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Research Article

John-Nirenberg Type Inequalities for the Morrey-Campanato Spaces

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We give John-Nirenberg type inequalities for the Morrey-Campanato spaces on $\mathbb{R}^n$.

Given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and a cube Q on $\mathbb{R}^n$, let f_Q denote the average of f on Q, $f_Q = (1/|Q|)\int_Q f(x)dx$. We say that f has bounded mean oscillation if there is a constant C such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \left|f(x) - f_Q\right| dx \le C. \tag{1}$$

The space of functions with this property is denoted by BMO. For $f \in \text{BMO}$, define the norm on BMO by

$$\|f\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|}\int_Q \left|f(x) - f_Q\right| dx. \tag{2}$$

John and Nirenberg [1] obtained the following well-known John-Nirenberg inequality for BMO.

Theorem 1. *Let $f \in \text{BMO}$ and $\|f\|_{\text{BMO}} \neq 0$. Then there exist positive constants C_1 and C_2, depending only on the dimension, such that for all cube Q and any $\lambda > 0$,*

$$\left|\left\{x \in Q : \left|f(x) - f_Q\right| > \lambda\right\}\right| \le C_1 e^{-C_2\lambda/\|f\|_{\text{BMO}}}|Q|. \tag{3}$$

Suppose f is a locally integrable function on $\mathbb{R}^n$, Q is a cube, and s is a nonnegative integer; let $P_Q f(x)$ be the unique polynomial of degree at most s such that

$$\int_Q (f(x) - P_Q(f)(x)) x^\alpha dx = 0 \tag{4}$$

for all $0 \le |\alpha| \le s$. Moreover, for any $x \in Q$,

$$\left|P_Q(f)(x)\right| \le \frac{A}{|Q|} \int_Q \left|f(x)\right| dx, \tag{5}$$

where the constant A is independent of f and Q. Clearly, $A \ge 1$.

For $\beta \ge 0$, $s \ge 0$, $1 \le q < \infty$, we will say that a locally integrable function $f(x)$ belongs to the Morrey-Campanato spaces $L(\beta, q, s)$ if

$$\|f\|_{L(\beta,q,s)} = \sup_Q |Q|^{-\beta} \left\{ \frac{1}{|Q|} \int_Q \left|f(x) - P_Q(f)(x)\right|^q dx \right\}^{1/q} < \infty, \tag{6}$$

where Q is a cube. Then if $f - g$ is a polynomial of degree at most s, g also satisfies (6) and $\|f\|_{L(\beta,q,s)} = \|g\|_{L(\beta,q,s)}$. If this is the case we say that f and g are equivalent, the quotient space divided by such equivalence classes will be denoted by $L(\beta, q, s)$, and (6) defines its norm.

These spaces played an important role in the study of partial differential equations and they were studied extensively. Reader is referred, in particular, to [2–4]. Recently, Deng et al. [5] and Duong and Yan [6] gave several new characterizations for the Morrey-Campanato spaces.

As noted in [2], for $\beta = 0$ and $1 \le q \le \infty$, these spaces are variants of the BMO space. For $\beta > 0$ and $s \ge [n\beta]$, the spaces $L(\beta, q, s)$ are variants of the homogeneous Lipschitz spaces $\dot{\Lambda}_{n\beta}(\mathbb{R}^n)$ which are duals of certain Hardy spaces. See also [1].

In [7], we proved a John-Nirenberg-type inequality for homogeneous Lipschitz spaces $\dot{\Lambda}_\alpha(\mathbb{R}^n)$, $0 < \alpha < 1$. In this note, we will show that a similar inequality is also true for the Morrey-Campanato spaces $L(\beta, q, s)$ on $\mathbb{R}^n$, where β is nonnegative, $1 \le q \le \infty$, and the integer $s \ge 0$. Our main result can be stated as follows.

Theorem 2 (John-Nirenberg-type inequality). *Given $\beta \ge 0$ and $s \ge 0$, let $f \in L(\beta, 1, s)$ and $\|f\|_{L(\beta,1,s)} \ne 0$. Then there exist positive constants C_1 and C_2, depending only on the dimension, such that for all cube Q and any $\lambda > 0$,*

$$\left|\left\{x \in Q : |Q|^{-\beta} \left|f(x) - P_Q(f)(x)\right| > \lambda\right\}\right| \le C_1 e^{-C_2 \lambda / \|f\|_{L(\beta,1,s)}} |Q|. \tag{7}$$

Proof. Let Q be a fixed cube and let λ_0 be some positive real number which will be determined later. Applying the Calderon-Zygmund decomposition to the function $|Q|^{-\beta} |f(x) - P_Q(f)(x)|$ at height λ_0 to obtain a family of subcubes $\{Q_j\}$ of Q with disjoint interiors such that

$$|Q|^{-\beta} \left|f(x) - P_Q(f)(x)\right| \le \lambda_0 \quad \text{a.e. } Q \setminus \bigcup_{j=1}^{\infty} Q_j, \tag{8}$$

$$\lambda_0 < \frac{1}{\left|Q_j\right|} \int_{Q_j} |Q|^{-\beta} \left|f(x) - P_Q(f)(x)\right| dx \le 2^n \lambda_0 \quad \text{for any } j, \tag{9}$$

$$\sum_{j=1}^{\infty} \left|Q_j\right| \le \frac{1}{\lambda_0} \int_Q |Q|^{-\beta} \left|f(x) - P_Q(f)(x)\right| dx. \tag{10}$$

By (5), for any $x \in Q_j$, we get

$$\begin{aligned} \left|P_Q(f)(x) - P_{Q_j}(f)(x)\right| &= \left|P_{Q_j}\left(P_Q(f) - P_{Q_j}(f)\right)(x)\right| \\ &\le \frac{A}{|Q_j|}\int_{Q_j}\left|P_Q(f)(y) - P_{Q_j}(f)(y)\right|dy. \end{aligned} \tag{11}$$

Thus for any $x \in Q_j$, by (9) we have

$$\begin{aligned} |Q|^{-\beta}\left|P_Q(f)(x) - P_{Q_j}(f)(x)\right| &\le \frac{A}{|Q_j|}\int_{Q_j}|Q|^{-\beta}\left|P_Q(f)(y) - P_{Q_j}(f)(y)\right|dy \\ &\le \frac{A}{|Q_j|}\int_{Q_j}|Q|^{-\beta}\left|f(y) - P_Q(f)(y)\right|dy + \frac{A}{|Q_j|}\int_{Q_j}|Q_j|^{-\beta}\left|f(y) - P_{Q_j}(f)(y)\right|dy \\ &\le A2^n\lambda_0 + A\|f\|_{L(\beta,1,s)}. \end{aligned} \tag{12}$$

Denote $b = A2^n\lambda_0 + A\|f\|_{L(\beta,1,s)} > \lambda_0$. For any $x \in Q_j$, we have

$$\begin{aligned} |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| &\le |Q|^{-\beta}\left|P_Q(f)(x) - P_{Q_j}(f)(x)\right| + \left|Q_j\right|^{-\beta}\left|f(x) - P_{Q_j}(f)(x)\right| \\ &\le b + \left|Q_j\right|^{-\beta}\left|f(x) - P_{Q_j}(f)(x)\right|. \end{aligned} \tag{13}$$

Then for any $\lambda > 0$, we have

$$\left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda + b\right\} \subset \left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda_0\right\} \subset \bigcup_{j=1}^{\infty} Q_j. \tag{14}$$

By (13) and (14),

$$\begin{aligned} \left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda + b\right\} &\subset \bigcup_{j=1}^{\infty}\left\{x \in Q_j : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda + b\right\} \\ &\subset \bigcup_{j=1}^{\infty}\left\{x \in Q_j : \left|Q_j\right|^{-\beta}\left|f(x) - P_{Q_j}(f)(x)\right| > \lambda\right\}. \end{aligned} \tag{15}$$

For any $\lambda > 0$, we set

$$F_f(\lambda) = \sup_Q \frac{1}{|Q|}\left|\left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda\right\}\right|. \tag{16}$$

Clearly, $F_f(\lambda)$ is a decreasing function on $[0,\infty)$ and $F_f(0) \le 1$. Using (10), we have

$$\begin{aligned} \frac{1}{|Q|}\left|\left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda + b\right\}\right| &\le F_f(\lambda)\frac{1}{|Q|}\sum_{j=1}^{\infty}\left|Q_j\right| \\ &\le F_f(\lambda)\frac{1}{\lambda_0|Q|}\int_Q |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right|dx. \end{aligned} \tag{17}$$

So for any $\lambda \geq 0$, we get $F_f(\lambda + b) \leq \lambda_0^{-1}\|f\|_{L(\beta,1,s)} F_f(\lambda)$. Taking $\lambda_0 = e\|f\|_{L(\beta,1,s)}$, then $b = A(2^n e + 1)\|f\|_{L(\beta,1,s)}$ is also a fixed positive number and for any $\lambda \geq 0$,

$$F_f(\lambda + b) \leq \frac{1}{e} F_f(\lambda). \tag{18}$$

By induction argument for any $k \geq 1$, we get

$$F_f\big((k+1)b\big) \leq e^{-k} F_f(b). \tag{19}$$

Thus, for $\lambda \in (kb, (k+1)b]$, we have

$$F_f(\lambda) \leq F_f(kb) \leq e^{-k} F_f(0) \leq e e^{-\lambda/b}. \tag{20}$$

Notice that this inequality is also true for $\lambda \in [0, b]$, due to $F_f(\lambda) \leq F_f(0) = 1 \leq e e^{-\lambda/b}$. Thus, for any $\lambda \geq 0$, we have

$$\frac{1}{|Q|}\left|\left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda\right\}\right| \leq e e^{-\lambda/b}. \tag{21}$$

This concludes the proof of the theorem. □

Corollary 1. *Given $\beta \geq 0$, $s \geq 0$. For all $q \in [1, \infty)$, the spaces $L(\beta, q, s)$ coincide, and the norms $\|\cdot\|_{L(\beta,q,s)}$ are equivalent, namely,*

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \left[|Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right|\right]^q dx\right)^{1/q} \approx \sup_Q \frac{1}{|Q|}\int_Q |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| dx. \tag{22}$$

Proof. It will suffice to prove that $\|f\|_{L(\beta,q,s)} \leq C_q \|f\|_{L(\beta,1,s)}$ for any $1 < q < \infty$. In fact, by (7),

$$\begin{aligned}\int_Q \left(|Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right|\right)^q dx &\leq q\int_0^\infty \lambda^{q-1}\left|\left\{x \in Q : |Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right| > \lambda\right\}\right| d\lambda \\ &\leq C_1 q|Q|\int_0^\infty \lambda^{q-1} e^{-C_2\lambda/\|f\|_{L(\beta,1,s)}} d\lambda\end{aligned} \tag{23}$$

make the change of variables $\mu = C_2\lambda/\|f\|_{L(\beta,1,s)}$, then we get

$$\begin{aligned}\frac{1}{|Q|}\int_Q \left(|Q|^{-\beta}\left|f(x) - P_Q(f)(x)\right|\right)^q dx &\leq C_1 q\left(\frac{\|f\|_{L(\beta,1,s)}}{C_2}\right)^q \int_0^\infty \mu^{q-1} e^{-\mu} d\mu \\ &= C_1 q C_2^{-q}\Gamma(q)\left(\|f\|_{L(\beta,1,s)}\right)^q\end{aligned} \tag{24}$$

which yields the desired inequality. □

As a consequence of the proof of Corollary 1, we get two additional results.

Corollary 2. *Given $\beta \geq 0$, $s \geq 0$, $1 \leq q < \infty$, if $f \in L(\beta, q, s)$, then there exists $\lambda > 0$ such that for any cube Q,*

$$\frac{1}{|Q|}\int_Q e^{\lambda|Q|^{-\beta}|f(x) - P_Q(f)(x)|} dx < \infty. \tag{25}$$

Corollary 3. *Given $\beta \geq 0$, $s \geq 0$, $1 \leq q < \infty$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, suppose there exist constants C_1, C_2, and K such that for any cube Q and $\lambda > 0$,*

$$|\{x \in Q : |Q|^{-\beta}|f(x) - P_Q(f)(x)| > \lambda\}| \leq C_1 e^{-C_2\lambda/K}|Q|. \tag{26}$$

Then $f \in L(\beta, q, s)$.

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