

Research Article

New Inequalities Similar to Hardy-Hilbert Inequality and their Applications

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Received 25 January 2007; Revised 7 July 2007; Accepted 22 November 2007

Recommended by Lars-Erik Persson

Two classes of new inequalities similar to Hardy-Hilbert inequality are showed by introducing some parameters a, b, c and two real functions $\phi(x)$ and $\psi(x)$. Some applications are obtained.

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1. Introduction

The following inequality is well known as Hardy-Hilbert inequality:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{1/q}, \quad (1.1)$$

where $\pi/\sin(\pi/p)$ is the best value (see Hardy et al. [1]).

Integral analogues of (1.1) are the following inequalities:

$$\iint_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left(\int_0^{\infty} f^2(x) dx \int_0^{\infty} g^2(y) dy \right)^{1/2}, \quad (1.2)$$
$$\int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} dx \right)^2 dy \leq \pi^2 \int_0^{\infty} f^2(x) dx,$$

where π is the best value (cf., [1, Chapter 9]).

In recent years, Gao [2], Yang [3–5], Yang and Debnath [6], Kuang [7], and Kuang and Debnath [8] gave some distinct improvements and generalizations of (1.1)-(1.2).

Yang and Rassias [9] gave a new inequality with a best constant factor similar to (1.1) as

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{n=2}^{\infty} n^{p-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=2}^{\infty} n^{q-1} b_n^q \right\}^{1/q}, \tag{1.3}$$

where $\pi/\sin(\pi/p)$ is the best possible.

In this paper, we have two major objectives. One is motivated by [10], to give a generalization of (1.3) by introducing two real functions $\phi(x)$ and $\psi(x)$. The other is to build a class of new inequalities similar to Hardy-Hilbert inequality (1.2) by introducing some parameters a, b , and c .

2. Some lemmas

First, we give the β function $B(m, n)$:

$$B\left(\frac{1}{p}, \frac{1}{q}\right) = \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/q} du, \tag{2.1}$$

where $p > 1, 1/p + 1/q = 1$.

LEMMA 2.1. *Let $b > a \geq 1 - c$, and*

$$\omega(a, b, x) = \int_a^b \frac{1}{(y+c)\ln(x+c)(y+c)} \left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2} dy, \tag{2.2}$$

provided the generalized integral exists. Then

$$\omega(a, b, x) \leq \pi - 4 \arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}}; \tag{2.3}$$

$$\omega(0, b, x) = \lim_{a \rightarrow 0} \omega(a, b, x) \leq \pi - 4 \arctan \sqrt[4]{\frac{\ln c}{\ln(b+c)}}; \tag{2.4}$$

$$\omega(a, \infty, x) = \lim_{b \rightarrow \infty} \omega(a, b, x) \leq \pi - 2 \arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(x+c)}}. \tag{2.5}$$

Proof. Putting $u = \ln(y+c)/\ln(x+c)$, we have

$$\begin{aligned} \omega(a, b, x) &= \int_a^b \frac{1}{(y+c)\ln(x+c)(y+c)} \left(\frac{\ln(x+c)}{\ln(y+c)}\right)^{1/2} dy = \int_{\ln(a+c)/\ln(x+c)}^{\ln(b+c)/\ln(x+c)} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/2} du \\ &= \int_0^{\infty} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/2} du - \int_{\ln(b+c)/\ln(x+c)}^{\infty} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/2} du - \int_0^{\ln(a+c)/\ln(x+c)} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/2} du \\ &= \pi - \left(\int_0^{\ln(x+c)/\ln(b+c)} \frac{1}{1+v} \left(\frac{1}{v}\right)^{1/2} dv + \int_0^{\ln(a+c)/\ln(x+c)} \frac{1}{1+u} \left(\frac{1}{u}\right)^{1/2} du \right) \\ &= \pi - \left(2 \arctan \sqrt{\frac{\ln(x+c)}{\ln(b+c)}} + 2 \arctan \sqrt{\frac{\ln(a+c)}{\ln(x+c)}} \right). \end{aligned} \tag{2.6}$$

Since $\arctan x$ is strictly increasing, then

$$\begin{aligned} \omega(a, b, x) &= \pi - 2 \arctan \frac{\sqrt{\ln(x+c)/\ln(b+c)} + \sqrt{\ln(a+c)/\ln(x+c)}}{1 - \sqrt{\ln(a+c)/\ln(b+c)}} \\ &\leq \pi - 2 \arctan \frac{2\sqrt{\ln(a+c)/\ln(b+c)}}{1 - \sqrt{\ln(a+c)/\ln(b+c)}} = \pi - 4 \arctan \sqrt{\frac{\ln(a+c)}{\ln(b+c)}}. \end{aligned} \tag{2.7}$$

Relation (2.3) is valid. By (2.3) as $a \rightarrow 0$, we have

$$\omega(0, b, x) = \lim_{a \rightarrow 0} \omega(a, b, x) \leq \pi - 4 \arctan \sqrt{\frac{\ln c}{\ln(b+c)}}. \tag{2.8}$$

Relation (2.4) is valid. Similarly, (2.5) is also valid. The lemma is proved. □

LEMMA 2.2. Let $0 < \alpha < 1$, $0 \leq c < 1$, $g(s) \in C^1[c, 1]$, $g(s) > 0$, $g'(s) > 0$ for all $s \in [c, 1]$, and $F(x) = \int_c^x (s^{-\alpha}/g(s)) ds$ for all $x \in [c, 1]$. Then

$$F(x) \geq \frac{x^{1-\alpha} - c^{1-\alpha}}{1 - c^{1-\alpha}} F(1). \tag{2.9}$$

Proof. Let $\tau = s^{1-\alpha}$, then

$$F(x) = \int_c^x \frac{s^{-\alpha}}{g(s)} ds = \frac{1}{1-\alpha} \int_{c^{1-\alpha}}^{x^{1-\alpha}} \frac{1}{g(\tau^{1/(1-\alpha)})} d\tau. \tag{2.10}$$

Let $G(y) = (1/1-\alpha) \int_{c^{1-\alpha}}^y (1/g(\tau^{1/(1-\alpha)})) d\tau$. Since $G'(y) > 0$, $G''(x) \leq 0$ in $[c^{1-\alpha}, 1]$, and $G(y)$ is concave in $[c^{1-\alpha}, 1]$, then

$$\begin{aligned} G(y) &= G\left(\frac{1-y}{1-c^{1-\alpha}} c^{1-\alpha} + \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}\right) = G((1-\lambda)c^{1-\alpha} + \lambda) \quad \left(\lambda = \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}}\right) \\ &\geq \frac{1-y}{1-c^{1-\alpha}} G(c^{1-\alpha}) + \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}} G(1) = \frac{y-c^{1-\alpha}}{1-c^{1-\alpha}} G(1). \end{aligned} \tag{2.11}$$

Thus

$$F(x) = G(x^{1-\alpha}) \geq \frac{x^{1-\alpha} - c^{1-\alpha}}{1 - c^{1-\alpha}} F(1). \tag{2.12}$$

The lemma is proved. □

Let

$$F_{1,r}(x) = \int_0^{\ln(a+c)/\ln(x+c)} \frac{u^{-1/r}}{1+u} du, \quad F_{2,r}(x) = \int_0^{\ln(x+c)/\ln(b+c)} \frac{u^{-1/r}}{1+u} du, \tag{2.13}$$

where $r > 1$, $1 - c \leq a \leq x \leq b$.

If $g(s) = 1 + s$ and $\alpha = 1/r$ in Lemma 2.2, we get the following.

LEMMA 2.3. Let $1 - c < a \leq x \leq b < +\infty$, $p > 1$, $1/p + 1/q = 1$. Then

$$\begin{aligned} F_{1,q}(x) + F_{2,p}(x) &\geq \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p} \Phi(q) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q} \Phi(p) \\ &\geq \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq} (q\Phi(q))^{1/q} (p\Phi(p))^{1/p}; \end{aligned} \tag{2.14}$$

$$\begin{aligned} F_{1,p}(x) + F_{2,q}(x) &\geq \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/q} \Phi(p) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/p} \Phi(q) \\ &\geq \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq} (q\Phi(q))^{1/q} (p\Phi(p))^{1/p}, \end{aligned} \tag{2.15}$$

where $\Phi(r) = \int_0^1 (u^{-1/r}/1+u) du$.

Proof. For $1 - c < a \leq x \leq b < +\infty$, by Lemma 2.2, we have

$$\begin{aligned} F_{1,q}(x) + F_{2,p}(x) &\geq \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p} \Phi(q) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q} \Phi(p), \\ F_{1,p}(x) + F_{2,q}(x) &\geq \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/q} \Phi(p) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/p} \Phi(q). \end{aligned} \tag{2.16}$$

Let $\alpha = 1/p$, $\beta = 1/q$, $p_1 = 1 + \alpha/\beta$, $q_1 = 1 + \beta/\alpha$, then

$$\frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{\alpha}{p_1} + \frac{\beta}{q_1} = \frac{2\alpha\beta}{\alpha + \beta}, \quad \alpha + \beta = 1. \tag{2.17}$$

By Young inequality, we get

$$\begin{aligned} &\left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{1/p} \Phi(q) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{1/q} \Phi(p) \\ &= \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^\alpha \Phi(q) + \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^\beta \Phi(p) \\ &= \frac{1}{p_1} \left(p_1^{1/p_1} \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{\alpha/p_1} (\Phi(q))^{1/p_1}\right)^{p_1} + \frac{1}{q_1} \left(q_1^{1/q_1} \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{\beta/q_1} (\Phi(p))^{1/q_1}\right)^{q_1} \\ &\geq \left(p_1^{1/p_1} \left(\frac{\ln(a+c)}{\ln(x+c)}\right)^{\alpha/p_1} (\Phi(q))^{1/p_1}\right) \left(q_1^{1/q_1} \left(\frac{\ln(x+c)}{\ln(b+c)}\right)^{\beta/q_1} (\Phi(p))^{1/q_1}\right) \\ &= \left(1 + \frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} \left(1 + \frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)} \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{\alpha\beta/(\alpha+\beta)} \times (\Phi(q))^{\beta/(\alpha+\beta)} (\Phi(p))^{\alpha/(\alpha+\beta)} \\ &= \left(\frac{\ln(a+c)}{\ln(b+c)}\right)^{1/pq} (q\Phi(q))^{1/q} (p\Phi(p))^{1/p}. \end{aligned} \tag{2.18}$$

Then (2.14) is valid.

In the same way, (2.15) can be obtained. This completes the proof. □

LEMMA 2.4. Let $p > 1$, $1/p + 1/q = 1$, $\phi(x)$ and $\psi(x)$ are continuously differentiable functions on (a, b) , $\phi(a) \geq 1$, $\phi'(x) > 0$, $\psi(a) \geq 1$, $\psi'(x) > 0$, $\inf_x \phi'(x) \neq 0$, and $\inf_x \psi'(x) \neq 0$, provided that the generalized integral exists. Then

$$\begin{aligned} & \int_a^b \frac{1}{\psi(y) \ln \phi(x) \psi(y)} \left(\frac{\ln \phi(x)}{\ln \psi(y)} \right)^{1/q} dy \\ & \leq \frac{1}{\inf \{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(a)}{\ln \psi(b)} \right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q} \right), \end{aligned} \quad (2.19)$$

where Φ is as in Lemma 2.3.

Proof. Putting $u = \ln \psi(y) / \ln \phi(x)$, by Lemma 2.2 and the proof of Lemma 2.3, we have

$$\begin{aligned} & \int_a^b \frac{1}{\psi(y) \ln \phi(x) \psi(y)} \left(\frac{\ln \phi(x)}{\ln \psi(y)} \right)^{1/q} dy \\ & = \int_{\ln \psi(a) / \ln \phi(x)}^{\ln \psi(b) / \ln \phi(x)} \frac{1}{1+u} \left(\frac{1}{u} \right)^{1/q} \frac{1}{\psi'(y)} du \\ & \leq \frac{1}{\inf \{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \int_0^{\ln \psi(a) / \ln \phi(x)} \frac{1}{1+u} \left(\frac{1}{u} \right)^{1/q} du - \int_0^{\ln \phi(x) / \ln \psi(b)} \frac{1}{1+u} \left(\frac{1}{u} \right)^{1/p} du \right) \\ & \leq \frac{1}{\inf \{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(a)}{\ln \phi(x)} \right)^{1/p} \Phi(q) - \left(\frac{\ln \phi(x)}{\ln \psi(b)} \right)^{1/q} \Phi(p) \right) \\ & \leq \frac{1}{\inf \{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(a)}{\ln \psi(b)} \right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q} \right). \end{aligned} \quad (2.20)$$

The lemma is proved. □

Remark 2.5. When $a = 1$, and $b = \infty$, we get

$$\begin{aligned} \int_1^\infty \frac{1}{\psi(y) \ln \phi(x) \psi(y)} \left(\frac{\ln \phi(x)}{\ln \psi(y)} \right)^{1/q} dy & \leq \frac{1}{\inf \{\psi'(y)\}} \left(\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(1)}{\ln \phi(x)} \right)^{1/p} \Phi(q) \right) \\ & \leq \frac{1}{\inf \{\psi'(y)\}} \frac{\pi}{\sin(\pi/p)}. \end{aligned} \quad (2.21)$$

3. Main results

Now, we introduce main results.

THEOREM 3.1. *Let $-c \leq a < b < +\infty$, f, g are integrable nonnegative functions on $[a, b]$ such that $0 < \int_a^b (x+c)f^2(x)dx < \infty$ and $0 < \int_a^b (y+c)g^2(y)dy < \infty$. Then*

$$\begin{aligned} & \iint_a^b \frac{f(x)g(y)}{\ln(x+c)(y+c)} dx dy \\ & \leq \left(\pi - 4 \arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}} \right) \left(\int_a^b (x+c)f^2(x)dx \int_a^b (y+c)g^2(y)dy \right)^{1/2}. \end{aligned} \tag{3.1}$$

Proof. By Cauchy-Schwarz inequality and (2.3), we have

$$\begin{aligned} & \iint_a^b \frac{f(x)g(y)}{\ln(x+c)\ln(y+c)} dx dy \\ & = \iint_a^b \frac{f(x)}{(\ln(x+c)(y+c))^{1/2}} \left(\frac{\ln(x+c)}{\ln(y+c)} \right)^{1/4} \left(\frac{x+c}{y+c} \right)^{1/2} \\ & \quad \times \frac{g(y)}{(\ln(x+c)(y+c))^{1/2}} \left(\frac{\ln(y+c)}{\ln(x+c)} \right)^{1/4} \left(\frac{y+c}{x+c} \right)^{1/2} dx dy \\ & \leq \left[\iint_a^b \frac{f^2(x)}{\ln(x+c)(y+c)} \left(\frac{\ln(x+c)}{\ln(y+c)} \right)^{1/2} \frac{x+c}{y+c} dx dy \right]^{1/2} \\ & \quad \times \left[\iint_a^b \frac{g^2(y)}{\ln(x+c)(y+c)} \left(\frac{\ln(y+c)}{\ln(x+c)} \right)^{1/2} \frac{y+c}{x+c} dx dy \right]^{1/2} \\ & = \left[\int_a^b (x+c)f^2(x) \left(\int_a^b \frac{1}{(y+c)\ln(x+c)(y+c)} \left(\frac{\ln(x+c)}{\ln(y+c)} \right)^{1/2} dy \right) dx \right]^{1/2} \\ & \quad \times \left[\int_a^b (y+c)g^2(y) \left(\int_a^b \frac{1}{(x+c)\ln(x+c)(y+c)} \left(\frac{\ln(y+c)}{\ln(x+c)} \right)^{1/2} dx \right) dy \right]^{1/2} \\ & \leq \left(\pi - 4 \arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}} \right) \left(\int_a^b (x+c)f^2(x)dx \int_a^b (y+c)g^2(y)dy \right)^{1/2}. \end{aligned} \tag{3.2}$$

Then relation (3.1) is valid. Theorem 3.1 is proved. □

In a similar way to the proof of Theorem 3.1, we can prove the following theorem.

THEOREM 3.2. Let $1 - c \leq a < b < +\infty$, f is an integrable nonnegative function on $[a, b]$, such that $0 < \int_a^b (x+c)f^2(x)dx < \infty$, then

$$\int_a^b \left(\int_a^b \frac{f(x)}{\ln(x+c)(y+c)} dx \right)^2 dy \leq \left(\pi - 4 \arctan \sqrt[4]{\frac{\ln(a+c)}{\ln(b+c)}} \right)^2 \int_a^b (x+c)f^2(x)dx. \tag{3.3}$$

Remark 3.3. Specially, when $a = 0$, $c = 1$, and $b = \infty$ in Theorems 3.1 and 3.2, we get

$$\begin{aligned} \iint_0^\infty \frac{f(x)g(y)}{\ln(x+1)(y+1)} dx dy &\leq \pi \left(\int_0^\infty (x+1)f^2(x)dx \right)^{1/2} \left(\int_0^\infty (y+1)g^2(y)dy \right)^{1/2}; \\ \int_0^\infty \left(\int_0^\infty \frac{f(x)}{\ln(x+1)(y+1)} dx \right)^2 dy &\leq \pi^2 \int_0^\infty (x+1)f^2(x)dx. \end{aligned} \tag{3.4}$$

THEOREM 3.4. Let $p > 1$, $1/p + 1/q = 1$, f, g are integrable nonnegative functions on $[a, b]$, such that $0 < \int_a^b \phi^{p-1}(x)f^p(x)dx < \infty$ and $0 < \int_a^b \psi^{q-1}(y)g^q(y)dy < \infty$. Then

$$\begin{aligned} \iint_a^b \frac{f(x)g(y)}{\ln \phi(x)\psi(y)} dx dy &\leq \frac{[\pi/\sin(\pi/p) - \phi_1]^{1/p} [\pi/\sin(\pi/p) - \phi_2]^{1/q}}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \\ &\quad \times \left\{ \int_a^b \phi^{p-1}(x)f^p(x)dx \right\}^{1/p} \left\{ \int_a^b \psi^{q-1}(y)g^q(y)dy \right\}^{1/q}; \\ \int_a^b \frac{1}{\psi(y)} \left(\int_a^b \frac{f(x)}{\ln \phi(x)\psi(y)} dx \right)^p dy &\leq \left(\frac{[\pi/\sin(\pi/p) - \phi_1]^{1/p} [\pi/\sin(\pi/p) - \phi_2]^{1/q}}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \right)^p \\ &\quad \times \int_a^b \phi^{p-1}(x)f^p(x)dx, \end{aligned} \tag{3.5}$$

where the $\phi(x)$ and $\psi(y)$ are as in Lemma 2.4 ($\phi_1 = (\ln \psi(a)/\ln \psi(b))^{1/pq}(p\Phi(p))^{1/p} \times (q\Phi(q))^{1/q}$, $\phi_2 = (\ln \phi(a)/\ln \phi(b))^{1/pq}(p\Phi(p))^{1/p}(q\Phi(q))^{1/q}$).

Proof. By Hölder inequality and (2.19), we have

$$\begin{aligned} &\iint_a^b \frac{f(x)g(y)}{\ln \phi(x)\psi(y)} dx dy \\ &= \iint_a^b \left[\frac{f(x)}{[\ln \phi(x)\psi(y)]^{1/p}} \left(\frac{\ln \phi(x)}{\ln \psi(y)} \right)^{1/pq} \frac{\phi(x)^{1/q}}{\psi(y)^{1/p}} \right] \\ &\quad \times \left[\frac{g(y)}{[\ln \phi(x)\psi(y)]^{1/q}} \left(\frac{\ln \psi(y)}{\ln \phi(x)} \right)^{1/pq} \frac{\psi(y)^{1/p}}{\phi(x)^{1/q}} \right] dx dy \end{aligned}$$

$$\begin{aligned}
 &\leq \left\{ \iint_a^b \frac{f^p(x)}{\ln \phi(x)\psi(y)} \left(\frac{\ln \phi(x)}{\ln \psi(y)} \right)^{1/q} \frac{\phi(x)^{p-1}}{\psi(y)} dx dy \right\}^{1/p} \\
 &\quad \times \left\{ \iint_a^b \frac{g^q(y)}{\ln \phi(x)\psi(y)} \left(\frac{\ln \psi(y)}{\ln \phi(x)} \right)^{1/p} \frac{\psi(y)^{q-1}}{\phi(x)} dx dy \right\}^{1/q} \\
 &= \left\{ \int_a^b \omega(\phi, \psi, q, x) f^p(x) dx \right\}^{1/p} \left\{ \int_a^b \omega(\psi, \phi, p, y) g^q(y) dy \right\}^{1/q} \\
 &\leq \frac{1}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \\
 &\quad \times \left\{ \int_a^b \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(a)}{\ln \psi(b)} \right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q} \right] \phi^{p-1}(x) f^p(x) dx \right\}^{1/p} \\
 &\quad \times \left\{ \int_a^b \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \phi(a)}{\ln \phi(b)} \right)^{1/pq} (p\Phi(p))^{1/p} (q\Phi(q))^{1/q} \right] \psi^{q-1}(y) g^q(y) dy \right\}^{1/q} \\
 &\leq \frac{[\pi/\sin(\pi/p) - \phi_3]^{1/p} [\pi/\sin(\pi/p) - \phi_4]^{1/q}}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \\
 &\quad \times \left\{ \int_a^b \phi^{p-1}(x) f^p(x) dx \right\}^{1/p} \left\{ \int_a^b \psi^{q-1}(y) g^q(y) dy \right\}^{1/q}
 \end{aligned} \tag{3.7}$$

Hence (3.5) is valid.

Let $g(y) = (1/\psi(y))(\int_a^b (f(x)/\ln \phi(x)\psi(y))dx)^{p-1} > 0$ ($y \in (a, b)$). By (4.2), we have

$$\begin{aligned}
 0 &< \int_a^b \psi(y)^{q-1} g^q(y) dy = \int_a^b \frac{1}{\psi(y)} \left(\int_a^b \frac{f(x)}{\ln \phi(x)\psi(y)} dx \right)^p dy = \iint_a^b \frac{f(x)g(y)}{\ln \phi(x)\psi(y)} dx dy \\
 &\leq \frac{\pi/\sin(\pi/p)}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \left\{ \int_a^b \phi^{p-1}(x) f^p(x) dx \right\}^{1/p} \left\{ \int_a^b \psi^{q-1}(y) g^q(y) dy \right\}^{1/q}.
 \end{aligned} \tag{3.8}$$

Then we find

$$\begin{aligned}
 &\int_a^b \frac{1}{\psi(y)} \left(\int_a^b \frac{f(x)}{\ln \phi(x)\psi(y)} dx \right)^p dy \\
 &= \int_a^b \psi(y)^{q-1} g^q(y) dy \leq \left(\frac{\pi/\sin(\pi/p)}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \right)^p \int_a^b \phi^{p-1}(x) f^p(x) dx.
 \end{aligned} \tag{3.9}$$

Since $0 < \int_a^b \phi^{p-1}(x) f^p(x) dx$, it follows that $0 < \int_a^b \psi(y)^{q-1} g^q(y) dy < \infty$. Still by (3.5), we have (3.6). The theorem is proved. \square

Remark 3.5. Specially when $a = 1$ and $b = \infty$, we get

$$\begin{aligned} & \iint_1^\infty \frac{f(x)g(y)}{\ln\phi(x)\psi(y)} dx dy \\ & \leq \frac{1}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \\ & \quad \times \left\{ \int_1^\infty \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\psi(1)}{\ln\phi(x)} \right)^{1/p} \Phi(q) \right] \phi^{p-1}(x)f^p(x)dx \right\}^{1/p} \\ & \quad \times \left\{ \int_1^\infty \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln\phi(1)}{\ln\psi(y)} \right)^{1/q} \Phi(p) \right] \psi^{q-1}(y)g^q(y)dy \right\}^{1/q} \end{aligned} \tag{3.10}$$

$$\begin{aligned} & \leq \frac{\pi/\sin(\pi/p)}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \\ & \quad \times \left\{ \int_1^\infty \phi^{p-1}(x)f^p(x)dx \right\}^{1/p} \left\{ \int_1^\infty \psi^{q-1}(y)g^q(y)dy \right\}^{1/q}; \\ & \int_1^\infty \frac{1}{\psi(y)} \left(\int_1^\infty \frac{f(x)}{\ln\phi(x)\psi(y)} dx \right)^p dy \\ & \leq \left(\frac{\pi/\sin(\pi/p)}{(\inf\{\psi'(y)\})^{1/p}(\inf\{\phi'(x)\})^{1/q}} \right)^p \int_1^\infty \phi^{p-1}(x)f^p(x)dx, \end{aligned} \tag{3.11}$$

where Φ is as in Lemma 2.3.

By Theorem 3.4, we have the following corollary.

COROLLARY 3.6. Let $1 - c \leq a < b < +\infty$, $p > 1$, $1/p + 1/q = 1$, f, g are integrable nonnegative functions on $[a, b]$, such that $0 < \int_a^b (x+c)^{p-1} f^p(x)dx < \infty$ and $0 < \int_a^b (y+c)^{q-1} g^q(y)dy < \infty$. Then

$$\begin{aligned} & \iint_a^b \frac{f(x)g(y)}{\ln(x+c)(y+c)} dx dy \leq \left(B\left(\frac{1}{p}, \frac{1}{q}\right) - \left(\frac{\ln(a+c)}{\ln(b+c)} \right)^{1/pq} (q\Phi(q))^{1/q} (p\Phi(p))^{1/p} \right) \\ & \quad \times \left(\int_a^b (x+c)^{p-1} f^p(x)dx \right)^{1/p} \left(\int_a^b (y+c)^{q-1} g^q(y)dy \right)^{1/q}, \end{aligned} \tag{3.12}$$

where Φ is as in Lemma 2.3.

In what follows, we give the associated discrete inequalities. The proofs should be omitted.

THEOREM 3.7. Let $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$, $\{b_n\}$ are nonnegative real sequences, such that $0 < \sum_{n=2}^{\infty} \phi^{p-1}(n)a_n^p < \infty$, $0 < \sum_{n=2}^{\infty} \psi^{q-1}(n)b_n^q < \infty$. Then

$$\begin{aligned} & \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{\ln \phi(m) \psi(n)} \\ & \leq \frac{1}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \\ & \quad \times \left\{ \sum_{m=2}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \psi(1)}{\ln \phi(m)} \right)^{1/p} \Phi(q) \right] \phi^{p-1}(m) a_m^p \right\}^{1/p} \\ & \quad \times \left\{ \sum_{n=2}^{\infty} \left[\frac{\pi}{\sin(\pi/p)} - \left(\frac{\ln \phi(1)}{\ln \psi(n)} \right)^{1/q} \Phi(p) \right] \psi^{q-1}(n) b_n^q \right\}^{1/q} \\ & \leq \frac{\pi/\sin(\pi/p)}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \left\{ \sum_{m=2}^{\infty} \phi^{p-1}(m) a_m^p \right\}^{1/p} \left\{ \sum_{n=2}^{\infty} \psi^{q-1}(n) b_n^q \right\}^{1/q}. \end{aligned} \tag{3.13}$$

where $\phi(x)$ and $\psi(y)$ are as in Lemma 2.4, and Φ is as in Lemma 2.3.

THEOREM 3.8. Let $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$ is nonnegative real sequence, such that $0 < \sum_{n=2}^{\infty} \phi^{p-1}(n)a_n^p < \infty$. Then

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{1}{\psi(n)} \left(\sum_{m=2}^{\infty} \frac{a_m}{\ln \phi(m) \psi(n)} \right)^p \\ & \leq \left(\frac{\pi/\sin(\pi/p)}{(\inf \{\psi'(y)\})^{1/p} (\inf \{\phi'(x)\})^{1/q}} \right)^p \sum_{m=2}^{\infty} \phi^{p-1}(m) a_m^p, \end{aligned} \tag{3.14}$$

where $\phi(x)$ and $\psi(y)$ are as in Lemma 2.4.

Remark 3.9. When $\phi(x) = x$ and $\psi(y) = y$, then inequalities (3.10), (3.11), (3.13), and (3.14) change to (2.4), (2.10), (3.3), and (3.4) in [10], respectively, hence inequalities (3.10), (3.11), (3.13), and (3.14) are generalizations of related results in [10].

4. Some corollaries

By Theorems 3.4, 3.7, and 3.8, some inequalities can also be obtained.

For example, we take $\phi(x)$ and $\psi(y)$ as

$$\phi(x) = e^x, \quad \psi(y) = e^y, \tag{4.1}$$

then by Theorems 3.4, 3.7, and 3.8, we get the following corollaries.

COROLLARY 4.1. Let $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$, $\{b_n\}$ are nonnegative real sequences, such that $0 < \sum_{n=2}^{\infty} e^{(p-1)n} a_n^p < \infty$, $0 < \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q < \infty$. Then

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi/\sin(\pi/p)}{e} \left\{ \sum_{m=2}^{\infty} e^{(p-1)m} a_m^p \right\}^{1/p} \left\{ \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q \right\}^{1/q}; \quad (4.2)$$

$$\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} \frac{a_m}{m+n} \right)^p \leq \left(\frac{\pi/\sin(\pi/p)}{e} \right)^p \sum_{m=2}^{\infty} e^{(p-1)m} a_m^p.$$

COROLLARY 4.2. Let $p > 1$, $1/p + 1/q = 1$, f , g are integrable nonnegative functions on $[a, b]$, such that $0 < \int_0^{\infty} e^{(p-1)t} f^p(t) dt < \infty$, $0 < \int_0^{\infty} e^{(q-1)t} g^q(t) dt < \infty$. Then

$$\iint_1^{\infty} \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi/\sin(\pi/p)}{e} \left\{ \int_1^{\infty} e^{(p-1)x} f^p(x) dx \right\}^{1/p} \left\{ \int_1^{\infty} e^{(q-1)y} g^q(y) dy \right\}^{1/q};$$

$$\int_1^{\infty} \frac{1}{y} \left(\int_1^{\infty} \frac{f(x)}{x+y} dx \right)^p dy \leq \left(\frac{\pi/\sin(\pi/p)}{e} \right)^p \int_1^{\infty} e^{(p-1)x} f^p(x) dx. \quad (4.3)$$

We take $\phi(x)$ and $\psi(y)$ as

$$\phi(x) = x^2, \quad \psi(y) = e^y. \quad (4.4)$$

Then we have the following corollary.

COROLLARY 4.3. Let $p > 1$, $1/p + 1/q = 1$, $\{a_m\}$, $\{b_n\}$ are nonnegative real sequences, such that $0 < \sum_{n=2}^{\infty} n^{2(p-1)} a_n^p < \infty$, $0 < \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q < \infty$. Then

$$\sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \frac{a_m b_n}{2 \ln m + n} \leq \frac{\pi/\sin(\pi/p)}{2^{1/q} e^{1/p}} \left\{ \sum_{m=2}^{\infty} m^{2(p-1)} a_m^p \right\}^{1/p} \left\{ \sum_{n=2}^{\infty} e^{(q-1)n} b_n^q \right\}^{1/q}; \quad (4.5)$$

$$\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} \frac{a_m}{2 \ln m + n} \right)^p \leq \left(\frac{\pi/\sin(\pi/p)}{2^{1/q} e^{1/p}} \right)^p \sum_{m=2}^{\infty} m^{2(p-1)} a_m^p.$$

COROLLARY 4.4. Let $p > 1$, $1/p + 1/q = 1$, f , g are integrable nonnegative functions on $[a, b]$, such that $0 < \int_1^{\infty} x^{2(p-1)} f^p(x) dx < \infty$, $0 < \int_0^{\infty} e^{(q-1)x} g^q(x) dx < \infty$. Then

$$\iint_1^{\infty} \frac{f(x)g(y)}{2 \ln x + y} dx dy \leq \frac{\pi/\sin(\pi/p)}{2^{1/q} e^{1/p}} \left\{ \int_1^{\infty} x^{2(p-1)} f^p(x) dx \right\}^{1/p} \left\{ \int_1^{\infty} e^{(q-1)y} g^q(y) dy \right\}^{1/q};$$

$$\int_1^{\infty} \frac{1}{y} \left(\int_1^{\infty} \frac{f(x)}{2 \ln x + y} dx \right)^p dy \leq \left(\frac{\pi/\sin(\pi/p)}{2^{1/q} e^{1/p}} \right)^p \int_1^{\infty} x^{2(p-1)} f^p(x) dx. \quad (4.6)$$

Remark 4.5. Inequalities (4.2)–(4.6) are also new results.

Acknowledgment

The authors thank the referees for their help and patience in improving the paper. This work is supported by the Natural Science Foundation of China, Project no. 10771181 and the Natural Science Foundation of Jiangsu Higher Education Bureau, Project no. 07KJD110206.

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