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Research Article

New Means of Cauchy's Type

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We will introduce new means of Cauchy's type $M_{r,l}^{s}(f,\mu)$ defined, for example, as $M_{r,l}^{s}(f,\mu) = ((l(l-s)/r(r-s))(M_r^r(f,\mu)-M_s^r(f,\mu)/M_l^l(f,\mu)-M_s^l(f,\mu)))^{1/(r-l)}$, in the case when $l \neq r \neq s$, $l, r \neq 0$. We will show that this new Cauchy's mean is monotonic, that is, the following result. *Theorem*. Let $t, r, u, v \in \mathbb{R}$, such that $t \leq v$, $r \leq u$. Then for $M_{r,l}^{s}(f,\mu)$, one has $M_{t,r}^{s} \leq M_{v,u}^{s}$. We will also give some related comparison results.

1. Introduction

Let Ω be a convex set equipped with a probability measure μ. Then for a strictly monotonic continuous function f, the integral power mean of order $r \in \mathbb{R}$ is defined as follows:

$$M_r(f,\mu) = \begin{cases} \left(\int_\Omega (f(u))^r d\mu(u)\right)^{1/r}, & r \neq 0, \\ \exp\left(\int_\Omega \log(f(u)) d\mu(u)\right), & r = 0. \end{cases} \tag{1.1}$$

Throughout our present investigation, we tacitly assume, without further comment, that all the integrals involved in our results exist.

The following inequality for differences of power means was obtained (see [1, Remark 8]):

$$\left|\frac{r(r-s)}{l(l-s)}\right| m \leq \left|\frac{M_r^r(f,\mu)-M_s^r(f,\mu)}{M_l^l(f,\mu)-M_s^l(f,\mu)}\right| \leq \left|\frac{r(r-s)}{l(l-s)}\right| M, \tag{1.2}$$

where $r, l, s \in \mathbb{R}$, $l \neq r \neq s$, $r, l \neq 0$ and where m and M are, respectively, the minimum and the maximum values of the function x^{r-l} on the image of $f(u)$ $(u \in \Omega)$.

Let us note that (1.2) was obtained as consequence of the following result (see, e.g., [1, Corollary 1]).

Theorem 1.1. *Let $r, s, l \in \mathbb{R}$, and let Ω be a convex set equipped with a probability measure μ. Then,*

$$\frac{M_r^r(f,\mu) - M_s^r(f,\mu)}{M_l^l(f,\mu) - M_s^l(f,\mu)} = \frac{r(r-s)}{l(l-s)}\eta^{r-l} \tag{1.3}$$

for some η in the image of $f(u)$ $(u \in \Omega)$, provided that the denominator on the left-hand side of (1.3) *is non-zero.*

We can also note that from (1.3) we can get the following form of (1.2):

$$\inf_{u\in\Omega} f(u) \leq \left(\frac{l(l-s)}{r(r-s)} \frac{M_r^r(f,\mu) - M_s^r(f,\mu)}{M_l^l(f,\mu) - M_s^l(f,\mu)}\right)^{1/(r-l)} \leq \sup_{u\in\Omega} f(u), \tag{1.4}$$

where $r, l, s \in \mathbb{R}$, $r \neq l \neq s$, $r, l \neq 0$. Moreover, (1.4) suggests introducing a new mean of Cauchy type. We will prove in Section 3 a comparison theorem for these means. Finally we will, in Section 4, give some applications.

2. New Cauchy's mean

From (1.4), we can define a new mean $M_{r,l}^s$ as follows:

$$M_{r,l}^s(f,\mu) = \left(\frac{l(l-s)}{r(r-s)} \frac{M_r^r(f,\mu) - M_s^r(f,\mu)}{M_l^l(f,\mu) - M_s^l(f,\mu)}\right)^{1/(r-l)}, \quad l \neq r \neq s,\ l, r \neq 0. \tag{2.1}$$

Now by taking $\lim_{l\to 0} M_{r,l}^s(f,\mu)$, we will get

$$\begin{aligned} M_{r,0}^s(f,\mu) &= M_{0,r}^s(f,\mu) = \lim_{l\to 0} M_{r,l}^s(f,\mu) \\ &= \left(\frac{s\left[M_r^r(f,\mu) - M_s^r(f,\mu)\right]}{r(r-s)\left[\log M_s(f,\mu) - \log M_0(f,\mu)\right]}\right)^{1/r}, \quad r \neq s,\ r, s \neq 0. \end{aligned} \tag{2.2}$$

Now by taking $\lim_{r\to s} M_{r,l}^s(f,\mu)$, we will get

$$\begin{aligned} \lim_{r\to s} M_{r,l}^s(f,\mu) &= M_{s,l}^s(f,\mu) = M_{l,s}^s(f,\mu) \\ &= \left(\frac{l(l-s)}{s} \frac{\left[\int f(u)^s \log f(u)\,d\mu(u) - M_s^s(f,\mu)\log M_s(f,\mu)\right]}{M_l^l(f,\mu) - M_s^l(f,\mu)}\right)^{1/(s-l)}, \quad l \neq s,\ l, s \neq 0. \end{aligned} \tag{2.3}$$

By similar way, we can calculate all the cases for $r, s, l \in \mathbb{R}$. Finally, we get the following definition of $M^s_{r,l}(f,\mu)$:

$$
\begin{aligned}
M^s_{r,l}(f,\mu) &= \left(\frac{l(l-s)}{r(r-s)}\frac{M^r_r(f,\mu)-M^r_s(f,\mu)}{M^l_l(f,\mu)-M^l_s(f,\mu)}\right)^{1/(r-l)}, \quad l \neq r \neq s,\ l,r \neq 0;\\
M^s_{r,0}(f,\mu) &= M^s_{0,r}(f,\mu) = \left(\frac{s\left[M^r_r(f,\mu)-M^r_s(f,\mu)\right]}{r(r-s)\left[\log M_s(f,\mu)-\log M_0(f,\mu)\right]}\right)^{1/r}, \quad r \neq s, \quad r,s \neq 0;\\
M^s_{s,l}(f,\mu) &= M^s_{l,s}(f,\mu) = \left(\frac{l(l-s)}{s}\frac{\int f(u)^s \log f(u)\,d\mu(u) - M^s_s(f,\mu)\log M_s(f,\mu)}{M^l_l(f,\mu)-M^l_s(f,\mu)}\right)^{1/(s-l)},\\
&\qquad\qquad l \neq s,\ l,s \neq 0;\\
M^s_{s,0}(f,\mu) &= M^s_{0,s}(f,\mu) = \left(\frac{\int f(u)^s \log f(u)\,d\mu(u) - M^s_s(f,\mu)\log M_s(f,\mu)}{\log M_s(f,\mu)-\log M_0(f,\mu)}\right)^{1/s}, \quad s \neq 0;\\
M^0_{r,l}(f,\mu) &= \left(\frac{l^2\left(M^r_r(f,\mu)-M^r_0(f,\mu)\right)}{r^2\left(M^l_l(f,\mu)-M^l_0(f,\mu)\right)}\right)^{1/(r-l)}, \quad l,r \neq 0;\\
M^0_{r,0}(f,\mu) &= M^0_{0,r}(f,\mu) = \left(\frac{2\left[M^r_r(f,\mu)-M^r_0(f,\mu)\right]}{r^2\left[M^2_2(\log f,\mu)-M^2_1(\log f,\mu)\right]}\right)^{1/r}, \quad r \neq 0;\\
M^s_{t,t} &= \exp\left(-\frac{2t-s}{t(t-s)}+\frac{\int f^t \log f\,d\mu(u) - M^t_s(f,\mu)\log M_s(f,\mu)}{M^t_t(f,\mu)-M^t_s(f,\mu)}\right), \quad t \neq s;\\
M^0_{t,t} &= \exp\left(-\frac{2}{t}+\frac{\int f^t \log f\,d\mu(u) - M^t_0(f,\mu)\log M_0(f,\mu)}{M^t_t(f,\mu)-M^t_0(f,\mu)}\right), \quad t \neq 0;\\
M^0_{0,0} &= \exp\left(\frac{1}{3}\frac{\int (\log f)^3 d\mu(u) - \left(\log M_0(f,\mu)\right)^3}{\int (\log f)^2 d\mu(u) - \left(\log M_0(f,\mu)\right)^2}\right),\\
M^s_{s,s} &= \exp\left(-\frac{1}{s}+\frac{\int f^s (\log f)^2 d\mu(u) - M^s_s(f,\mu)\left(\log M_s(f,\mu)\right)^2}{2\left(\int f^s \log f\,d\mu(u) - \left(M^s_s(f,\mu)\log M_s(f,\mu)\right)\right)}\right), \quad s \neq 0;\\
M^s_{0,0} &= \exp\left(\frac{1}{s}+\frac{\int (\log f)^2 d\mu(u) - \left(\log M_s(f,\mu)\right)^2}{2\left(\int \log f\,d\mu(u) - \log M_s(f,\mu)\right)}\right), \quad s \neq 0.
\end{aligned}
\tag{2.4}
$$

3. Monotonicity of new means

In this section, we will prove the monotonicity of (2.4). We need the following lemmas for log-convex function.

Lemma 3.1. *Let f be log-convex function and if $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}. \tag{3.1}$$

Proof. In [2, page 3] we have the following result for convex function f, with $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}. \tag{3.2}$$

Putting $f = \log f$, we get

$$\log\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2-x_1)} \leq \log\left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2-y_1)}, \tag{3.3}$$

after applying exponential function we get (3.1). □

The following two lemmas are proved (for functionals) in [3] (Theorem 4 and Lemma 2, for Lemma 3.2 see also [4, Theorem 1]).

Lemma 3.2. *Let us consider Λ_t defined as*

$$\Lambda_t(g,\mu) = \begin{cases} \dfrac{M_t^t(g,\mu) - M_1^t(g,\mu)}{t(t-1)}, & t \neq 0,1; \\ \log M_1(g,\mu) - \log M_0^t(g,\mu), & t = 0; \\ \displaystyle\int g \log g\mu - M_0(g,\mu)\log M_0(g,\mu), & t = 1. \end{cases} \tag{3.4}$$

Then, Λ_t is a log-convex function.

Lemma 3.3. *Let us consider Λ_t defined as*

$$\Lambda_t = \begin{cases} \dfrac{1}{t^2}\left(M_t^t(f,\mu) - M_0^t(f,\mu)\right), & t \neq 0; \\ \dfrac{1}{2}\left(M_2^2(\log f,\mu) - M_1^2(\log f,\mu)\right), & t = 0. \end{cases} \tag{3.5}$$

Then, Λ_t is a log-convex function.

Theorem 3.4. *Let $t, r, u, v \in \mathbb{R}$, such that, $t \leq v,\ r \leq u$. Then for (2.4), we have*

$$M_{t,r}^s \leq M_{v,u}^s. \tag{3.6}$$

Proof

Case 1 ($s \neq 0$). Let us consider Λ_t defined as in Lemma 3.2. Λ_t is a continuous and log-convex. So, Lemma 3.1 implies that for $t, r, u, v \in \mathbb{R}$, such that, $t \leq v,\ r \leq u,\ t \neq r,\ v \neq u$, we have

$$\left(\frac{\Lambda_t}{\Lambda_r}\right)^{1(t-r)} \leq \left(\frac{\Lambda_v}{\Lambda_u}\right)^{1/(v-u)}. \tag{3.7}$$

For $s > 0$ by substituting $g = f^s,\ t = t/s,\ r = r/s,\ u = u/s,\ v = v/s \in \mathbb{R}$, such that, $t/s \leq v/s,\ r/s \leq u/s,\ t \neq r,\ v \neq u$, in (3.4), we get

$$\Lambda_{t,s}(f,\mu) = \begin{cases} \dfrac{s^2}{t(1-s)}\left[M_t^t(f,\mu) - M_s^t(f,\mu)\right], & t \neq 0, s; \\ s\left(\log M_s(f,\mu) - \log M_0(f,\mu)\right), & t = 0; \\ s\left(\displaystyle\int f^s \log f - M_0^s(f,\mu)\log M_0(f,\mu)\right), & t = s. \end{cases} \tag{3.8}$$

And (3.7) becomes

$$\left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{1(t-r)} \leq \left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{1/(v-u)}. \tag{3.9}$$

From (3.9), we get our required result.

Now when $s < 0$ by substituting $g = f^s,\ t = t/s,\ r = r/s,\ u = u/s,\ v = v/s \in \mathbb{R}$, such that, $v/s \leq t/s,\ u/s \leq r/s,\ t \neq r,\ v \neq u$, in (3.4) we get (3.8).

And (3.7) becomes

$$\left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{s/(v-u)} \leq \left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{s/(t-r)}. \tag{3.10}$$

Now $s < 0$, from (3.10), by raising power $-s$, we get

$$\left(\frac{\Lambda_{t,s}}{\Lambda_{r,s}}\right)^{1/(t-r)} \leq \left(\frac{\Lambda_{v,s}}{\Lambda_{u,s}}\right)^{1/(v-u)}. \tag{3.11}$$

From (3.11), we get our required result.

Case 2 ($s = 0$). In this case, we can get our result by taking limit $s \to 0$ in (3.8) and also in this case we can consider Λ_t defined as in Lemma 3.3.

Λ_t is log-convex function. So, Lemma 3.1 implies that for $t, r, u, v \in \mathbb{R}$, such that, $t \leq v,\ r \leq u,\ t \neq r,\ v \neq u$, we have

$$\left(\frac{\Lambda_t}{\Lambda_r}\right)^{1/(t-r)} \leq \left(\frac{\Lambda_v}{\Lambda_u}\right)^{1/(v-u)}. \tag{3.12}$$

Therefore, we have for $t, r, u, v \in \mathbb{R}$, such that, $t \leq v,\ r \leq u,\ t \neq r,\ v \neq u$:

$$M_{t,r}^0 \leq M_{v,u}^0, \tag{3.13}$$

which completes the proof. □

4. Further consequences and applications

In this section, we will represent the various applications of our previous definition of a new Cauchy mean and monotonicity of this above defined a new Cauchy mean.

4.1. *Tobey and Stolarsky-Tobey means*

Let E_{n-1} represent the $(n-1)$-dimensional Euclidean simplex given by

$$E_{n-1} = \left\{ (u_1, u_2, \dots, u_{n-1}) : u_i \geq 0,\ 1 \leq i \leq n-1,\ \sum_{i=1}^{n-1} u_i \leq 1 \right\}, \tag{4.1}$$

and set $u_n = 1 - \sum_{i=1}^{n-1} u_i$. Moreover, with $u = (u_1, \dots, u_n)$, let $\mu(u)$ be a probability measure on E_{n-1}. The power mean of order p ($p \in \mathbb{R}$) of the positive n-tuple $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, with the weights $u = (u_1, \dots, u_n)$, is defined by

$$\overline{M}_p(x, \mu) = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} u_i x_i^p \right)^{1/p}, & p \neq 0; \\ \displaystyle\prod_{i=1}^{n} x_i^{u_i}, & p = 0. \end{cases} \tag{4.2}$$

Then, the Tobey mean $L_{p,r}(x;\mu)$ is defined as follows:

$$L_{p,r}(x;\mu) = M_r(\overline{M}_p(x,\mu);\mu), \tag{4.3}$$

where $M_r(g, \mu)$ denotes the integral power mean, in which Ω is now the $(n-1)$-dimensional Euclidean simplex E_{n-1}. We note that, since $\overline{M}_p(x,\mu)$ is a mean we have $\min\{x_i\} \leq \overline{M}_p(x,\mu) \leq \max\{x_i\}$. Now setting $f(x,\mu) = \overline{M}_p(x,\mu)$ in (2.4) we get

$$\Gamma_{p,r,l}^{s}(x,\mu) = \left(\frac{l(l-s)}{r(r-s)} \frac{L_{p,r}^{r}(x,\mu) - L_{p,s}^{r}(x,\mu)}{L_{p,l}^{l}(x,\mu) - L_{p,s}^{l}(x,\mu)} \right)^{1/(r-l)}, \quad l \neq r \neq s,\ l, r \neq 0;$$

$$\Gamma_{p,r,0}^{s}(x,\mu) = \Gamma_{p,0,r}^{s}(x,\mu) = \left(\frac{s\left[L_{p,r}^{r}(x,\mu) - L_{p,s}^{r}(x,\mu)\right]}{r(r-s)\left[\log L_{p,s}(x,\mu) - \log L_{p,0}(x,\mu)\right]} \right)^{1/r}, \quad r \neq s,\ r, s \neq 0;$$

$$\Gamma_{p,s,l}^{s}(x,\mu) = \Gamma_{p,l,s}^{s}(x,\mu) = \left(\frac{l(l-s)}{s} \frac{\int \overline{M}_p(x,\mu)^s \log d\mu(u) - L_{p,s}^{s}(x,\mu) \log L_{p,s}(x,\mu)}{L_{p,l}^{l}(x,\mu) - L_{p,s}^{l}(x,\mu)} \right)^{1/(s-l)},$$

$$l \neq s,\ l, s \neq 0;$$

$$\Gamma_{p,s,0}^{s}(x,\mu) = \Gamma_{p,0,s}^{s}(x,\mu) = \left(\frac{\int \overline{M}_p(x,\mu)^s \log \overline{M}_p(x,\mu) d\mu(u) - L_{p,s}^{s}(x,\mu) \log L_{p,s}(x,\mu)}{\log L_{p,s}(x,\mu) - \log L_{p,0}(x,\mu)} \right)^{1/s}, \quad s \neq 0;$$

$$\Gamma^0_{p,r,l}(x,\mu)=\left(\frac{l^2\left(L^r_{p,r}(x,\mu)-L^r_{p,0}(x,\mu)\right)}{r^2\left(L^l_{p,l}(x,\mu)-L^l_{p,0}(x,\mu)\right)}\right)^{1/(r-l)},\quad l,r\neq 0;$$

$$\Gamma^0_{p,r,0}(x,\mu)=\Gamma^0_{p,0,r}(x,\mu)=\left(\frac{2\left[L^r_{p,r}(x,\mu)-L^r_{p,0}(x,\mu)\right]}{r^2\left[M_2^2\left(\log\overline{M}_p(x,\mu),\mu\right)-M_1^2\left(\log\overline{M}_p(x,\mu),\mu\right)\right]}\right)^{1/r},\quad r\neq 0;$$

$$\Gamma^s_{p,t,t}(x,\mu)=\exp\left(-\frac{2t-s}{t(t-s)}+\frac{\int\overline{M}_p(x,\mu)^t\log\overline{M}_p(x,\mu)d\mu(u)-L^t_{p,s}(x,\mu)\log L_{p,s}(x,\mu)}{L^t_{p,t}(x,\mu)-L^t_{p,s}(x,\mu)}\right),\quad t\neq s;$$

$$\Gamma^0_{p,t,t}(x,\mu)=\exp\left(-\frac{2}{t}+\frac{\int\overline{M}_p(x,\mu)^t\log\overline{M}_p(x,\mu)d\mu(u)-L^t_{p,0}(x,\mu)\log L_{p,0}(x,\mu)}{L^t_{p,t}(x,\mu)-L^t_{p,0}(x,\mu)}\right),\quad t\neq 0;$$

$$\Gamma^0_{p,0,0}(x,\mu)=\exp\left(\frac{1}{3}\frac{\int\left(\log\overline{M}_p(x,\mu)\right)^3d\mu(u)-\left(\log L_{p,0}(x,\mu)\right)^3}{\int\left(\log\overline{M}_p(x,\mu)\right)^2d\mu(u)-\left(\log L_{p,0}(x,\mu)\right)^2}\right),$$

$$\Gamma^s_{p,s,s}(x,\mu)=\exp\left(-\frac{1}{s}+\frac{\int\overline{M}_p(x,\mu)^s\left(\log\overline{M}_p(x,\mu)\right)^2d\mu(u)-L^s_{p,s}(x,\mu)\left(\log L_{p,s}(x,\mu)\right)^2}{2\left(\int\overline{M}_p(x,\mu)^s\log\overline{M}_p(x,\mu)d\mu(u)-\left(L^s_{p,s}(x,\mu)\log L_{p,s}(x,\mu)\right)\right)}\right),\quad s\neq 0;$$

$$\Gamma^s_{p,0,0}(x,\mu)=\exp\left(\frac{1}{s}+\frac{\int\left(\log\overline{M}_p(x,\mu)\right)^2d\mu(u)-\left(\log L_{p,s}(x,\mu)\right)^2}{2\left(\int\log\overline{M}_p(x,\mu)d\mu(u)-\log M_s(x,\mu)\right)}\right),\quad s\neq 0. \tag{4.4}$$

Theorem 4.1. *Let $t,r,u,v\in\mathbb{R}$, such that, $t<v,\ r<u$. Then for* (4.4)*, we have*

$$\Gamma^s_{p,t,r}\leq\Gamma^s_{p,v,u}. \tag{4.5}$$

Proof. It is a simple consequence of Theorem 3.4. □

Pečarić and Šimić (see [5, Definition 1]) introduced the Stolarsky-Tobey mean $\varepsilon_{p,q}(x,\mu)$ defined by

$$\varepsilon_{p,q}(x,\mu)=L_{p,q-p}(x,\nu)=M_{q-p}\left(\overline{M}_p(x,\mu);\mu\right), \tag{4.6}$$

where $L_{p,r}(x,\nu)$ is the Tobey mean already introduced above.

For the Stolarsky-Tobey mean and (2.4), we get the following:

$$\Upsilon^s_{p,r,l}(x,\mu)=\left(\frac{l(l-s)}{r(r-s)}\frac{\varepsilon^r_{p,p+r}(x,\mu)-\varepsilon^r_{p,p+s}(x,\mu)}{\varepsilon^l_{p,p+l}(x,\mu)-\varepsilon^l_{p,p+s}(x,\mu)}\right)^{1/(r-l)},\quad l\neq r\neq s,\ l,r\neq 0;$$

$$\Upsilon^s_{p,r,0}(x,\mu)=\Upsilon^s_{p,0,r}(x,\mu)=\left(\frac{s\left[\varepsilon^r_{p,p+r}(x,\mu)-\varepsilon^r_{p,p+s}(x,\mu)\right]}{r(r-s)\left[\log\varepsilon_{p,p+s}(x,\mu)-\log\varepsilon_{p,p}(x,\mu)\right]}\right)^{1/r},\quad r\neq s,\ r,s\neq 0;$$

$$\Upsilon^s_{p,s,l}(x,\mu)=\Upsilon^s_{p,l,s}(x,\mu)=\left(\frac{l(l-s)}{s}\frac{\int\overline{M}_p(x,\mu)^s\log d\mu(u)-\varepsilon^s_{p,p+s}(x,\mu)\log\varepsilon_{p,p+s}(x,\mu)}{\varepsilon^l_{p,p+l}(x,\mu)-\varepsilon^l_{p,p+s}(x,\mu)}\right)^{1/(s-l)},$$
$$l\neq s,\ l,s\neq 0;$$

$$\Upsilon^s_{p,s,0}(x,\mu)=\Upsilon^s_{p,0,s}(x,\mu)=\left(\frac{\int\overline{M}_p(x,\mu)^s\log\overline{M}_p(x,\mu)d\mu(u)-\varepsilon^s_{p,p+s}(x,\mu)\log\varepsilon_{p,p+s}(x,\mu)}{\log\varepsilon_{p,p+s}(x,\mu)-\log\varepsilon_{p,p}(x,\mu)}\right)^{1/s},$$
$$s\neq 0;$$

$$\Upsilon^0_{p,r,l}(x,\mu)=\left(\frac{l^2(\varepsilon^r_{p,p+r}(x,\mu)-\varepsilon^r_{p,p}(x,\mu))}{r^2(\varepsilon^l_{p,p+l}(x,\mu)-\varepsilon^l_{p,p}(x,\mu))}\right)^{1/(r-l)},\quad l,r\neq 0;$$

$$\Upsilon^0_{p,r,0}(x,\mu)=\Upsilon^0_{p,0,r}(x,\mu)=\left(\frac{2[\varepsilon^r_{p,p+r}(x,\mu)-\varepsilon^r_{p,p}(x,\mu)]}{r^2\left[M^2_2(\log\overline{M}_p(x,\mu),\mu)-M^2_1(\log\overline{M}_p(x,\mu),\mu)\right]}\right)^{1/r},\quad r\neq 0;$$

$$\Upsilon^s_{p,t,t}(x,\mu)=\exp\left(-\frac{2t-s}{t(t-s)}+\frac{\int\overline{M}_p(x,\mu)^t\log\overline{M}_p(x,\mu)d\mu(u)-M^t_s\log\varepsilon_{p.p+s}(x,\mu)}{\varepsilon^t_{p,p+t}(x,\mu)-\varepsilon^t_{p,p+s}(x,\mu)}\right),\quad t\neq s;$$

$$\Upsilon^0_{p,t,t}(x,\mu)=\exp\left(-\frac{2}{t}+\frac{\int\overline{M}_p(x,\mu)^t\log\overline{M}_p(x,\mu)d\mu(u)-\varepsilon^t_{P,p}(x,\mu)\log\varepsilon_{p,p}(x,\mu)}{\varepsilon^t_{p,p+t}(x,\mu)-\varepsilon^t_{p,p}(x,\mu)}\right),\quad t\neq 0;$$

$$\Upsilon^0_{p,0,0}(x,\mu)=\exp\left(\frac{1}{3}\frac{\int(\log\overline{M}_p(x,\mu))^3d\mu(u)-(\log\varepsilon_{p,p}(x,\mu))^3}{\int(\log\overline{M}_p(x,\mu))^2d\mu(u)-(\log\varepsilon_{p,p}(x,\mu))^2}\right),$$

$$\Upsilon^s_{p,s,s}(x,\mu)=\exp\left(-\frac{1}{s}+\frac{\int\overline{M}_p(x,\mu)^s(\log\overline{M}_p(x,\mu))^2d\mu(u)-\varepsilon^s_{p,p+s}(x,\mu)(\log\varepsilon_{p.p+s}(x,\mu))^2}{2(\int\overline{M}_p(x,\mu)^s\log\overline{M}_p(x,\mu)d\mu(u)-(\varepsilon^s_{p.p+s}(x,\mu)\log\varepsilon_{p,p+s}(x,\mu)))}\right),$$
$$s\neq 0;$$

$$\Upsilon^s_{p,0,0}(x,\mu)=\exp\left(\frac{1}{s}+\frac{\int(\log\overline{M}_p(x,\mu))^2d\mu(u)-(\log\varepsilon_{p,p+s}(x,\mu))^2}{2(\int\log\overline{M}_p(x,\mu)d\mu(u)-\log\varepsilon_{p.p+s}(x,\mu)))}\right),\quad s\neq 0. \tag{4.7}$$

Theorem 4.2. *Let $t,r,u,v\in\mathbb{R}$, such that, $t<v,\ r<u$. Then for* (4.7)*, we have*

$$\Upsilon^s_{p,t,r}\leq\Upsilon^s_{p,v,u}. \tag{4.8}$$

Proof. It is a simple consequence of Theorem 3.4. □

4.2. The complete symmetric mean

The rth complete symmetric polynomial mean (the complete symmetric mean) of the positive real n-tuple x is defined by (see [6, pages 332,341])

$$Q_n^{[r]}(x) = \left(q_n^{[r]}(x)\right)^{1/r} = \left(\frac{c_n^{[r]}(x)}{\binom{n+r-1}{r}}\right)^{1/r}, \tag{4.9}$$

where $c_n^{[0]}(x) = 1$ and $c_n^{[r]}(x) = \sum_{j=1}^{n}(\prod_{i=1}^{n} x_i^{i_j})$ and the sum is taken over all $\binom{n+r-1}{r}$ nonnegative integer n-tuples $(i_1, \ldots, i_n)$ with $\sum_{j=1}^{n} i_j = r \ (r \neq 0)$. The complete symmetric polynomial mean can also be written in an integral form as follows:

$$Q_n^{[r]} = \left(\int_{E_{n-1}}\left(\sum_{i=1}^{n} x_i u_i\right)^r d\mu(u)\right)^{1/r}, \tag{4.10}$$

where μ represents a probability measure such that $d\mu(u) = (n-1)!du_1 \cdots du_{n-1}$. We can see this as a special case of the integral power mean $M_r(f,\mu)$, where $f(u) = \sum_{i=1}^{n} x_i u_i$, μ is a probability measure as above, and Ω is the above defined $(n-1)$-dimensional simplex E_{n-1}. Thus from (2.4), we have the following result:

$$\Theta_{n,r,l}^{s}(x,\mu) = \left(\frac{l(l-s)}{r(r-s)}\frac{\left(Q_n^{[r]}\right)^r(x,\mu) - \left(Q_n^{[s]}\right)^r(x,\mu)}{\left(Q_n^{[l]}\right)^l(x,\mu) - \left(Q_n^{[s]}\right)^l(x,\mu)}\right)^{1/(r-l)}, \quad l \neq r \neq s,\ l,r \in \mathbb{N}. \tag{4.11}$$

A simple consequence of Theorem 3.4 is the following result.

Theorem 4.3. *Let $t, r, u, v \in \mathbb{N}$, such that, $t < v$, $r < u$. Then for* (4.11), *we have*

$$\Theta_{n,t,r}^{s} \leq \Theta_{n,v,u}^{s}. \tag{4.12}$$

4.3. Whiteley means

Let x be a positive real n-tuple, $s \in \mathbb{R}$ $(s \neq 0)$ and $r \in \mathbb{N}$. Then, the sth function of degree r is defined by the following generating function (see [6, pages 341–344]):

$$\sum_{r=0}^{\infty} t_n^{[r,s]}(x)t^r = \begin{cases} \prod_{i=1}^{n}(1 + x_i t)^s, & s > 0, \\ \prod_{i=1}^{n}(1 - x_i t)^s, & s < 0. \end{cases} \tag{4.13}$$

The Whiteley mean is now defined by

$$\mathcal{W}_n^{[r,s]}(x) = \left(w_n^{[r,s]}(x)\right)^{1/r} = \begin{cases} \left(\dfrac{t_n^{[r,s]}(x)}{\binom{nr}{s}}\right)^{1/r}, & s > 0, \\ \left(\dfrac{t_n^{[r,s]}(x)}{(-1)^r\binom{nr}{s}}\right)^{1/r}, & s < 0. \end{cases} \tag{4.14}$$

For $s < 0$, the Whiteley mean can be further generalized if we slightly change the definition of $t_n^{[r,s]}(x)$ and define $h_n^{[r,\sigma]}(x)$ as follows:

$$\sum_{r=0}^{\infty} h_n^{[r,\sigma]}(x)t^r = \prod_{i=1}^{n} \frac{1}{(1-x_i t)^{\sigma_i}}, \tag{4.15}$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$; $\sigma \in \mathbb{R}_+$; $i = 1, \ldots, n$. The following generalization of the Whiteley mean for $s < 0$ is defined by (see [7, Lemma 2.3])

$$\mathscr{H}_n^{[r,\sigma]}(x) = \left(\frac{h_n^{[r,\sigma]}(x)}{\binom{\sum_{i=1}^n \sigma_i + r - 1}{r}} \right)^{1/r}. \tag{4.16}$$

If we denote by μ a measure on the simplex $\Delta_{n-1} = \{(u_1, \ldots, u_n) : u_i \geq 0,\ i = 1, \ldots, n-1,\ \sum_{i=1}^n u_i \leq 1\}$ such that

$$d\mu(u) = \frac{\Gamma(\sum_{i=1}^n \sigma_i)}{\prod_{i=1}^n \Gamma(\sigma_i)} \prod_{i=1}^{n} u_i^{\sigma_i - 1} du_1 \cdots du_{n-1}, \tag{4.17}$$

where $u_n = 1 - \sum_{i-1}^{n-1}$, then we have μ as a probability measure and we can also write the mean $\mathscr{H}_n^{[r,\sigma]}(x)$ in integral form as follows:

$$\mathscr{H}_n^{[r,\sigma]}(x) = \left(\int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i \right)^r d\mu(u) \right)^{1/r}. \tag{4.18}$$

Finally, just as we did above in this investigation, we can develop the following analogous definition:

$$\mathfrak{H}_{n,r,l}^{s}(x,\mu) = \left(\frac{l(l-s)}{r(r-s)} \frac{\left(\mathscr{H}_n^{[r,\sigma]}\right)^r(x,\mu) - \left(\mathscr{H}_n^{[s,\sigma]}\right)^r(x,\mu)}{\left(\mathscr{H}_n^{[l,\sigma]}\right)^l(x,\mu) - \left(\mathscr{H}_n^{[s,\sigma]}\right)^l(x,\mu)} \right)^{1/(r-l)}, \quad l \neq r \neq s,\ l, r \in \mathbb{N}. \tag{4.19}$$

A simple consequence of Theorem 3.4 is the following result.

Theorem 4.4. *Let $t, r, u, v \in \mathbb{N}$, such that, $t < v$, $r < u$. Then for (4.19), we have*

$$\mathfrak{H}_{n,t,r}^{s} \leq \mathfrak{H}_{n,v,u}^{s}. \tag{4.20}$$

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