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Research Article

Hilbert's Type Linear Operator and Some Extensions of Hilbert's Inequality

Yongjin Li, Zhiping Wang, and Bing He

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The norm of a Hilbert's type linear operator $T: L^2(0,\infty)\to L^2(0,\infty)$ is given. As applications, a new generalizations of Hilbert integral inequality, and the result of series analogues are given correspondingly.

1. Introduction

At the close of the 19th century a theorem of great elegance and simplicity was discovered by D. Hilbert.

Theorem 1.1 (Hilbert's double series theorem). *The series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n} \tag{1.1}$$

is convergent whenever $\sum_{n=1}^{\infty} a_n^2$ is convergent.

The Hilbert's inequalities were studied extensively; refinements, generalizations, and numerous variants appeared in the literature (see [1, 2]). Firstly, we will recall some Hilbert's inequalities. If $f(x), g(x)\ge 0$, $0<\int_0^\infty f^2(x)dx<\infty$ and $0<\int_0^\infty g^2(x)dx<\infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy<\pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.2}$$

where the constant factor π is the best possible. Inequality (1.2) is named of Hardy-Hilbert's integral inequality (see [3]). Under the same condition of (1.2), we have the

Hardy-Hilbert's type inequality (see [3], Theorem 319, Theorem 341) similar to (1.2):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dx\,dy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.3}$$

where the constant factor 4 is also the best possible. The corresponding inequalities for series are:

$$\begin{aligned}\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} &< \pi\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2};\\ \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} &< 4\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2},\end{aligned} \tag{1.4}$$

where the constant factors π and 4 are both the best possible.

Let H be a real separable Hilbert space, and $T : H \to H$ be a bounded self-adjoint semi-positive definite operator, then (see [4])

$$(x, Ty)^2 \le \frac{\|T\|^2}{2}\left[\|x\|^2\|y\|^2 + (x,y)^2\right], \tag{1.5}$$

where $x, y \in H$ and $\|x\| = \sqrt{(x,x)}$ is the norm of x.

Set $H = L^2(0,\infty) = \{f(x) : \int_0^\infty f^2(x)dx < \infty\}$ and define $T : L^2(0,\infty) \to L^2(0,\infty)$ as the following:

$$(Tf)(y) := \int_0^\infty \frac{1}{x+y} f(x)dx, \tag{1.6}$$

where $y \in (0,\infty)$. It is easy to see T is a bounded operator (see [5]). By (1.5), one has the sharper form of Hilbert's inequality as (see [4]),

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\,dy \le \frac{\pi}{\sqrt{2}}\left\{\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx + \left(\int_0^\infty f(x)g(x)dx\right)^2\right\}^{1/2}. \tag{1.7}$$

Recently, Yang [6, 7] studied the Hilbert's inequalities by the norm of some Hilbert's type linear operators.

The main purpose of this article is to study the norm of a Hilbert's type linear operator with the kernel $A\min\{x,y\} + B\max\{x,y\}$ and give some new generalizations of Hilbert's inequality. As applications, we also consider some particular results.

2. Main results and applications

LEMMA 2.1. *Define the weight function* $\varpi(x)$ *as*

$$\begin{aligned}\varpi(x) &:= \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dy, \quad x\in(0,\infty),\\ \varpi(y) &\triangleq \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx, \quad y\in(0,\infty).\end{aligned}\tag{2.1}$$

Then $\varpi(x)=\varpi(y)=D(A,B)$ *is a constant and* $0<D(A,B)<\infty$.

In particular, one has $D(1,1)=\pi$ *and* $D(1,0)=4$.

Proof. For fixed x, letting $t=y/x$, we get

$$\begin{aligned}\varpi(x) &= \int_0^\infty \frac{1}{A\ \min\{x,y\}+B\ \max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dy\\ &= \int_0^\infty \frac{1}{A\ \min\{1,t\}+B\ \max\{1,t\}}t^{-1/2}dt\\ &= \int_0^1 \frac{1}{At+B}t^{-1/2}dt+\int_1^\infty \frac{1}{A+Bt}t^{-1/2}dt\\ &= \frac{1}{\sqrt{AB}}\int_0^{A/B}\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+t}t^{-1/2}dt\\ &\le \frac{1}{\sqrt{AB}}\int_0^\infty\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_0^\infty \frac{1}{1+t}t^{-1/2}dt\\ &= \frac{2}{\sqrt{AB}}B\left(\frac{1}{2},\frac{1}{2}\right)<\infty.\end{aligned}\tag{2.2}$$

therefore $0<D(A,B)<\infty$. Moreover,

$$\begin{aligned}\varpi(y) &= \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx\\ &= \int_0^\infty \frac{1}{A\min\{1,t\}+B\max\{1,t\}}t^{-1/2}dt\\ &= \int_0^1 \frac{1}{At+B}t^{-1/2}dt+\int_1^\infty \frac{1}{A+Bt}t^{-1/2}dt\\ &= \frac{1}{\sqrt{AB}}\frac{A^{-1+(1/2)}}{B^{1/2}}\int_0^{A/B}\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+t}t^{-1/2}dt\\ &= \frac{1}{\sqrt{AB}}\int_0^{A/B}\frac{1}{1+u}u^{-1/2}du+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+u}u^{-1/2}du\end{aligned}\tag{2.3}$$

(setting $t=1/u$).

Thus $\varpi(y) = D(A,B)$. In particular:

$$D(1,1) = \int_0^\infty \frac{1}{x+y}\left(\frac{y}{x}\right)^{1/2} dx = \int_0^\infty \frac{1}{1+t} t^{-1/2} dt = \pi,$$
$$D(0,1) = \int_0^\infty \frac{1}{\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2} dx = \int_0^\infty \frac{1}{\max\{1,t\}} t^{-1/2} dt = 4. \tag{2.4}$$

□

THEOREM 2.2. *Let* $A \geq 0$, $B > 0$ *and* $T : L^2(0,\infty) \to L^2(0,\infty)$ *is defined as follows:*

$$(Tf)(y) := \int_0^\infty \frac{1}{A\min\{x,y\} + B\max\{x,y\}} f(x)dx \quad (y \in (0,\infty)). \tag{2.5}$$

Then $\|T\| = D(A,B)$, *and for any* $f(x), g(x) \geq 0$, $f,g \in L^2(0,\infty)$, *one has* $(Tf,g) < D(A,B)\|f\|\|g\|$, *that is,*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\} + B\max\{x,y\}} dx\, dy < D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.6}$$

where the constant factor $D(A,B)$ *is the best possible. In particular,*

(i) *for* $A = B = 1$, *it reduces to Hardy-Hilbert's inequality:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}; \tag{2.7a}$$

(ii) *for* $A = 0$, $B = 1$, *it reduces to Hardy-Hilbert's type inequality:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dx\, dy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{2.7b}$$

Proof. For $A > 0$, $B > 0$. Applying Hölder's inequality, we obtain

$$\begin{aligned}(Tf,g) &= \left(\int_0^\infty \frac{f(x)}{A\,\min\{x,y\} + B\,\max\{x,y\}} dx, g(y)\right)\\ &= \int_0^\infty \left(\int_0^\infty \frac{f(x)}{A\,\min\{x,y\} + B\,\max\{x,y\}} dx\right) g(y)dy\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy \\
&= \int_0^\infty \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left[f(x)\left(\frac{x}{y}\right)^{1/4}\right]\left[g(y)\left(\frac{y}{x}\right)^{1/4}\right]dx\,dy \\
&\leq \left\{\int_0^\infty \int_0^\infty \frac{f^2(x)}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dx\,dy\right\}^{1/2} \\
&\quad\times\left\{\int_0^\infty \int_0^\infty \frac{g^2(y)}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx\,dy\right\}^{1/2} \\
&= \left\{\int_0^\infty \varpi(x)f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty \varpi(y)g^2(y)dy\right\}^{1/2} \\
&= D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)dy\right\}^{1/2} \\
&= D(A,B)\|f\|\|g\|.
\end{aligned}
\tag{2.8}
$$

Thus $\|T\| \leq D(A,B)$ and the inequality (2.6) holds.

Assume that (2.8) takes the form of the equality, then there exist constants a and b, such that they are not both zero and (see [8])

$$af^2(x)\left(\frac{x}{y}\right)^{1/2} = bg^2(y)\left(\frac{y}{x}\right)^{1/2}. \tag{2.9}$$

Then, we have

$$af^2(x)x = bg^2(y)y \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.10}$$

Hence there exist a constant d, such that

$$af^2(x)x = bg^2(y)y = d \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.11}$$

Without losing the generality, suppose $a \neq 0$, then we obtain $f^2(x) = d/(ax)$, a.e. on $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.8) takes the form of strict inequality, we obtain (2.6).

For $\varepsilon > 0$ sufficiently small, set $f_\varepsilon(x) = x^{(-1-\varepsilon)/2}$, for $x \in [1,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,1)$. Then $g_\varepsilon(y) = y^{(-1-\varepsilon)/2}$, for $y \in [1,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,1)$. Assume that the constant factor $D(A,B)$ in (2.6) is not the best possible, then there exist a positive real number K

with $K < D(A,B)$, such that (2.6) is valid by changing $D(A,B)$ to K. On one hand,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy < K\left\{\int_0^\infty f_\varepsilon^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g_\varepsilon^2(x)dx\right\}^{1/2} = K/\varepsilon. \tag{2.12}$$

On the other hand, setting $t = y/x$, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy \\
&\quad= \int_1^\infty \int_1^\infty \frac{x^{(-1-\varepsilon)/2}y^{(-1-\varepsilon)/2}}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy \\
&\quad= \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx \\
&\quad= \int_1^\infty x^{-1-\varepsilon}\int_0^\infty \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx \\
&\qquad- \int_1^\infty x^{-1-\varepsilon}\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx.
\end{aligned} \tag{2.13}$$

For $x \geq 1$, we get

$$\begin{aligned}
&\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt \\
&\quad= \int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{At+B}dt \\
&\quad\leq \frac{1}{B}\int_0^{1/x} t^{(-1-\varepsilon)/2}dt \\
&\quad= \frac{1}{B}\frac{1}{1-(1+\varepsilon)/2}\left(\frac{1}{x}\right)^{1-(1+\varepsilon)/2} \\
&\quad\leq \frac{4}{B}x^{-1/4}
\end{aligned} \tag{2.14}$$

(setting $0 < \varepsilon < 1/2$).

Thus

$$\begin{aligned}
0 &< \int_1^\infty x^{-1-\varepsilon}\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx \\
&\leq \frac{4}{B}\int_1^\infty x^{-1-\varepsilon-1/4}dx \\
&\leq \frac{4}{B}\int_1^\infty x^{-1-1/4}dx = \frac{16}{B}.
\end{aligned} \tag{2.15}$$

Note that

$$\int_1^\infty x^{-1-\varepsilon}\int_0^{1/x}\frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx=O(1). \tag{2.16}$$

So the inequality $\int_0^\infty\int_0^\infty(f_\varepsilon(x)g_\varepsilon(y)/(A\min\{x,y\}+B\max\{x,y\}))dxdy=(1/\varepsilon)[D(A,B)+o(1)]-O(1)=(1/\varepsilon)[D(A,B)+o(1)]$. Thus we get $(1/\varepsilon)[D(A,B,p)+o(1)]\le K/\varepsilon$, that is, $D(A,B)\le K$ when ε is sufficiently small, which contradicts the hypothesis. Hence the constant factor $D(A,B)$ in (2.6) is the best possible and $\|T\|=D(A,B)$. This completes the proof. □

Theorem 2.3. *Suppose that* $f\ge 0$, $A\ge 0$, $B>0$ *and* $0<\int_0^\infty f^2(x)dx<\infty$. *Then*

$$\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy<D^2(A,B)\int_0^\infty f^2(x)dx, \tag{2.17}$$

where the constant factor $D^2(A,B)$ *is the best possible. Inequality (2.17) is equivalent to (2.6).*

Proof. Let $g(y)=\int_0^\infty(f(x)/(A\min\{x,y\}+B\max\{x,y\}))dx$, then by (2.6), we get

$$\begin{aligned}0&<\int_0^\infty g^2(y)dy\\&=\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy\\&=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\&\le D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)dy\right\}^{1/2}.\end{aligned} \tag{2.18}$$

Hence, we obtain

$$0<\int_0^\infty g^2(y)dy=D^2(A,B)\int_0^\infty f^2(x)dx<\infty. \tag{2.19}$$

By (2.6), both (2.18) and (2.19) take the form of strict inequality, so we have (2.17). On the other hand, suppose that (2.17) is valid. By Hölder's inequality, we find

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\&\quad=\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]g(y)dy\\&\quad\le\left\{\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}.\end{aligned} \tag{2.20}$$

By (2.17), we have (2.6). Thus (2.6) and (2.17) are equivalent.

If the constant $D^2(A,B)$ in (2.17) is not the best possible, by (2.20), we may get a contradiction that the constant factor in (2.6) is not the best possible. This completes the proof. □

It is easy to see that for $A = 1$, $B = 1$, the inequality (2.17) reduces to

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}dx\right]^2 dy < \pi^2 \int_0^\infty f^2(x)dx, \tag{2.21a}$$

and for $A = 0$, $B = 1$, the inequality (2.17) reduces to

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\right]^2 dy < 16 \int_0^\infty f^2(x)dx, \tag{2.21b}$$

where both the constant factors π^2 and 16 are the best possible.

3. The corresponding theorem for series

Theorem 3.1. *Suppose that* $a_n, b_n \geq 0$, $A \geq 0$, $B > 0$, *and* $0 < \sum_{n=1}^\infty a_n^2 < \infty$, $0 < \sum_{n=1}^\infty b_n^2 < \infty$. *Then*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{A\min\{m,n\} + B\max\{m,n\}} < D(A,B)\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}, \tag{3.1}$$

$$\sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{a_m}{A\min\{m,n\} + B\max\{m,n\}}\right]^2 < D^2(A,B)\sum_{n=1}^\infty a_n^2, \tag{3.2}$$

where the constant factor $D(A,B)$ *and* $D^2(A,B)$ *are both the best possible, (3.1) and (3.2) are equivalent. In particular,*

(i) *for* $A = 1$, $B = 1$, *it reduces to Hardy-Hilbert's inequality:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}; \tag{3.3a}$$

(ii) *for* $A = 0$, $B = 1$, *it reduces to Hardy-Hilbert's type inequality:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < 4\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}. \tag{3.3b}$$

Proof. Define the weight function $\omega(n)$ as

$$\omega(n) := \sum_{m=1}^\infty \frac{1}{A\min\{m,n\} + B\max\{m,n\}}\left(\frac{n}{m}\right)^{1/2}, \quad n \in N. \tag{3.4}$$

Then we obtain

$$\omega(n) < \varpi(n) = D(A,B). \tag{3.5}$$

Using the method similar to Theorem 2.2 and applying Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}} \le \left[\sum_{n=1}^{\infty}\omega(n)a_n^2\right]^{1/2}\left[\sum_{n=1}^{\infty}\omega(n)b_n^2\right]^{1/2}. \tag{3.6}$$

By (3.5), we obtain (3.1).

For $\varepsilon > 0$ sufficiently small, setting $\widetilde{a}_n = n^{-(1+\varepsilon)/2}$, $\widetilde{b}_n = n^{-(1+\varepsilon)/2}$, then

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{A\min\{m,n\}+B\max\{m,n\}} > \int_1^{\infty}\int_1^{\infty}\frac{f_\varepsilon(x)g_\varepsilon(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy,\\ \left\{\sum_{n=1}^{\infty}\widetilde{a}_n^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}\widetilde{b}_n^2\right\}^{1/2} = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < 1+\int_1^{\infty}\frac{1}{t^{1+\varepsilon}} = 1+\frac{1}{\varepsilon}.\end{gathered} \tag{3.7}$$

If the constant factor $D(A,B)$ in (3.1) is not the best possible, then applying the result of Theorem 2.2, we can get the contradiction. Let $b_n = \sum_{m=1}^{\infty}(a_m/(A\min\{m,n\}+B\max\{m,n\}))$ and we can obtain the following relation:

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m,n\}+B\max\{m,n\}}\right]^2\\ &\quad= \sum_{n=1}^{\infty}b_n^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}}.\end{aligned} \tag{3.8}$$

Applying (3.1) and the method similar to Theorem 2.3, we get (3.2), and (3.2) is equivalent to (3.1) with the best constant. □

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Yongjin Li: Institute of Logic and Cognition, Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China
Email address: stslyj@mail.sysu.edu.cn

Zhiping Wang: Department of Mathematics, Guangdong University of Finance, Zhaoqing Campus, Zhaoqing 526060, China
Email address: m001wzp@126.com

Bing He: Department of Mathematics, Guangdong Education College, Guangzhou 510303, China
Email address: hzs314@163.com