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Research Article

New Strengthened Carleman's Inequality and Hardy's Inequality

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In this note, new upper bounds for Carleman's inequality and Hardy's inequality are established.

1. Introduction

The following Carleman's inequality and Hardy's inequality are well known.

Theorem 1.1 (see [1, Theorem 334]). *Let* $a_n \geq 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{1.1}$$

Theorem 1.2 (see [1, Theorem 349]). *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} \left(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}$$

In [2–16], some refined work on Carleman's inequality and Hardy's inequality had been gained. It is observing that in [3] the authors obtained the following inequalities

$$\left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1/5}\right)^{1/2} < e < \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n+1/6}\right)^{1/2}. \tag{1.3}$$

From the inequality above, [3, 4] extended Theorems A and B to the following new results.

Theorem 1.3 (see [3, Theorem 1]). *Let* $a_n \geq 0 (n \in N)$ *and* $0 < \sum_{n=1}^{\infty} a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n + 1/5}\right)^{-1/2} a_n. \tag{1.4}$$

Theorem 1.4 (see [4, Theorem]). *Let* $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_n$, $a_n \geq 0\,(n \in N)$ *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$, *then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \lambda_n \left(1 + \frac{1}{\Lambda_n/\lambda_n + 1/5}\right)^{-1/2} a_n. \tag{1.5}$$

In this note, Carleman's inequality and Hardy's inequality are strengthened as follows.

Theorem 1.5. *Let* $a_n \geq 0\,(n \in N)$, $0 < \sum_{n=1}^{\infty} a_n < +\infty$, *and* $c \geq \sqrt{6}/4$. *Then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2cn + 4c/3 + 1/2}\right)^{c} a_n. \tag{1.6}$$

Theorem 1.6. *Let* $c \geq \sqrt{6}/4$, $0 < \lambda_{n+1} \leq \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$, $a_n \geq 0\,(n \in N)$, *and* $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$. *Then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/\Lambda_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2c\Lambda_n + (4c/3 + 1/2)\lambda_n}\right)^{c} \lambda_n a_n. \tag{1.7}$$

In order to prove two theorems mentioned above, we need introduce several lemmas first.

2. Lemmas

Lemma 2.1. *Let* $x > 0$ *and* $c \geq \sqrt{6}/4$. *Then inequality*

$$\left(1 + \frac{1}{x}\right)^{x} \left(1 + \frac{1}{2cx + 4c/3 - 1/2}\right)^{c} < e \tag{2.1}$$

or

$$\left(1 + \frac{1}{x}\right)^{x} < e \left(1 - \frac{1}{2cx + 4c/3 + 1/2}\right)^{c} \tag{2.2}$$

holds. Furthermore, $4c/3 - 1/2$ *is the best constant in inequality (2.1) or* $4c/3 + 1/2$ *is the best constant in inequality (2.2).*

Proof. (i) We construct a function as

$$f(x) = x \ln\left(1 + \frac{1}{x}\right) + c \ln\left(1 + \frac{1}{2cx + b}\right) - 1, \tag{2.3}$$

where $x \in (0,+\infty)$ and $b = 4c/3 - 1/2$. It is obvious that the existence of Lemma 2.1 can be ensured when proving $f(x) < 0$. We simply compute

$$\begin{aligned} f'(x) &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} + 2c^2\left(\frac{1}{2cx+b+1} - \frac{1}{2cx+b}\right), \\ f''(x) &= -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} + 4c^3\left(\frac{1}{(2cx+b)^2} - \frac{1}{(2cx+b+1)^2}\right) \\ &= -\frac{1}{x(x+1)^2} + \frac{4c^3(4cx+2b+1)}{(2cx+b)^2(2cx+b+1)^2} \\ &= -\frac{p(x)}{x(x+1)^2(2cx+b)^2(2cx+b+1)^2}, \end{aligned} \tag{2.4}$$

where $p(x) = (24b^2c^2 + 24bc^2 + 4c^2 - 16c^4 - 16bc^3 - 8c^3)x^2 + (8b^3c + 4b^2c + 4bc - 8bc^3 - 4c^3)x + b^2(b+1)^2$. Since $x \in (0,+\infty)$, $b = 4c/3 - 1/2$, and $c \geq \sqrt{6}/4$, we have

$$\begin{aligned} 24b^2c^2 + 24bc^2 + 4c^2 - 16c^4 - 16bc^3 - 8c^3 &\geq 0, \\ 8b^3c + 4b^2c + 4bc - 8bc^3 - 4c^3 &> 0, \\ b^2(b+1)^2 &> 0. \end{aligned} \tag{2.5}$$

From the above analysis, we easily get that $f''(x) < 0$ and $f'(x)$ is decreasing on $(0,+\infty)$. Meanwhile $f'(x) > \lim_{x\to+\infty} f'(x) = 0$ for $x \in (0,+\infty)$. Thus, $f(x)$ is increasing on $(0,+\infty)$, and $f(x) < \lim_{x\to+\infty} f(x) = 0$ for $x \in (0,+\infty)$.

(ii) The inequality (2.2) is equivalent to

$$\frac{e^{1/c}}{e^{1/c} - (1+1/x)^{x/c}} - 2cx < \frac{4}{3}c + \frac{1}{2}, \quad x > 0. \tag{2.6}$$

Let $g(t) = (1+t)^{1/(ct)}$ and $t > 0$. Then

$$\begin{aligned} g'(0^+) &= \lim_{t\to 0^+} \frac{(1+t)^{1/(ct)}}{c}\left[\frac{1}{t(1+t)} - \frac{\log(1+t)}{t^2}\right] = -\frac{e^{1/c}}{2c}, \\ g''(0^+) &= \lim_{t\to 0^+} \frac{(1+t)^{1/(ct)}}{c^2}\left[\frac{1}{t(1+t)} - \frac{\log(1+t)}{t^2}\right]^2 \\ &\quad + \lim_{t\to 0^+} \frac{(1+t)^{1/(ct)}\left[-3t^2 - 2t + 2(1+t^2)\log(1+t)\right]}{ct^3(1+t)^2} \\ &= \left(\frac{1}{4c^2} + \frac{2}{3c}\right)e^{1/c}. \end{aligned} \tag{2.7}$$

Using Taylor's formula, we have

$$g(t) = e^{1/c} - \frac{e^{1/c}}{2c}t + \frac{1}{2}\left(\frac{1}{4c^2} + \frac{2}{3c}\right)e^{1/c}t^2 + o(t^2). \tag{2.8}$$

When letting $x = 1/t$ and using (2.8) we find that

$$\begin{aligned}\lim_{x\to+\infty}\left[\frac{e^{1/c}}{e^{1/c}-(1+1/x)^{x/c}}-2cx\right] &= \lim_{t\to 0^+}\frac{te^{1/c}-2c\left[e^{1/c}-(1+t)^{1/(ct)}\right]}{t\left[e^{1/c}-(1+t)^{1/(ct)}\right]}\\ &= \lim_{t\to 0^+}\frac{(1/(4c)+2/3)e^{1/c}t^2+o(t^2)}{e^{1/c}t^2/(2c)+o(t^2)}\\ &= \frac{4}{3}c+\frac{1}{2}.\end{aligned} \tag{2.9}$$

Therefore, $4c/3+1/2$ is the best constant in (2.2). □

LEMMA 2.2. *The inequality*

$$\left(1+\frac{1}{n+1/5}\right)^{1/2} < \left(1+\frac{2}{3n+1}\right)^{3/4} \tag{2.10}$$

holds for every positive integer n.

Proof. Let

$$h(x) = \frac{1}{2}\ln\left(1+\frac{1}{x+1/5}\right)-\frac{3}{4}\ln\left(1+\frac{2}{3x+1}\right) \tag{2.11}$$

for $x\in[1,+\infty)$, then

$$h'(x) = \frac{x/5-7/25}{2(x+6/5)(x+1/5)(x+1)(3x+1)}. \tag{2.12}$$

Thus, $h(x)$ is decreasing on $[1,7/5)$. Since for $h(1)<0$, we have $h(x)<0$ on $[1,7/5)$. At the same time, $h(x)$ is increasing on $[7/5,+\infty)$, and we have $h(x) < \lim_{x\to+\infty}h(x)=0$ on $[7/5,+\infty)$. Hence $h(x) < 0$ on $[1,+\infty)$. By the definition of $h(x)$, it turns out that the inequality (2.10) is accrate.

In the same way we can prove the following result. □

LEMMA 2.3. *The inequality*

$$\left(1+\frac{2}{3n+1}\right)^{3/4} < \left(1+\frac{1}{(5/4)n+1/3}\right)^{5/8} \tag{2.13}$$

holds for every positive integer n.

Combining Lemmas 2.1, 2.2, and 2.3 gives

LEMMA 2.4. *The inequality*

$$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1/5}\right)^{1/2} < \left(1+\frac{1}{n}\right)^n\left(1+\frac{2}{3n+1}\right)^{3/4} < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{(5/4)n+1/3}\right)^{5/8} < e \tag{2.14}$$

holds for every positive integer n.

3. Proof of Theorem 1.5

By the virtue of the proof of article [3], we can testify Theorem 1.5. Assume that $c_n > 0$ for $n \in N$. Then applying the arithmetic-geometric average inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} &= \sum_{n=1}^{\infty}(c_1c_2\cdots c_n)^{-1/n}(c_1a_1c_2a_2\cdots c_na_n)^{1/n}\\ &\le \sum_{n=1}^{\infty}(c_1c_2\cdots c_n)^{-1/n}\frac{1}{n}\sum_{m=1}^{n}c_ma_m \\ &= \sum_{m=1}^{\infty}c_ma_m\sum_{n=m}^{\infty}\frac{1}{n}(c_1c_2\cdots c_n)^{-1/n}.\end{aligned} \tag{3.1}$$

Setting $c_m = (m+1)^m/m^{m-1}$, we have $c_1c_2\cdots c_n = (n+1)^n$ and

$$\begin{aligned}\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} &\le \sum_{m=1}^{\infty}c_ma_m\sum_{n=m}^{\infty}\frac{1}{n(n+1)}\\ &= \sum_{m=1}^{\infty}\frac{1}{m}c_ma_m \\ &= \sum_{m=1}^{\infty}\left(1+\frac{1}{m}\right)^m a_m.\end{aligned} \tag{3.2}$$

By (3.2) and (2.2), we obtain

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\frac{1}{2cn+4c/3+1/2}\right)^c a_n. \tag{3.3}$$

Thus, Theorem 1.5 is proved.

4. Proof of Theorem 1.6

Now, processing the proof of Theorem 1.6. Assume that $c_n > 0$ for $n \in N$. Using the arithmetic-geometric average inequality we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda n}\right)^{1/\Lambda_n} &= \sum_{n=1}^{\infty}\frac{\lambda_{n+1}}{\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/\Lambda_n}}\left[(c_1a_1)^{\lambda_1}(c_2a_2)^{\lambda_2}\cdots(c_na_n)^{\lambda_n}\right]^{1/\Lambda_n}\\ &\le \sum_{n=1}^{\infty}\frac{\lambda_{n+1}}{\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/\Lambda_n}}\frac{1}{\Lambda_n}\sum_{m=1}^{n}\lambda_mc_ma_m\\ &= \sum_{m=1}^{\infty}\lambda_mc_ma_m\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n\left(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}\right)^{1/\Lambda_n}}.\end{aligned} \tag{4.1}$$

Choosing $c_n = (1+\lambda_{n+1}/\Lambda_n)^{\Lambda_n/\lambda_n}\Lambda_n$, we get that

$$\begin{aligned}
\sum_{n=1}^{\infty}\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n} &\le \sum_{m=1}^{\infty}\left(1+\frac{\lambda_{m+1}}{\Lambda_m}\right)^{\Lambda_m/\lambda_m}\lambda_m a_m \\
&\le \sum_{m=1}^{\infty}\left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m}\lambda_m a_m \\
&< e\sum_{m=1}^{\infty}\left(1-\frac{1}{2c(\Lambda_m/\lambda_m)+4c/3+1/2}\right)^{c}\lambda_m a_m \\
&= e\sum_{m=1}^{\infty}\left(1-\frac{\lambda_m}{2c\Lambda_m+(4c/3+1/2)\lambda_m}\right)^{c}\lambda_m a_m,
\end{aligned} \tag{4.2}$$

from (4.1) and (2.2).

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