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Research Article

Some Geometric Properties of Sequence Spaces Involving Lacunary Sequence

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We introduce new sequence space involving lacunary sequence connected with Cesaro sequence space and examine some geometric properties of this space equipped with Luxemburg norm.

1. Introduction

Let $(X,\|\cdot\|)$ be a real Banach space and let $B(X)$(resp., $S(X)$) be the closed unit ball (resp., the unit sphere) of X. A point $x\in S(X)$ is an H-point of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\|\to 1$ as $n\to\infty$, weak convergence of (x_n) to x (write $x_n\stackrel{w}{\to}x$) implies that $\|x_n-x\|\to 0$ as $n\to\infty$. If every point in $S(X)$ is an H-point of $B(X)$, then X is said to have the property (H). A point $x\in S(X)$ is an extreme point of $B(X)$ if for any $y,z\in S(X)$ the equality $x=(y+z)/2$ implies $y=z$. A point $x\in S(X)$ is a locally uniformly rotund point of $B(X)$(LUR-point) if for any sequence (x_n) in $B(X)$ such that $\|x_n+x\|\to 2$ as $n\to\infty$, there holds $\|x_n-x\|\to 0$ as $n\to\infty$. A Banach space X is said to be rotund (R) if every point of $S(X)$ is an extreme point of $B(X)$. If every point of $S(X)$ is an LUR-point of $B(X)$, then X is said to be locally uniformly rotund (LUR). If X is LUR, then it has R-property. For these geometric notions and their role in mathematics, we refer to the monograph [1–10].

By a lacunary sequence $\theta=(k_r)$ where $k_0=0$, we will mean an increasing sequence of nonnegative integers with $k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$. We write $h_r=k_r-k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [12] as

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - \ell| = 0, \text{ for some } \ell \right\}. \tag{1.1}$$

It is well known that there exists very close connection between the space of lacunary strongly convergent sequences and the space of strongly Cesaro summability sequences. One can find this connection in [11–16]. Because of these connections, a lot of geometric property of Cesaro sequence spaces can generalize the lacunary sequence spaces.

Let w be the space of all real sequences. Let $p = (p_r)$ be a bounded sequence of the positive real numbers. We introduce the new sequence space $l(p,\theta)$ involving lacunary sequence as follows:

$$l(p,\theta) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r} < \infty \right\}. \tag{1.2}$$

Paranorm on $l(p,\theta)$ is given by

$$\|x\|_{l(p,\theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r} \right)^{1/H}, \tag{1.3}$$

where $H = \sup_r p_r$. If $p_r = p$ for all r, we will use the notation $l_p(\theta)$ in place of $l(p,\theta)$. The norm on $l_p(\theta)$ is given by

$$\|x\|_{l_p(\theta)} = \left(\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p} \right)^{1/p}. \tag{1.4}$$

It is easy to see that the space $l(p,\theta)$ with (1.3) is a complete paranormed space.

By using the properties of lacunary sequence in the space $l(p,\theta)$, we get the following sequences. If $\theta = (2^r)$, then $l(p,\theta) = \text{ces}(p)$. If $\theta = (2^r)$ and $p_r = p$ for all r, then $l(p,\theta) = \text{ces}_p$.

For $x \in l(p,\theta)$, let

$$\sigma(x) = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k\in I_r} |x_k| \right)^{p_r} \tag{1.5}$$

and define the Luxemburg norm on $l(p,\theta)$ by

$$\|x\| = \inf \left\{ \rho > 0 : \sigma\left(\frac{x}{\rho}\right) \le 1 \right\}. \tag{1.6}$$

The Luxemburg norm on $l_p(\theta)$ can be reduced to a usual norm on $l_p(\theta)$, that is, $\|x\|_{l_p(\theta)} = \|x\|$. To do this, we have

$$\begin{aligned}\|x\| &= \inf\left\{\rho>0:\sigma\left(\frac{x}{\rho}\right)\le 1\right\}\\ &= \inf\left\{\rho>0:\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{\rho}\right|\right)^{p}\le 1\right\}\\ &= \inf\left\{\rho>0:\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|x_k|\right)^{p}\le \rho^p\right\}\\ &= \inf\left\{\rho>0:\left(\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|x_k|\right)^{p}\right)^{1/p}\le \rho\right\}\\ &= \left(\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}|x_k|\right)^{p}\right)^{1/p} = \|x\|_{l_p(\theta)}.\end{aligned} \tag{1.7}$$

The main purpose of this work is to show that the space $l(p,\theta)$ equipped with Luxemburg norm is a modular space and to investigate the geometric structure of this space.

2. Main results

In this section, first we give some theorems which show the connection between $l(p,\theta)$ and $\text{ces}(p)$.

Theorem 2.1. *If* $\liminf q_r > 1$, *then* $\text{ces}(p) \subset l(p,\theta)$.

Proof. If $\liminf q_r > 1$, then there exists $\delta > 0$ such that $q_r > 1+\delta$ for all $r \ge 2$. Then for $x \in \text{ces}(p)$, we have

$$\begin{aligned}\sum_{r=2}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_i|\right)^{p_r} &= \sum_{r=2}^{\infty}\left(\frac{1}{h_r}\sum_{i=1}^{k_r}|x_i| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r}\\ &\le C\left[\sum_{r=2}^{\infty}\left(\frac{1}{h_r}\sum_{i=1}^{k_r}|x_i|\right)^{p_r}+\sum_{r=2}^{\infty}\left(\frac{1}{h_r}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r}\right]\\ &= C\left[\sum_{r=2}^{\infty}\left(\frac{k_r}{h_r}\frac{1}{k_r}\sum_{i=1}^{k_r}|x_i|\right)^{p_r}+\sum_{r=2}^{\infty}\left(\frac{k_{r-1}}{h_r}\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r}\right],\end{aligned} \tag{2.1}$$

where $C = \max(1, 2^{H-1})$. Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} < \frac{\delta}{1+\delta}, \qquad \frac{k_{r-1}}{h_r} < \frac{1}{\delta}. \tag{2.2}$$

By using (2.2), we have

$$\sum_{r=2}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x_i|\right)^{p_r} \le C\left[\sum_{r=2}^{\infty}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}|x_i|\right)^{p_r}+\sum_{r=2}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r}\right]. \tag{2.3}$$

Since $x \in \text{ces}(p)$, we get that

$$\sum_{r=2}^{\infty}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}|x_i|\right)^{p_r} < \infty,$$
$$\sum_{r=2}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r} < \infty. \tag{2.4}$$

So we obtain that $x \in l(p,\theta)$. □

Theorem 2.2. *If* $1 < \limsup q_r < \infty$ *, then* $l(p,\theta) \subset \text{ces}(p)$.

Proof. We suppose that $1 < \limsup q_r < \infty$, then there exists positive number K such that $1 < q_r < K$ for all $r \geq 2$. Then if m is any integer with $k_{r-1} < m \leq k_r$ and $x \in l(p,\theta)$, we can write

$$\left(\frac{1}{m}\sum_{i=1}^{m}|x_i|\right)^{p_r} \leq \left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}|x_i|\right)^{p_r},$$
$$\begin{aligned}\sum_{m=1}^{\infty}\left(\frac{1}{m}\sum_{i=1}^{m}|x_i|\right)^{p_m} &\leq C\left[\sum_{r=2}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r} + \sum_{r=2}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{i\in I_r}|x_i|\right)^{p_r}\right] \\ &\leq C\left[\sum_{r=2}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|x_i|\right)^{p_r} + \sum_{r=2}^{\infty}\left(\frac{h_r}{k_{r-1}}\frac{1}{h_r}\sum_{i\in I_r}|x_i|\right)^{p_r}\right].\end{aligned} \tag{2.5}$$

Since $h_r/k_{r-1} = (k_r - k_{r-1})/k_{r-1} = q_r - 1 < K - 1$, we get $l(p,\theta) \subset \text{ces}(p)$. □

Now we give some lemmas about convex modular on $l(p,\theta)$

Lemma 2.3. *The functional* σ *is a convex modular on* $l(p,\theta)$

Proof. It is clear that $\sigma(x) = 0 \Leftrightarrow x = 0$ and $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$. Let $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of $|t| \rightarrow |t|^{p_r}$ for every $r \in N$, we have

$$\begin{aligned}\sigma(\alpha x + \beta y) &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|\alpha x(i) + \beta y(i)|\right)^{p_r} \\ &\leq \alpha\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} + \beta\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|y(i)|\right)^{p_r} \\ &= \alpha\sigma(x) + \beta\sigma(y).\end{aligned} \tag{2.6}$$

□

Lemma 2.4. *For* $x \in l(p,\theta)$ *, the modular* σ *on* $l(p,\theta)$ *satisfies the following properties:*

(i) *if* $0 < a < 1$, *then* $a^H\sigma(x/a) \leq \sigma(x)$ *and* $\sigma(ax) \leq a\sigma(x)$;
(ii) *if* $a > 1$, *then* $\sigma(x) \leq a^H\sigma(x/a)$;
(iii) *if* $a \geq 1$, *then* $\sigma(x) \leq a\sigma(x/a) \leq \sigma(ax)$.

Proof. (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}\sigma(x) &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} = \sum_{r=1}^{\infty}\left(\frac{1}{h_r}a\sum_{i\in I_r}\left|\frac{x(i)}{a}\right|\right)^{p_r} \\ &= \sum_{r=1}^{\infty}a^{p_r}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)}{a}\right|\right)^{p_r} \geq a^H\sigma\left(\frac{x}{a}\right).\end{aligned} \tag{2.7}$$

The property $\sigma(ax) \leq a\sigma(x)$ follows from the convexity of σ

(ii) Let $a > 1$. Then we have

$$\begin{aligned}\sigma(x) &= \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r} = \sum_{r=1}^{\infty}\left(\frac{1}{h_r}a\sum_{i\in I_r}\left|\frac{x(i)}{a}\right|\right)^{p_r} \\ &= \sum_{r=1}^{\infty}a^{p_r}\left(\frac{1}{h_r}\sum_{i\in I_r}\left|\frac{x(i)}{a}\right|\right)^{p_r} \leq a^H\sigma\left(\frac{x}{a}\right).\end{aligned} \tag{2.8}$$

(iii) follows from the convexity of σ. □

By the following lemma, we give some connections between the modular σ and the Luxemburg norm on $l(p,\theta)$.

Lemma 2.5. *For any $x \in l(p,\theta)$,*

(i) *if* $\|x\| < 1$, *then* $\sigma(x) \leq \|x\|$;

(ii) *if* $\|x\| > 1$, *then* $\sigma(x) \geq \|x\|$;

(iii) $\|x\| = 1$ *if and only if* $\sigma(x) = 1$;

(iv) $\|x\| < 1$ *if and only if* $\sigma(x) < 1$;

(v) $\|x\| > 1$ *if and only if* $\sigma(x) > 1$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$. Then we obtain that $\varepsilon + \|x\| < 1$. By definition of norm, there exists $\rho > 0$ such that $\varepsilon + \|x\| > \rho$ and $\sigma(x/\rho) \leq 1$. By (i) and (iii) of Lemma 2.4, we have

$$\begin{aligned}\sigma(x) &\leq \sigma\left(\frac{(\varepsilon+\|x\|)x}{\rho}\right) = \sigma\left((\varepsilon+\|x\|)\frac{x}{\rho}\right) \\ &\leq (\varepsilon+\|x\|)\sigma\left(\frac{x}{\rho}\right) \leq \varepsilon+\|x\|.\end{aligned} \tag{2.9}$$

Hence, we obtain that $\sigma(x) \leq \|x\|$, and (i) is satisfied.

(ii) If $\|x\| > 1$, then $0 > 1 - \|x\|$ and $0 > (1 - \|x\|)/\|x\|$. Hence, we get that $(\|x\| - 1)/\|x\| > 0$. Let $\varepsilon > 0$ be such that $0 < \varepsilon < (\|x\| - 1)/\|x\|$. Since $(\|x\| - 1)/\|x\| > 0$ and $\|x\|(\varepsilon - 1) < -1$, we can write $-1/(\|x\|(\varepsilon - 1)) < 1 < 1/(\|x\|(\varepsilon - 1))$. By definition of $\|.\|$ and Lemma 2.4(i), we have

$$1 < \sigma\left(\frac{x}{(1-\varepsilon)\|x\|}\right) \leq \frac{1}{(1-\varepsilon)\|x\|}\sigma(x). \tag{2.10}$$

So $(1-\varepsilon)\|x\| \leq \sigma(x)$ for all $\varepsilon \in (0,(\|x\|-1)/\|x\|)$, which implies that $\|x\| \leq \sigma(x)$.

(iii) Assume that $\|x\| = 1$. Let $\varepsilon > 0$, then there exists $\rho > 0$ such that $1+\varepsilon > \rho > \|x\|$ and $\sigma(x/\rho) \le 1$. By Lemma 2.4(i), we have $\sigma(x) \le \rho^H \sigma(x/\rho) \le \sigma(x/\rho) \le \rho^H < (1+\varepsilon)^H$, so $(\sigma(x))^{1/H} \le 1+\varepsilon$ for all $\varepsilon > 0$ which implies that $\sigma(x) \le 1$. If $\sigma(x) < 1$, let $a \in (0,1)$ such that $\sigma(x) < a^H < 1$. From Lemma 2.4(i), we have $\sigma(x/a) \le (1/a^H)\sigma(x) \le 1$, hence $\|x\| \le a < 1$, which is a contradiction. Thus, we have $\sigma(x) = 1$.

Conversely, assume that $\sigma(x) = 1$, by the definition of $\|.\|$ we get that $\|x\| \le 1$. If $\|x\| \le 1$, then by (i), we have that $\sigma(x) < \|x\|$, which contradicts to our assumption, so we obtain that $\|x\| = 1$.

(iv) follows from (i) and (iii).

(v) follows from (iii) and (iv). □

Lemma 2.6. *For $x \in l(p,\theta)$,*

(i) *if $0 < a < 1$ and $\|x\| > a$, then $\sigma(x) > a^H$;*

(ii) *if $a \ge 1$ and $\|x\| < a$, then $\sigma(x) < a^H$.*

Proof. (i) We suppose that $0 < a < 1$ and $\|x\| > a$. Then $\|x/a\| > 1$. By Lemma 2.5(ii), we have $\sigma(x/a) > \|x/a\| > 1$. Hence, by Lemma 2.4(i), we get that $\sigma(x/a) \ge a^H \sigma(x/a) > a^H$.

(ii) We suppose that $a > 1$ and $\|x\| < a$. Then $\|x/a\| < 1$. By Lemma 2.5(i), $\sigma(x/a) < \|x/a\| < 1$. If $a = 1$, we have $\sigma(x) < 1$, by Lemma 2.4(ii), we obtain that $\sigma(x) < a^H \sigma(x/a) < a^H$. □

Lemma 2.7. *Let (x_n) be a sequence in $l(p,\theta)$,*

(i) *if $\lim_{n\to\infty}\|x_n\| = 1$, then $\lim_{n\to\infty}\sigma(x_n) = 1$;*

(ii) *if $\lim_{n\to\infty}\sigma(x_n) = 0$, then $\lim_{n\to\infty}\|x_n\| = 0$.*

Proof. (i) Suppose that $\lim_{n\to\infty}\|x_n\| = 1$. Let $\varepsilon \in (0,1)$. Then there exists n_0 such that $1-\varepsilon < \|x_n\| < 1+\varepsilon$ for all $n \ge n_0$. By Lemma 2.6, $(1-\varepsilon)^H < \|x_n\| < (1+\varepsilon)^H$ for all $n \ge n_0$, which implies that $\lim_{n\to\infty}\sigma(x_n) = 1$.

(ii) Suppose that $\|x_n\| \nrightarrow 0$. Then there is an $\varepsilon \in (0,1)$ and subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \varepsilon$ for all $k \in N$. By Lemma 2.6(i), we obtain that $\sigma(x_{n_k}) > \varepsilon^H$ for all $k \in N$. This implies $\sigma(x_{n_k}) \nrightarrow 0$ as $n \to \infty$. □

Lemma 2.8. *Let (x_n) be a sequence in $l(p,\theta)$. If $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in N$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. We put that

$$\sigma_0(x) = \sum_{r=1}^{r_0}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r}, \qquad \sigma_1(x) = \sum_{r=r_0+1}^{\infty}\left(\frac{1}{h_r}\sum_{i\in I_r}|x(i)|\right)^{p_r}. \tag{2.11}$$

Since $\sigma(x) < \infty$, there exists $r_0 \in N$ such that

$$\sigma_1(x) < \frac{\varepsilon}{3}\frac{1}{2^{H+1}}. \tag{2.12}$$

Again, since $\sigma(x_n) - \sigma_0(x_n) \to \sigma(x) - \sigma_0(x)$ as $n \to \infty$, there exists $n_0 \in N$ such that

$$\sigma_1(x_n) = \sigma(x_n) - \sigma_0(x_n) \le \sigma(x) - \sigma_0(x) + \frac{\varepsilon}{3}\frac{1}{2^{H+1}}. \tag{2.13}$$

Also since $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in N$, we can take

$$\sigma_0(x_n - x) \le \frac{\varepsilon}{3} \tag{2.14}$$

for all $n \ge n_0$. It follows from (2.12), (2.13), and (2.14) for all $n \ge n_0$,

$$\begin{aligned}
\sigma(x_n - x) &= \sigma_0(x_n - x) + \sigma_1(x_n - x) \le \frac{\varepsilon}{3} + 2^H(\sigma_1(x_n) + \sigma_1(x)) \\
&\le \frac{\varepsilon}{3} + 2^H\left(\sigma(x) - \sigma_0(x) + \frac{\varepsilon}{3}\frac{1}{2^H} + \sigma_1(x)\right) \\
&= \frac{\varepsilon}{3} + 2^H\left(2\sigma_1(x) + \frac{\varepsilon}{3}\frac{1}{2^H}\right) \\
&\le \frac{\varepsilon}{3} + 2^H\left(\frac{\varepsilon}{3}\frac{2}{2^{H+1}} + \frac{\varepsilon}{3}\frac{1}{2^H}\right) = \varepsilon.
\end{aligned} \tag{2.15}$$

This show that $\sigma(x_n - x) \to 0$ as $n \to \infty$. Hence, by Lemma 2.7(ii), we get that $\|x_n - x\| \to 0$ as $n \to \infty$. □

Theorem 2.9. *The space $l(p,\theta)$ has the property (H).*

Proof. Let $x \in S(l(p,\theta))$ and $x_n(i) \subseteq l(p,\theta)$ such that $\|x_n(i)\| \to 1$ and $x_n(i) \xrightarrow{w} x(i)$ as $n \to \infty$. From Lemma 2.5(iii), we get $\sigma(x) = 1$. So from Lemma 2.6(i), it follows that $\sigma(x_n) \to \sigma(x)$ as $n \to \infty$. Since mapping $\pi_i : l(p,\theta) \to R$ defined by $\pi_i(y_i) = y(i)$ is a continuous linear functional on $l(p,\theta)$. It follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in N$. □

Corollary 2.10. *For $1 \le p < \infty$, the space $l_p(\theta)$ with respect to Luxemburg norm has H-property.*

Proof. The proof is obtained directly form Theorem 2.9. □

Remark 2.11. For a bounded sequence of positive real numbers $p = (p_r)$ with $p_r \ge 1$ for all $r \in N$, the space $l(p,\theta)$ equipped with the Luxemburg norm is not rotund, so it is not *LUR*. In [9], it is shown that the space $ces(p)$ equipped with the Luxemburg norm is not rotund nor *LUR*. Since $ces(p) \subset l(p,\theta)$ from Theorem 2.1, we obtain that the space $l(p,\theta)$ has neither *R*-property nor *LUR* property. Furthermore, if we take lacunary sequence $\theta = (k_r) = \{2^{r-1}, r \text{ even}; 2^r, r \text{ odd}\}$ and $x = \{0,0,0,0,0,2,3,0,0,0,\dots\}$, $y = \{1,1,0,0,0,0,0,0,\dots\}$, we get that the space $l(p,\theta)$ is not rotund.

Indeed, we take $x, y \in S(l(p,\theta))$ such that $\sigma(x) = \sigma(y) = 1$. Since $\sigma((x+y)/2) \ne 1$, we have $\|(x+y)/2\| \ne 1$ by Lemma 2.5(iii). This shows that $l(p,\theta)$ is not rotund, so it is not *LUR*.

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