.15}$$

3. p-adic q-Genocchi measures

In this section we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x\log q)$. Let χ be a primitive Dirichlet's character with a conductor $d(=\text{odd}) \in \mathbb{N}$. For any positive integers N, k, and $d(=\text{odd})$, let $\mu_k = \mu_{k,q;G}$ be defined as

$$\mu_k(a+dp^N\mathbb{Z}_p) = (-1)^a[dp^N]_q^{k-1}\frac{[2]_q}{[2]_{q^{dp^N}}}G_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right). \tag{3.1}$$

By using Theorem 2.5 and (3.1), we show that

$$\sum_{i=0}^{p-1}\mu_k(a+idp^N+dp^{N+1}\mathbb{Z}_p) = \mu_k(a+dp^N\mathbb{Z}_p). \tag{3.2}$$

Therefore, we obtain the following theorem.

Theorem 3.1. *Let d be an odd positive integer. For any positive integers N, k, and let $\mu_k = \mu_{k,q;G}$ be defined as*

$$\mu_k(a+dp^N\mathbb{Z}_p) = (-1)^a[dp^N]_q^{k-1}\frac{[2]_q}{[2]_{q^{dp^N}}}G_{k,q^{dp^N}}\left(\frac{a}{dp^N}\right). \tag{3.3}$$

Then μ_k can be extended to a distribution on X.

From the definition of μ_k and (2.15) we note that

$$\int_X \chi(x)d\mu_k(x) = G_{k,\chi,q}. \tag{3.4}$$

By (2.1) and (2.3), it is not difficult to show that

$$G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} G_{k,q}. \tag{3.5}$$

From (3.1) and (3.5) we derive

$$d\mu_k(a) = \lim_{N\to\infty} \mu_k(a + dp^N \mathbb{Z}_p) = k[a]_q^{k-1} d\mu_{-q}(a). \tag{3.6}$$

Therefore, we obtain the following corollary.

Corollary 3.2. *Let k be a positive integer. Then,*

$$G_{k,\chi,q} = \int_X \chi(x) d\mu_k(x) = k \int_X \chi(x)[x]_q^{k-1} d\mu_{-q}(x). \tag{3.7}$$

Moreover,

$$G_{k,q} = k \int_X [x]_q^{k-1} d\mu_{-q}(x). \tag{3.8}$$

Remark 3.3. In the recent paper (see [1]), Cenkci et al. have studied q-Genocchi numbers and polynomials and p-adic q-Genocchi measures. Starting from T. Kim, L.-C. Jang, and H. K. Pak's construction of q-Genocchi numbers [7], they employed the method developed in a series of papers by Kim [see, e.g., [5, 14–16]] and they considerd another q-analogue of Genocchi numbers $G_k(q)$ as

$$G_k(q) = \frac{q(1+q)}{(1-q)^{k-1}} \sum_{m=0}^{k} \binom{k}{m} \frac{m(-1)^{m+1}}{1+q^m}, \tag{3.9}$$

which is easily derived from the generating function

$$F_q^{(G)}(t) = \sum_{k=0}^{\infty} G_k(q)\frac{t^k}{k!} = q(1+q)t\sum_{n=0}^{\infty}(-1)^n q^n e^{[n]t}. \tag{3.10}$$

However, these q-Genocchi numbers and generating function do not seem to be natural ones; in particular, these numbers cannot be represented as a nice Witt's type formula for the p-adic invariant integral on $\mathbb{Z}_p$ and the generating function does not seems to be simple and useful for deriving many interesting identities related to q-Genocchi numbers. By this reason, we consider q-Genocchi numbers and polynomials which are different. Our q-Genocchi numbers and polynomials to treat in this paper can be represented by p-adic q-fermionic integral on $\mathbb{Z}_p$ [9, 13] and this integral representation also can be considered as Witt's type formula for q-Genocchi numbers. These formulae are useful to study congruences and worthwhile identities for q-Genocchi numbers. By using the generating function of our q-Genocchi numbers, we can derive many properties and identities as same as ordinary Genocchi numbers which were well known.

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