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Research Article

Some Characterizations of Ideal Points in Vector Optimization Problems

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Several relations between (proper) ideal points and (weakly, positive proper, general positive) efficient points are derived in real linear spaces. Moreover, some sufficient conditions for the existence of proper ideal points and positive proper efficient points are proved under certain assumptions.

1. Introduction and preliminaries

Ideal points and efficient elements play an important role in the investigation of multiobjective optimization problems (see, e.g., [1–9] and references therein). Recently, some relations between ideal points and efficient elements have been studied by many authors (see [3, 4]). In [3], the authors derived some sufficient conditions for the existence of ideal points in normed vector spaces. The algebra closure and vector closure were investigated in [1, 2], which are weaker than topological closure.

In this paper, we present some relations between (proper) ideal points and (weakly, positive proper, general positive) efficient points in real linear spaces. We also derive some sufficient conditions for the existence of proper ideal points and positive proper efficient points.

Let X be a real linear space and A a nonempty subset of X. A is said to be a cone if $\lambda A \subset A$ for all $\lambda > 0$. A is called a convex cone if A is a cone and $A + A \subset A$. A is called a pointed cone if A is a cone and $A \cap (-A) = \{0\}$.

The algebraic interior and relative algebraic interior of $A \subset X$ are defined by, respectively,

$$\begin{aligned} \operatorname{cor}(A) &= \{x \in A : \forall h \in X, \exists \lambda' > 0 : \forall \lambda \in [0, \lambda'], x + \lambda h \in A\}, \\ \operatorname{icr}(A) &= \{x \in A : \forall h \in \operatorname{span}(A - A), \exists \lambda' > 0 : \forall \lambda \in [0, \lambda'], x + \lambda h \in A\}. \end{aligned} \tag{1.1}$$

It is clear that $\operatorname{cor}(A) \subset \operatorname{icr}(A)$. The set A is called solid (resp., relatively solid) if $\operatorname{cor}(A) \neq \emptyset$ (resp., $\operatorname{icr}(A) \neq \emptyset$). If $A \subset X$ is a convex cone, then $\operatorname{icr}(A) \neq \emptyset$, $\operatorname{icr}(A) \cup \{0\}$ is a convex cone, $\operatorname{icr}(A) + A = \operatorname{icr}(A)$, and $\operatorname{icr}(\operatorname{icr}(A)) = \operatorname{icr}(\operatorname{icr}(A) \cup \{0\}) = \operatorname{icr}(A)$. If A is a convex pointed cone, then $0 \notin \operatorname{icr}(A)$.

Denote by X^* the algebraic dual of X. The positive dual and positive proper dual cone of $K \subset X$ are defined by, respectively,

$$\begin{aligned} K^+ &= \left\{ l \in X^* : \langle l, x \rangle \geq 0,\ \forall x \in K \right\}, \\ K^{+i} &= \left\{ l \in X^* : \langle l, x \rangle > 0,\ \forall x \in K \setminus \{0\} \right\}. \end{aligned} \tag{1.2}$$

It is obvious that $K^{+i} \subset K^+$.

Definition 1.1. Let A be a nonempty subset of X.

(1) The vector closure of A is defined by

$$\begin{aligned} \operatorname{vcl}(A) &= \left\{ b \in X : \exists x \in X : \forall \lambda' > 0,\ \exists \lambda \in (0, \lambda'],\ b + \lambda x \in A \right\} \\ &= \left\{ b \in X : \exists x \in X : \exists \{\lambda_n\}_{n \in N} \subset R_+ : \lambda_n \longrightarrow 0,\ b + \lambda_n x \in A,\ \forall n \in N \right\}. \end{aligned} \tag{1.3}$$

(2) A is called (vectorially) closed if $A = \operatorname{vcl}(A)$.

It is clear that $A \subset \operatorname{vcl}(A)$; if A is closed, then $A - x_0$ is also closed; if A is a cone, then $\operatorname{vcl}(A)$ is also a cone.

Throughout this paper, we always suppose that K is a closed, convex, and pointed cone of X.

Definition 1.2. (1) A point $x_0 \in X$ is called an ideal point of A if there exists a closed, convex and pointed cone $P \subset X$ such that $K \subset P$ and $A \subset x_0 + P$.

(2) If $x_0 \in X$ is an ideal point of A and $x_0 \in A$, then one says that x_0 is a proper ideal point of A.

Definition 1.3. (1) A point $x_0 \in A$ is said to be an efficient point of A if $(A - x_0) \cap (-K) = \{0\}$. Furthermore, a point $x_0 \in A$ is said to be a weakly efficient point of A if $(A - x_0) \cap (-\operatorname{icr} K) = \emptyset$, where $\operatorname{icr} K \neq \emptyset$.

(2) A point $x_0 \in A$ is said to be a positive proper efficient point of A if there is $l \in K^{+i}$ such that $\langle l, x_0 \rangle \leq \langle l, x \rangle$ for all $x \in A$. If there exists $l \in K^+ \setminus \{0\}$ such that $\langle l, x_0 \rangle \leq \langle l, x \rangle$ for all $x \in A$, then the point x_0 is called general positive efficient point of A.

We denote by $I(A)$ and $PI(A)$ the set of all ideal points of A and the set of all proper ideal points of A, and we denote by $\min(A)$, $W\min(A)$, $\operatorname{Pos}(A)$, and $\operatorname{GPos}(A)$ the set of all efficient points, the set of all weakly efficient points, the set of all positive proper efficient points, and the set of all general positive efficient points of A, respectively.

Lemma 1.4 (see [1]). *Let A be a convex subset of X and $K \subset X$ a closed, convex, and pointed cone. Then*

(1) $\mathrm{cone}(\mathrm{vcl}(A)) \subset \mathrm{vcl}(\mathrm{cone}(A))$.
Furthermore, if A is relatively solid, then
(2) $\mathrm{vcl}\,(A)$ *is closed and convex;*
(3) $\mathrm{icr}\,(A) = \mathrm{icr}\,(\mathrm{vcl}\,(A))$;
(4) *for any $a \in \mathrm{icr}\,(A)$ and $b \in \mathrm{vcl}\,(A)$, one has $[a, b) \subset \mathrm{icr}\,(A)$.*

Lemma 1.5 (see [2]). *Let S and T be cones of a real line space X with $\mathrm{icr}\,(S) \neq \varnothing$ and $\mathrm{icr}\,(T) \neq \varnothing$. If $T \cap \mathrm{icr}\,(S) = \varnothing$, then there exists $l \in X^* \setminus \{0\}$ such that $\langle l, s\rangle \leq 0 \leq \langle l, t\rangle$ for all $s \in S$ and $t \in T$. Furthermore, one has $\langle l, s\rangle < 0$ for all $s \in \mathrm{icr}\,(S)$ or $\langle l, t\rangle > 0$ for all $t \in \mathrm{icr}\,(T)$.*

2. Main results

In this section, we prove several relations between the proper ideal points and efficient points and derive some existence theorems of the proper ideal points.

We first consider the following proposition.

Proposition 2.1. *Let A be a convex subset of X and K a closed, convex, and pointed cone of X. Then the following assertions hold:*

(1) $PI(A) \subset \min(A) \cap I(A)$;

(2) *for every $x_0 \in I(A)$, there exist $l_0 \in K^+ \setminus \{0\}$ such that $\langle l_0, x_0\rangle \leq \langle l_0, x\rangle$ for all $x \in A$;*

(3) $\mathrm{pos}(A) \cup \min(A) \cup pI(A) \subset \mathrm{Gpos}(A)$.

Proof 1. (1) Let $x_0 \in PI(A)$. Then $x_0 \in A$ and there exists a closed convex and pointed cone $P \subset X$ such that $K \subset P$ and $A \subset x_0 + P$. If $x_0 \notin \min(A)$, then there exists $x_1 \in A \backslash \{x_0\}$, $c_1 \in K \setminus \{0\} \subset P \setminus \{0\}$, $k_1 \in P \setminus \{0\}$, such that $x_1 = x_0 + k_1$ and $x_0 = x_1 + c_1$. Consequently, $c_1 + k_1 = 0$, which is impossible since P is a pointed cone. Thus $x_0 \in \min(A)$ and so $PI(A) \subset \min(A)$. It is clear that $PI(A) \subset I(P)$ always holds. Therefore, $PI(A) \subset \min(A) \cap I(A)$.

(2) Let $x_0 \in I(A)$. Then there exists a closed convex and pointed cone $P \subset X$ such that $K \subset P$ and $A \subset x_0 + P$, that is, $A - x_0 \subset P$. If $x_0 \notin A$, then $(A - x_0) \cap (-P) = \varnothing$ since P is a pointed cone. It follows from $-K \subset -P$ that $(A - x_0) \cap (-K) = \varnothing$. If $x_0 \in A$, then $x_0 \in PI(A)$ and, from (1), we have $x_0 \in \min(A)$, that is, $(A - x_0) \cap (-K) = \{0\}$. Since A is convex, so is $A - x_0$. Hence, from the separation theorem, it follows that there is $l_0 \in X^* \setminus \{0\}$ such that

$$\langle l_0, x - x_0\rangle \geq 0 \geq \langle l_0, -k\rangle, \quad \forall x \in A,\ k \in K. \tag{2.1}$$

It follows that

$$\langle l_0, x\rangle \geq \langle l_0, x_0\rangle, \quad \forall x \in A, \qquad \langle l_0, k\rangle \geq 0, \quad \forall k \in K. \tag{2.2}$$

Therefore, we have $l_0 \in K^+ \setminus \{0\}$, which implies the desired conclusion.

(3) From definitions, one has $\mathrm{pos}(A) \subset \mathrm{Gpos}(A)$. Then, from (1), it suffices to prove $\min(A) \subset \mathrm{Gpos}(A)$. Let $x_0 \in \min(A)$. Then $x_0 \in A$ and $(A - x_0) \cap (-K) = \{0\}$. Since A is convex, so is $A - x_0$. The rest of the proof is similar to that in (2). This completes the proof. □

Proposition 2.2. *Let $A \subset X$ and $l \in X^* \setminus \{0\}$. Then the following assertions hold:*

(1) *if $\langle l, a\rangle \geq \alpha$ for all $a \in A$, where $\alpha \in R$, then $\langle l, a\rangle \geq \alpha$ for all $a \in \operatorname{vcl} A$;*

(2) *assume A is convex and $\operatorname{icr} A \neq \emptyset$. If $\langle l, a\rangle \geq \alpha$ for all $a \in \operatorname{icr} A$, where $\alpha \in R$, then $\langle l, a\rangle \geq \alpha$ for all $a \in \operatorname{vcl} A$.*

Proof 2. (1) Suppose that $\langle l, a\rangle \geq \alpha$ for all $a \in A$. Let $b \in \operatorname{vcl} A$. Then there exist $x \in X$ and $\{\lambda_n\}_{n \in N} \subset R_+$ with $\lambda_n \to 0$ such that $b + \lambda_n x \in A$ for each $n \in N$. Consequently, we obtain

$$\langle l, b\rangle + \lambda_n \langle l, x\rangle = \langle l, b + \lambda_n x\rangle \geq \alpha, \tag{2.3}$$

which yields $\langle l, b\rangle \geq \alpha$ as $n \to +\infty$.

(2) Assume that $\langle l, a\rangle \geq \alpha$ for all $a \in \operatorname{icr} A$. Let $b \in \operatorname{vcl} A$ and $a \in \operatorname{icr} A$. For $\lambda \in (0,1]$, Lemma 1.4(4) implies that $\lambda a + (1-\lambda)b \in \operatorname{icr} A$. It follows that

$$\lambda\langle l, a\rangle + (1-\lambda)\langle l, b\rangle = \langle l, \lambda a + (1-\lambda)b\rangle \geq \alpha \tag{2.4}$$

and so $\langle l, b\rangle \geq \alpha$ by taking $\lambda \to 0$. This completes the proof. □

By using Proposition 2.2, we establish the relation between weakly efficient points and general positive efficient points.

Proposition 2.3. *Let A be a convex subset of X. Then $W\min(A) \subset \operatorname{Gpos}(A)$.*

Proof 3. Note that $\operatorname{icr} K \neq \emptyset$. Let $x_0 \in W\min(A)$. Then $x_0 \in A$ and $(A - x_0) \cap (-\operatorname{icr} K) = \emptyset$. By assumption, $A - x_0$ is convex. From separation theorems, there exists $l \in X^* \setminus \{0\}$ such that

$$\langle l, x - x_0\rangle \geq 0 \geq \langle l, -k\rangle, \quad \forall x \in A,\ k \in \operatorname{icr} K. \tag{2.5}$$

Therefore, $\langle l, k\rangle \geq 0$ for all $k \in \operatorname{icr} K$. Now, Proposition 2.2 yields $\langle l, k\rangle \geq 0$ for all $k \in \operatorname{vcl} K$, and thus $\langle l, k\rangle \geq 0$ for all $k \in K$ since $K \subset \operatorname{vcl} K$. This establishes $l \in K^+ \setminus \{0\}$ and $\langle l, x\rangle \geq \langle l, x_0\rangle$ for all $x \in A$, that is, $x_0 \in \operatorname{Gpos}(A)$. This completes the proof. □

We say that $A \subset X$ satisfies the property $\mathcal{P}$ if, for any $x \in A$ and $\lambda \in [0,1]$, one has $\lambda x \in A$.

Remark that if $A \subset X$ is a cone, then it satisfies the property $\mathcal{P}$.

Proposition 2.4. *Let A be a nonempty subset of X. If A satisfies the property $\mathcal{P}$, then $\operatorname{cone}(\operatorname{vcl}(A)) = \operatorname{vcl}(\operatorname{cone}(A))$.*

Proof 4. Suppose that A satisfies the property $\mathcal{P}$. The inclusion $\operatorname{cone}(\operatorname{vcl}(A)) \subset \operatorname{vcl}(\operatorname{cone}(A))$ follows immediately from Lemma 1.4(1). Thus we only need to prove $\operatorname{cone}(\operatorname{vcl}(A)) \supset \operatorname{vcl}(\operatorname{cone}(A)))$. For this, let $b \in \operatorname{vcl}(\operatorname{cone}(A))$. Then there exists $x \in X$ such that, for any $\lambda' > 0$, there exists $\lambda \in (0, \lambda']$ such that $b + \lambda x \in \operatorname{cone} A$. Consequently, there are $b_\lambda \in A$ and $t_\lambda > 0$ such that $b + \lambda x = t_\lambda b_\lambda$. It follows that

$$\frac{1}{\alpha_\lambda} b + \frac{\lambda}{\alpha_\lambda} x = \frac{t_\lambda}{\alpha_\lambda} b_\lambda, \tag{2.6}$$

where $\alpha_\lambda \geq \max(1, t_\lambda)$. Since A satisfies the property $\mathcal{P}$, then $(t_\lambda/\alpha_\lambda)b_\lambda \in A$. Since $\lambda/\alpha_\lambda \in (0, \lambda']$, it follows that $(1/\alpha_\lambda)b = -(\lambda/\alpha_\lambda)x + (t_\lambda/\alpha_\lambda)b_\lambda \in \operatorname{vcl}(A)$ and so $b \in \operatorname{cone}(\operatorname{vcl}(A))$, which yields $\operatorname{cone}(\operatorname{vcl}(A)) \supset \operatorname{vcl}(\operatorname{cone}(A))$. This completes the proof. □

Corollary 2.5. *Let A be a cone of X. Then* $\operatorname{cone}(\operatorname{vcl}(A)) = \operatorname{vcl}(A)$.

Proposition 2.6. *Suppose that D is a cone of X and B is a closed subset of X such that B satisfies the property $\mathcal{P}$ and $B + D$ is closed. Then* $\operatorname{cone} B + D = \operatorname{vcl}(\operatorname{cone} B + D)$.

Proof 5. It is obvious that $\operatorname{cone} B + D \subset \operatorname{vcl}(\operatorname{cone} B + D)$ holds. We next prove that the converse inclusion is also true. Since B is closed, from Proposition 2.4, one has $\operatorname{cone} B = \operatorname{vcl}(\operatorname{cone} B)$ and as a consequence, we only need to show $\operatorname{vcl}(\operatorname{cone} B + D) \subset \operatorname{vcl}(\operatorname{cone} B) + D$. For this, let $b \in \operatorname{vcl}(\operatorname{cone} B + D)$. Then there exists $x \in X$ such that, for any $\lambda' > 0$, there is $\lambda \in (0, \lambda']$ such that $b + \lambda x \in \operatorname{cone} B + D$. Thus there are $t_\lambda > 0$, $b_\lambda \in B$, and $d_\lambda \in D$ such that $b + \lambda x = t_\lambda b_\lambda + d_\lambda$. As the proof in Proposition 2.4, we can prove

$$\frac{1}{\alpha_\lambda} b = -\frac{\lambda}{\alpha_\lambda} x + \frac{t_\lambda}{\alpha_\lambda} b_\lambda + \frac{1}{\alpha_\lambda} d_\lambda \in \operatorname{vcl}(B + D) = B + D \tag{2.7}$$

and so

$$\begin{aligned} b &= -\lambda x + t_\lambda b_\lambda + d_\lambda \\ &= \alpha_\lambda \left(-\frac{\lambda}{\alpha_\lambda} x + \frac{t_\lambda}{\alpha_\lambda} b_\lambda + \frac{1}{\alpha_\lambda} d_\lambda \right) \\ &\in \operatorname{cone}(B + D) \\ &\subset \operatorname{cone}(B) + \operatorname{cone}(D) \\ &= \operatorname{cone}(B) + D \\ &\subset \operatorname{vcl}(\operatorname{cone}(B)) + D, \end{aligned} \tag{2.8}$$

where $\alpha_\lambda \geq \max(1, t_\lambda)$. This implies that $\operatorname{vcl}(\operatorname{cone} B + D) \subset \operatorname{vcl}(\operatorname{cone} B) + D$. This completes the proof. □

Under certain assumptions, we prove that an efficient point is also a proper ideal point.

Theorem 2.7. *Let A be a convex subset of X and K a closed convex and pointed cone of X. Let $x_0 \in$* $\min(A)$. *Then* $K^+(x_0, A) = \{l \in K^+ : \langle l, x \rangle \geq \langle l, x_0 \rangle$, *for all* $x \in A\}$ *is nonempty. Moreover, if $K^+(x_0, A)$ separates points of X (i.e., if $\langle l, p \rangle = 0$ for all $l \in K^+(x_0, A)$, then $p = 0$), then $x_0 \in PI(A)$.*

Proof 6. As the proof of Proposition 2.1(3), we can show $K^+(x_0, A) \neq \varnothing$. Let

$$P = \{x \in X : \langle l, x \rangle \geq 0, \ \forall l \in K^+(x_0, A)\}. \tag{2.9}$$

Then it is easy to prove that P is a cone, $K \subset P$, and $A \subset x_0 + P$.

We next prove that P is closed convex and pointed cone. Suppose that $K^+(x_0, A)$ separates points of X. Obviously, P is convex. Let $p \in \operatorname{vcl}(P)$. For any $b \in P$, it follows from the definition of P that

$$\langle l, b \rangle \geq 0, \quad \forall l \in K^+(x_0, A), \tag{2.10}$$

and, from Proposition 2.2(1), we have

$$\langle l, p\rangle \geq 0, \quad \forall l \in K^{+}(x_0, A), \tag{2.11}$$

which yields $p \in P$ and so $P = \operatorname{vcl}(P)$, that is, P is closed.

Now, we show that P is a pointed cone. For this, let $p \in P \cap \{-P\}$. Then

$$\langle l, p\rangle = 0, \quad \forall l \in K^{+}(x_0, A). \tag{2.12}$$

Since $K^{+}(x_0, A)$ separates points of X, one has $p = 0$. Thus P is a pointed cone. Since P is a closed, convex, and pointed cone such that

$$K \subset P, \qquad A \subset x_0 + P, \tag{2.13}$$

we get $x_0 \in PI(A)$. This completes the proof. □

Theorem 2.8. *Let A be a closed convex subset of X, K a closed, convex, and pointed cone of X. Let $x_0 \in \min(A)$. If $B + K$ is closed,* cone B *is a pointed cone, and B satisfies the property $\mathcal{D}$, then $x_0 \in PI(A)$, where $B = A - x_0$.*

Proof 7. Since A is closed and convex, so is B. Then cone(B) is also convex. In fact, for any $x_1, x_2 \in \operatorname{cone}(B)$, there exist $t_1, t_2 > 0$, $b_1, b_2 \in B$ such that $x_1 = t_1 b_1$ and $x_2 = t_2 b_2$. Then, for any $t \in (0,1)$,

$$\begin{aligned}
tx_1 + (1-t)x_2 &= tt_1 b_1 + (1-t)t_2 b_2 \\
&= \left(tt_1 + (1-t)t_2\right)\left(\frac{tt_1}{tt_1 + (1-t)t_2} b_1 + \frac{(1-t)t_2}{tt_1 + (1-t)t_2} b_2\right) \\
&\in \left(tt_1 + (1-t)t_2\right) B \\
&\subset \operatorname{cone} B,
\end{aligned} \tag{2.14}$$

which implies the convexity of cone B.

Let $P = \operatorname{cone} B + K$. Since $0 \in \operatorname{cone} B \cap K$, we obtain $K \subset P$ and $B \subset P$. Since $B + K$ is closed and B satisfies the property $\mathcal{D}$, it follows from Proposition 2.6 that one has $P = \operatorname{vcl}(P)$, that is, P is closed. It is clear that P is convex since cone B and K are convex.

Now, we will show that P is a pointed cone. Let $b \in P \cap (-P)$. Then there exist $k_1, k_2 \in K$, $x_1, x_2 \in A$, and $\alpha_1, \alpha_2 > 0$ such that $b = \alpha_1(x_1 - x_0) + k_1 = -(\alpha_2(x_2 - x_0) + k_2)$. It follows that

$$\alpha_1 x_1 + \alpha_2 x_2 - (\alpha_1 + \alpha_2)x_0 = \alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0) = -k_1 - k_2 \tag{2.15}$$

and so

$$-K \ni -\frac{k_1 + k_2}{\alpha_1 + \alpha_2} = \frac{\alpha_1 x_1}{\alpha_1 + \alpha_2} + \frac{\alpha_2 x_2}{\alpha_1 + \alpha_2} - x_0 \in A - x_0. \tag{2.16}$$

Since $x_0 \in \min(A)$, it follows that $k_1 = k_2 = 0$ and thus $b = \alpha_1(x_1 - x_0) = -\alpha_2(x_2 - x_0) \in \operatorname{cone}(B) \cap (-\operatorname{cone}(B))$. Since cone$(B)$ is a pointed cone, one has $b = 0$ and hence P is a pointed cone. This yields $x_0 \in PI(A)$. This completes the proof. □

Corollary 2.9. *Let A be a closed convex subset of X, K a closed, convex, and pointed cone of X. If $B+K$ is closed,* cone B *is a pointed cone, and B satisfies the property $\mathcal{P}$, then* $\min(A) = PI(A)$.

Proof 8. The conclusion follows immediately from Proposition 2.1(1) and Theorem 2.8. □

From Proposition 2.1(1), we know that a proper ideal point must be an efficient point. But a proper ideal point is not a positive efficient point.

Next, we give a sufficient condition for proper ideal points being positive efficient points.

Theorem 2.10. *Let $x_0 \in A$. If the closed, convex, and pointed cones K and P satisfy $K \subset \mathrm{icr}\,(P) \cup \{0\}$ and $A \subset x_0 + P$, then $x_0 \in \mathrm{pos}(A)$.*

Proof 9. Since K is a closed, convex, and pointed cone, so does $-K$. It is easy to check that $\mathrm{icr}(P)$ is a convex pointed cone. Since $-K \subset -\mathrm{icr}(P) \cup \{0\}$ and $0 \notin \mathrm{icr}(P)$, one has

$$(-K) \cap \mathrm{icr}(P) = \varnothing. \tag{2.17}$$

From Lemma 1.5, it follows that there exists $l \in X^* \setminus \{0\}$ such that

$$\langle l, x\rangle > 0, \quad \forall x \in \mathrm{icr}(P), \qquad \langle l, x\rangle \geq 0, \quad \forall x \in P. \tag{2.18}$$

Since $K \setminus \{0\} \subset \mathrm{icr}(P)$, it follows that $\langle l, k\rangle > 0$ for all $k \in K \setminus \{0\}$, which implies $l \in K^{+i}$. Since $A - x_0 \subset P$, one has $\langle l, x\rangle \geq \langle l, x_0\rangle$ for all $x \in A$, which yields $x_0 \in \mathrm{pos}(A)$. This completes the proof. □

Let $x_0 \in \mathrm{Gpos}(A)$. Then there exists $l_0 \in K^+ \setminus \{0\}$ such that $\langle l_0, x\rangle \geq \langle l_0, x_0\rangle$ for all $x \in A$. Since $l_0 \neq 0$, there exists x_1 such that $-\infty < \langle l_0, x_1\rangle < 0$. For any given $\lambda > 0$, let

$$x_\lambda = x_0 + \lambda x_1. \tag{2.19}$$

Theorem 2.11. *Let $x_0 \in \mathrm{Gpos}(A)$. Let $A \subset X$ and K a closed, convex, and pointed cone of X. If $K^+(x_\lambda, A) = \{l \in K^+ : \langle l, x\rangle \geq \langle l, x_\lambda\rangle$, for all $x \in A\} \neq \varnothing$ separates points of X, where x_λ is given as above, and $\sup_{x\in A}|\langle l, x\rangle| < +\infty$ for each $l \in K^+$, then $x_0 \in \mathrm{vcl}\,(I(A))$.*

Proof. Set

$$\alpha = -\langle l_0, x_1\rangle > 0. \tag{2.20}$$

Then it follows that

$$\langle l_0, x_\lambda\rangle + \lambda\alpha = \langle l_0, x_\lambda\rangle - \lambda\langle l_0, x_1\rangle = \langle l_0, x_0\rangle \leq \langle l_0, x\rangle, \quad \forall x \in A. \tag{2.21}$$

Note that $\sup_{x\in A}|\langle l, x\rangle| < +\infty$ for each $l \in K^+$. Then there exists $l \in K^+ \setminus \{0\}$ such that

$$\left|\langle l - l_0, x\rangle\right| \leq \frac{\lambda\alpha}{2}, \quad \forall x \in A, \quad \left|\langle l - l_0, x_\lambda\rangle\right| \leq \frac{\lambda\alpha}{2}. \tag{2.22}$$

Therefore, we have

$$\langle l, x\rangle \geq \langle l_0, x\rangle - \frac{\lambda\alpha}{2} \geq \langle l_0, x_\lambda\rangle + \frac{\lambda\alpha}{2} \geq \langle l, x_\lambda\rangle, \quad \forall x \in A. \tag{2.23}$$

Then it follows that

$$K^{+}(x_\lambda, A) = \{l \in K^{+} : \langle l, x\rangle \geq \langle l, x_\lambda\rangle, \forall x \in A\} \neq \emptyset. \tag{2.24}$$

Let

$$P = \{x \in X : \langle l, x\rangle \geq 0, \forall l \in K^{+}(x_\lambda, A)\}. \tag{2.25}$$

Then it is easy to prove that P is a cone, $K \subset P$, and $A \subset x_\lambda + P$. As in the proof of Theorem 2.7, we can show that P is a closed, convex, and pointed cone. Consequently, $x_\lambda = x_0 + \lambda x_1 \in I(A)$ for each $\lambda > 0$, which implies that $x_0 \in \mathrm{vcl}(I(A))$. This completes the proof. □

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