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Research Article

Sufficient Conditions for Univalence of an Integral Operator

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In this paper we have introduced an integral general operator. For this general operator which is a generalization of more known integral operators we have demonstrated some univalence properties.

1. Introduction and preliminaries

Let U be the unit disk of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.1}$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U,

$$A_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \cdots,\ z \in U\} \tag{1.2}$$

with $A_1 = A$, and

$$S = \{f \in A : f \text{ is univalent in } U\}. \tag{1.3}$$

Lemma 1.1 (see [1]). *If the function f is regular in the unit disc U,*

$$\begin{gathered} f(z) = z + a_2z^2 + \cdots, \\ (1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad \forall z \in U, \end{gathered} \tag{1.4}$$

then the function f is univalent in U.

Definition 1.2 (St. Ruscheweyh [2]). For $f \in A,\ n \in \mathbb{N} \cup \{0\}$, let R^n be the operator defined by $R^n : A \to A$,

$$\begin{aligned} R^0 f(z) &= f(z), \\ R^1 f(z) &= z f'(z) \\ &\vdots \\ (n+1)R^{n+1} f(z) &= z\left[R^n f(z)\right]' + nR^n f(z), \quad z \in U. \end{aligned} \tag{1.5}$$

Remark 1.3. If $f \in A$

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.6}$$

then

$$R^n f(z) = z + \sum_{j=1}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad z \in U, \tag{1.7}$$

with

$$R^n f(0) = 0, \qquad \left[R^n f(0)\right]' = 1. \tag{1.8}$$

Lemma 1.4 ([3, Schwarz's lemma], [4, Lemma 4.26, page 103]). *If the analytic function $f(z)$ is regular in U with $f(0) = 0$ and $|f(z)| < 1$ for all $z \in U$, then*

$$\left|f(z)\right| \le |z|, \quad \forall z \in U, \tag{1.9}$$

and $|f'(0)| \le 1$.

The equality holds if and only if $f(z) = cz,\ z \in U,\ |c| = 1$.

2. Main results

By using the Ruscheweyh differential operator given by Definition 1.2, we introduce the following integral operator.

Definition 2.1. Let $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1, 2, 3, \dots, m\},\ \alpha_i \in \mathbb{C}$. Define the integral operator $I(f_1, f_2, \dots, f_m) : A^m \to A$,

$$I(f_1, f_2, \dots, f_m)(z) = \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt, \quad z \in U, \tag{2.1}$$

where $f_i(z) \in A$ and R^n is the Ruscheweyh differential operator.

Remark 2.2. (i) For $n = 0,\ m = 1,\ \alpha_1 = 1,\ \alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$,

$$R^0 f(z) = f(z) \in A, \tag{2.2}$$

we obtain Alexander integral operator introduced in 1915 in [5]:

$$I(z) = \int_0^z \frac{f(t)}{t} dt, \quad z \in U. \tag{2.3}$$

(ii) For $n = 0,\ m = 1,\ \alpha_1 = \alpha \in [0,1],\ \alpha_2 = \alpha_3 = \cdots = \alpha_m = 0,\ R^0 f(z) = f(z) \in S$, and we obtain the integral operator

$$I_\alpha(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\alpha dt \tag{2.4}$$

studied in [6].

(iii) For $n = 1,\ m = 1,\ \alpha_1 = \gamma \in \mathbb{C},\ |\gamma| \le 1/4,\ \alpha_2 = \cdots = \alpha_m = 0,\ R^1 f(z) = zf'(z) \in S$, we obtain the integral operator

$$F_\gamma(z) = \int_0^z \left[f'(t)\right]^\gamma dt \tag{2.5}$$

studied in [7, 8].

(iv) For $n = 0,\ m \in \mathbb{N} \cup \{0\},\ \alpha_i \in \mathbb{C},\ i \in \{1,2,\ldots,m\},\ R^0 f(z) = f(z) \in S$, and we obtain the integral operator

$$F(z) = \int_0^z \left[\frac{f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{f_m(t)}{t}\right]^{\alpha_m} dt \tag{2.6}$$

studied in [9].

(v) For $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1,2,\ldots,m\},\ \alpha_i > 0$, we obtain the integral operator $F_m : A^m \to A$,

$$F_m(f_1, f_2, \ldots, f_m)(z) = \int_0^z \left[\frac{R^n f_1(t)}{t}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(t)}{t}\right]^{\alpha_m} dt \tag{2.7}$$

studied in [10].

(vi) For $n = 0,\ m = 1,\ \alpha_1 = \gamma,\ \alpha_2 = \cdots = \alpha_m = 0,\ R^0 f(z) = f(z)$, and we obtain the integral operator

$$F_\gamma(z) = \int_0^z \left[\frac{f(t)}{t}\right]^\gamma dt \tag{2.8}$$

studied in [11, 12].

Theorem 2.3. *Let $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1,2,\ldots,m\},\ \alpha_i \in \mathbb{C},\ f_i \in A$. If*

$$\left|\frac{z(R^n f_i(z))'}{R^n f_i(z)} - 1\right| \le 1, \quad |\alpha_1| + |\alpha_2| + \cdots + |\alpha_m| \le 1, \quad z \in U, \tag{2.9}$$

then $I(f_1, f_2, \ldots, f_m)(z)$ given by (2.1) *is univalent.*

Proof. Since $f_i \in A,\ i \in \{1,2,\dots,m\}$, from Remark 1.3 we have

$$\frac{R^n f_i(z)}{z} = \frac{z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_{j,i} z^j}{z} = 1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_{j,i} z^{j-1}, \qquad \frac{R^n f_i(z)}{z} \neq 0, \quad z \in U. \tag{2.10}$$

For $z = 0$, we have

$$\left[\frac{R^n f_1(z)}{z}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(z)}{z}\right]^{\alpha_m} = 1. \tag{2.11}$$

By differentiating (2.1), we obtain

$$I'(f_1, f_2, \dots, f_m)(z) = \left[\frac{R^n f_1(z)}{z}\right]^{\alpha_1} \cdots \left[\frac{R^n f_m(z)}{z}\right]^{\alpha_m}, \quad z \in U, \qquad I'(f_1, f_2, \dots, f_m)(0) = 1. \tag{2.12}$$

Using (2.12), we obtain

$$\log I'(f_1, f_2, \dots, f_m)(z) = \alpha_1 \left[\log R^n f_1(z) - \log z\right] + \cdots + \alpha_m \left[\log R^n f_m(z) - \log z\right], \quad z \in U. \tag{2.13}$$

By differentiating (2.13), we have

$$\frac{I''(f_1, f_2, \dots, f_m)(z)}{I'(f_1, f_2, \dots, f_m)(z)} = \alpha_1 \left[\frac{(R^n f_1(z))'}{R^n f_1(z)} - \frac{1}{z}\right] + \cdots + \alpha_m \left[\frac{(R^n f_m(z))'}{R^n f_m(z)} - \frac{1}{z}\right], \quad z \in U \tag{2.14}$$

and after a short calculus we obtain

$$\frac{zI''(f_1, f_2, \dots, f_m)(z)}{I'(f_1, f_2, \dots, f_m)(z)} = |\alpha_1| \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1\right] + \cdots + |\alpha_m| \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1\right], \quad z \in U. \tag{2.15}$$

We multiply the modulus of (2.15) by $(1 - |z|^2)$ and we obtain

$$\begin{aligned}
&(1 - |z|^2)\left|\frac{zI''(f_1, f_2, \dots, f_m)(z)}{I'(f_1, f_2, \dots, f_m)(z)}\right| \\
&\quad = (1 - |z|^2)\left|\alpha_1 \left[\frac{z(R^n f_1(z))'}{R^n f_1(z)} - 1\right] + \cdots + \alpha_m \left[\frac{z(R^n f_m(z))'}{R^n f_m(z)} - 1\right]\right| \\
&\quad \leq (1 - |z|^2)\left[|\alpha_1|\left|\frac{z(R^n f_1(z))}{R^n f_1(z)} - 1\right| + \cdots + |\alpha_m|\left|\frac{z(R^n f_m(z))}{R^n f_m(z)} - 1\right|\right] \\
&\quad \leq \left[|\alpha_1| + \cdots + |\alpha_m|\right](1 - |z^2|) \leq |\alpha_1| + \cdots + |\alpha_m| \leq 1.
\end{aligned} \tag{2.16}$$

From Lemma A, we have $I(f_1, f_2, \dots, f_m)(z) \in S$. □

Remark 2.4. (i) For $n = 0,\ R^n f_i(z) = f_i(z) \in S$, we obtain Theorem 2.3 from [9].

(ii) For $\alpha_i \in \mathbb{R},\ \alpha_i > 0$, Theorem 2.3 can be rewritten as follows.

Corollary 2.5. *Let $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1, 2, \dots, m\},\ \alpha_i > 0$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_m \leq 1$. If $f_i \in A$ satisfy*

$$\left| \frac{z\left(R^n f_i(z)\right)'}{R^n f_i(z)} - 1 \right| \leq 1, \quad z \in U, \tag{2.17}$$

then the integral operator given by (2.1) *is univalent.*

Theorem 2.6. *Let $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1, 2, \dots, m\},\ \alpha_i \in \mathbb{C}$. If $f_i \in A$ satisfy*

(i) $|\alpha_1| + \cdots + |\alpha_m| \leq 1/3$,

(ii) $|R^n f_i(z)| \leq 1$,

(iii) $|z^2 (R^n f_i(z))' / (R_i^n f_i(z))^2 - 1| < 1$

for all $z \in U$, then the integral operator given by (2.1) *is univalent.*

Proof. Using (2.14), we obtain

$$\left| \frac{z\left[I(f_1, \dots, f_m)(z)\right]}{\left[I(f_1, \dots, f_m)(z)\right]'} \right|'' = |\alpha_1| \left| \frac{z\left(R^n f_1(z)\right)'}{R^n f_1(z)} - 1 \right| + \dots + |\alpha_m| \left| \frac{z\left(R^n f_m(z)\right)'}{R^n f_m(z)} - 1 \right|. \tag{2.18}$$

We multiply (2.18) by $(1 - |z|^2)$, use Schwarz's lemma, and obtain

$$\begin{aligned}
&(1 - |z|^2) \left| \frac{zT''(z)}{T'(z)} \right| \\
&\quad = (1 - |z|^2) |\alpha_1| \left| \frac{z(R^n f_1(z))}{R^n f_1(z)} - 1 \right| + \cdots + (1 - |z|^2) |\alpha_m| \left| \frac{z\left(R^n f_m(z)\right)'}{R^n f_1(z)} - 1 \right| \\
&\quad = (1 - |z|^2) |\alpha_1| \left| \frac{z\left(R^n f_1(z)\right)'}{R^n f_1(z)} \right| + (1 - |z|^2) |\alpha_1| + \cdots + (1 - |z|^2) |\alpha_m| \left| \frac{z\left(R^n f_m(z)\right)'}{R^n f_1(z)} \right| \\
&\qquad + (1 - |z|^2) |\alpha_m| \\
&\quad = (1 - |z|^2) |\alpha_1| \left[\left| \frac{z\left(R^n f_1(z)\right)'}{R^n f_1(z)} \right| + \cdots + \left| \frac{z\left(R^n f_m(z)\right)'}{R^n f_1(z)} \right| \right] + (1 - |z|^2) \left[|\alpha_1| + \cdots + |\alpha_m| \right] \\
&\quad = \left(1 - |z|^2\right) \left[|\alpha_1| \left| \frac{z^2\left(R^n f_1(z)\right)'}{\left(R^n f_1(z)\right)^2} \right| \frac{|R^n f_1|}{|z|} + \cdots + |\alpha_m| \left| \frac{z^2\left(R^n f_m(z)\right)'}{\left(R^n f_m(z)\right)^2} \right| \frac{|R^n f_m|}{|z|} \right] \\
&\qquad + (1 - |z|^2) \left[|\alpha_1| + \cdots + |\alpha_m| \right]
\end{aligned}$$

$$
\begin{aligned}
&\le \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2}\right| + \cdots + |\alpha_m|\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2}\right|\right] + \left(1-|z|^2\right)\left[|\alpha_1| + \cdots + |\alpha_m|\right] \\
&= \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2}\right| - |\alpha_1| + |\alpha_1|\right] \\
&\quad + \cdots + \left(1-|z|^2\right)\left[|\alpha_m|\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2}\right| - |\alpha_m| + |\alpha_m|\right] + \left(1-|z|^2\right)\left[|\alpha_1| + \cdots + |\alpha_m|\right] \\
&= \left(1-|z|^2\right)\left[|\alpha_1|\left|\frac{z^2(R^n f_1(z))'}{(R^n f_1(z))^2} - 1\right| + \cdots + |\alpha_m|\left|\frac{z^2(R^n f_m(z))'}{(R^n f_m(z))^2} - 1\right|\right] \\
&\quad + \left(1-|z|^2\right)\left(|\alpha_1| + \cdots + |\alpha_m|\right) + \left(1-|z|^2\right)\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&\le \left(1-|z|^2\right)\left(|\alpha_1| + |\alpha_1| + \cdots + |\alpha_m|\right) + 2\left(1-|z|^2\right)\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&= 3\left(1-|z|^2\right)\left(|\alpha_1| + \cdots + |\alpha_m|\right) \\
&\le 3\left(|\alpha_1| + \cdots + |\alpha_m|\right).
\end{aligned} \tag{2.19}
$$

From (2.19) and condition (i), we have

$$
\left(1-|z|^2\right)\left|\frac{zF''(z)}{F'(z)}\right| \le 1 \tag{2.20}
$$

for all $z \in U$.

By Lemma A, it follows that the integral operator $I(f_1, f_2, \ldots, f_m)(z)$ is univalent. □

Remark 2.7. For $n = 0,\ m = 1,\ \alpha_1 = \alpha \in \mathbb{C},\ |\alpha| \le 1/3,\ \alpha_2 = \cdots = \alpha_m = 0$, the result was obtained in [11, Theorem 1].

For $\alpha_i \in \mathbb{R},\ \alpha_i > 0$, Theorem 2.6 can be rewritten as follows.

Corollary 2.8. *Let $n, m \in \mathbb{N} \cup \{0\},\ i \in \{1, 2, \ldots, m\},\ \alpha_i > 0$. If $f_i \in A$ satisfy*

(i) $\alpha_1 + \alpha_2 + \cdots + \alpha_n \le 1/3$,

(ii) $|R^n f_i(z)| \le 1$,

(iii) $|z^2(R^n f_i(z))'/(R^n f_i(z))^2 - 1| < 1$

for all $z \in U$, then the integral operator given by (2.1) *is univalent.*

References

[1] J. Becker, "Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen," *Journal für die reine und angewandte Mathematik*, vol. 255, pp. 23–43, 1972.

[2] St. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.

[3] Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975.
[4] P. Hamburg, P. Mocanu, and N. Negoescu, *Analiză matematică (Funcţii complexe)*, Editura Didactică şi Pedagogică, Bucureşti, Romania, 1982.
[5] J. W. Alexander, "Functions which map the interior of the unit circle upon simple regions," *Annals of Mathematics*, vol. 17, no. 1, pp. 12–22, 1915.
[6] S. S. Miller, P. T. Mocanu, and M. O. Reade, "Starlike integral operators," *Pacific Journal of Mathematics*, vol. 79, no. 1, pp. 157–168, 1978.
[7] Y. J. Kim and E. P. Merkes, "On an integral of powers of a spirallike function," *Kyungpook Mathematical Journal*, vol. 12, pp. 249–252, 1972.
[8] N. N. Pascu and V. Pescar, "On the integral operators of Kim-Merkes and Pfaltzgraff," *Mathematica*, vol. 32(55), no. 2, pp. 185–192, 1990.
[9] D. Breaz and N. Breaz, "Two integral operators," *Studia Universitatis Babeş-Bolyai. Mathematica*, vol. 47, no. 3, pp. 13–19, 2002.
[10] G. I. Oros and G. Oros, "A convexity property for an integral operator F_m," in preparation.
[11] V. Pescar and S. Owa, "Sufficient conditions for univalence of certain integral operators," *Indian Journal of Mathematics*, vol. 42, no. 3, pp. 347–351, 2000.
[12] V. Pescar, "On some integral operations which preserve the univalence," *The Punjab University. Journal of Mathematics*, vol. 30, pp. 1–10, 1997.