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Research Article

On Multivariate Grüss Inequalities

Chang-Jian Zhao[1] and Wing-Sum Cheung[2]

[1] *Department of Information and Mathematics Sciences, College of Science, China Jiliang University, Hangzhou 310018, China*
[2] *Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong*

Correspondence should be addressed to Chang-Jian Zhao, chjzhao@163.com

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The main purpose of the present paper is to establish some new Grüss integral inequalities in n independent variables. Our results in special cases yield some of the recent results on Pachpatte's, Mitrinović's, and Ostrowski's inequalities, and provide new estimates on such types of inequalities.

1. Introduction

The well-known Grüss integral inequality [1] can be stated as follows (see [2, page 296]):

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right| \leq \frac{1}{4}(P-p)(Q-q), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ such that $p \leq f(x) \leq P$, $q \leq g(x) \leq Q$, for all $x \in [a,b]$, where p, P, q, Q are real constants.

Many generalizations, extensions, and variants of this inequality (1.1) have appeared in the literature, see [1–8] and the references given therein. The main purpose of the present paper is to establish several multivariate Grüss integral inequalities. Our results provide a new estimates on such type of inequalities.

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $D = \{(x_1,\ldots,x_n) : a_i \leq x_i \leq b_i\,(i = 1,\ldots,n)\}$. For a function $u(x) : \mathbb{R}^n \to \mathbb{R}$, we denote the first-order partial derivatives by $(\partial u(x)/\partial x_i)$ $(i = 1, \ldots, n)$ and $\int_D u(x)dx$ the n-fold integral $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \ldots, x_n)dx_1 \cdots dx_n$.

For continuous functions $p(x)$, $q(x) : D \to \mathbb{R}$ which are differentiable on D and $w(x) : D \to [0, \infty)$ an integrable function such that $\int_D w(x)dx > 0$, we use the notation

$$G[w, p, q]_n := \int_D w(x)p(x)q(x)dx - \frac{\left(\int_D w(x)p(x)dx\right)\left(\int_D w(x)q(x)dx\right)}{\int_D w(x)dx} \tag{2.1}$$

to simplify the details of presentation. Furthermore, if $\sum_{i=1}^{n}(\partial h/\partial x_i)\cdot(x_i - y_i) \neq 0$, for any $x, y \in D$, we use the abbreviations

$$\begin{aligned} &G[\Sigma_c, w, g, h]_n \\ &\quad := \frac{\int_D\left(\int_D\left(\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)/\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)\right)w(y)dy\right)g(x)h(x)w(x)dx}{\int_D w(y)dy} \\ &\qquad - \frac{\int_D\left(\int_D\left(\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)/\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)\right)w(y)h(y)dy\right)g(x)w(x)dx}{\int_D w(y)dy}, \\ &G[\Sigma_d, w, f, h]_n \\ &\quad := \frac{\int_D\left(\int_D\left(\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)/\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)\right)w(y)dy\right)f(x)h(x)w(x)dx}{\int_D w(y)dy} \\ &\qquad - \frac{\int_D\left(\int_D\left(\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)/\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)\right)w(y)h(y)dy\right)f(x)w(x)dx}{\int_D w(y)dy}. \end{aligned} \tag{2.2}$$

It is clear that if

$$\frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} = 1, \tag{2.3}$$

then $G[\Sigma_c, w, g, h]_n = G[w, g, h]_n$ and $G[\Sigma_d, w, f, h]_n = G[w, f, h]_n$.

Our main results are established in the following theorems.

Theorem 2.1. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be continuous functions on D. If f, g are differentiable on the interior of D and $w(x) : D \to [0, \infty)$ an integrable function such that $\int_D w(x)dx > 0$. If $\sum_{i=1}^{n}(\partial h/\partial x_i)\cdot(x_i - y_i) \neq 0$, for every $x \in D$, then*

$$\left|G[w, f, g]_n\right| \leq \frac{1}{2}\left\{\left|G[\Sigma_c, w, g, h]_n\right| + \left|G[\Sigma_d, w, f, h]_n\right|\right\}. \tag{2.4}$$

Proof. Let $x, y \in D$ with $x \neq y$. From the n-dimensional version of the Cauchy's mean value theorem (see [9]), we have

$$\begin{aligned} f(x) - f(y) &= \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)}(h(x) - h(y)), \\ g(x) - g(y) &= \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)}(h(x) - h(y)), \end{aligned} \tag{2.5}$$

where $c = (y_1+\alpha(x_1-y_1),\dots,y_n+\alpha(x_n-y_n))$ and $d = (y_1+\beta(x_1-y_1),\dots,y_n+\beta(x_n-y_n))$ $(0 < \alpha < 1,\ 0 < \beta < 1)$. Multiplying both sides of (2.5) by $g(x)$ and $f(x)$, respectively, and adding, we get

$$2f(x)g(x) - g(x)f(y) - f(x)g(y) = \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i-y_i)}(g(x)h(x)-g(x)h(y)) + \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i-y_i)}(f(x)h(x)-f(x)h(y)). \tag{2.6}$$

Multiplying both sides of (2.6) by $w(y)$ and integrating the resulting identity with respect to y over D, we have

$$\begin{aligned}
&2\left(\int_D w(y)dy\right)f(x)g(x) - g(x)\int_D w(y)f(y)dy - f(x)\int_D w(y)g(y)dy \\
&\quad= \left(\int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i-y_i)}w(y)dy\right)g(x)h(x) \\
&\qquad- g(x)\int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i-y_i)}w(y)h(y)dy \\
&\qquad+ \left(\int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i-y_i)}w(y)dy\right)f(x)h(x) \\
&\qquad- f(x)\int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i-y_i)}w(y)h(y)dy.
\end{aligned} \tag{2.7}$$

Next, multiplying both sides of (2.7) by $w(x)$ and integrating the resulting identity with respect to x over D, we have

$$\begin{aligned}
&2\left(\int_D w(y)dy\right)\int_D w(x)f(x)g(x)dx - \left(\int_D w(x)g(x)dx\right)\left(\int_D w(y)f(y)dy\right) \\
&\quad- \left(\int_D w(x)f(x)dx\right)\left(\int_D w(y)g(y)dy\right) \\
&\quad= \int_D\left(\int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i-y_i)}w(y)dy\right)g(x)h(x)w(x)dx \\
&\qquad- \int_D\left(\int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i-y_i)}w(y)h(y)dy\right)g(x)w(x)dx \\
&\qquad+ \int_D\left(\int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i-y_i)}w(y)dy\right)f(x)h(x)w(x)dx \\
&\qquad- \int_D\left(\int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i-y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i-y_i)}w(y)h(y)dy\right)f(x)w(x)dx.
\end{aligned} \tag{2.8}$$

From (2.8), it is easy to observe that

$$\left|G[w,f,g]_n\right| \le \frac{1}{2}\left\{\left|G[\Sigma_c,w,g,h]_n\right| + \left|G[\Sigma_d,w,f,h]_n\right|\right\}. \tag{2.9}$$

The proof is complete. □

Remark 2.2. When $n = 1$, we have $D = [a_1, b_1]$ and

$$\frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{f'(c)}{h'(c)}, \qquad \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} = \frac{g'(d)}{h'(d)}, \tag{2.10}$$

where $c = y_1 + \alpha(x_1 - y_1)$, $0 < \alpha < 1$, and $d = y_1 + \beta(x_1 - y_1)$, $0 < \beta < 1$. In this case, (2.4) reduces to the following inequality which was given by Pachpatte in [8]:

$$\left|G[w,f,g]\right| \le \frac{1}{2}\left\{\left\|\frac{f'}{h'}\right\|_\infty \left|G[w,g,h]\right| + \left\|\frac{g'}{h'}\right\|_\infty \left|G[w,f,h]\right|\right\}, \tag{2.11}$$

where $f(x), g(x), h(x) : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable in (a,b), $w : [a,b] \to [0,\infty)$ is an integrable function with $\int_a^b w(x)dx > 0$, $\|\cdot\|_\infty$ is the sup norm, and

$$G[w,p,q] := \int_a^b w(x)p(x)q(x)dx - \frac{\left(\int_a^b w(x)p(x)dx\right)\left(\int_a^b w(x)q(x)dx\right)}{\int_a^b w(x)dx}. \tag{2.12}$$

Remark 2.3. If

$$\frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} = 1, \tag{2.13}$$

we have $G[\Sigma_c, w, g, h]_n = G[w,f,g]_n$ and $G[\Sigma_d, w, f, h]_n = G[w,f,h]_n$. In this case, (2.4) reduces to the following interesting inequality:

$$\left|G[w,f,g]_n\right| \le \frac{1}{2}\left\{\left|G[w,g,h]_n\right| + \left|G[w,f,h]_n\right|\right\}. \tag{2.14}$$

Remark 2.4. If $h(x) = \sum_{i=1}^{n} x_i$, then (2.5) reduces to the following results, respectively,

$$f(x) - f(y) = \sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i), \quad g(x) - g(y) = \sum_{i=1}^{n}\frac{\partial g(d)}{\partial x_i}(x_i - y_i). \tag{2.15}$$

Furthermore, letting $w(y) = 1$, (2.7) reduces to

$$\begin{aligned} &\left| f(x)g(x) - \frac{1}{2M}g(x)\int_D f(y)dy - \frac{1}{2M}f(x)\int_D g(y)dy \right| \\ &\quad \le \frac{1}{2M}\sum_{i=1}^{n}\left(\left|g(x)\right|\left\|\frac{\partial f}{\partial x_i}\right\|_\infty + \left|f(x)\right|\left\|\frac{\partial g}{\partial x_i}\right\|_\infty\right)E_i(x), \end{aligned} \tag{2.16}$$

where $M = \text{mes}D := \prod_{i=1}^{n}(b_i - a_i)$, and $E_i(x) := \int_D |x_i - y_i| dy$. This is precisely a new inequality established by Pachpatte in [6]. If, in addition, $g(x) \equiv 1$, then inequality (2.16) reduces to the inequality established by Mitrinović in [2], which is in turn a generalization of the well-known Ostrowski inequality.

Theorem 2.5. *Let f, g, h be as in Theorem 2.1. Then,*

$$\begin{aligned}
\left|G[w,f,g]_n\right| \le{}& \frac{1}{\left(\int_D w(y)dy\right)^2} \\
&\times \Bigg| \int_D \Bigg(w(x)h^2(x) \int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \\
&\qquad \cdot \int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy \Bigg) dx \\
&+ \int_D \Bigg(w(x) \int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy \\
&\qquad \cdot \int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy \Bigg) dx \\
&- 2\int_D \Bigg(w(x)h(x) \int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \\
&\qquad \cdot \int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy \Bigg) dx \Bigg|.
\end{aligned} \tag{2.17}$$

Proof. Multiplying both sides of (2.5) by $w(y)$ and integrate the resulting identities with respect to y on D, we get, respectively,

$$\begin{aligned}
&\left(\int_D w(y)dy\right) f(x) - \int_D w(y)f(y)dy \\
&\quad = h(x)\int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy - \int_D \frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy, \\
&\left(\int_D w(y)dy\right) g(x) - \int_D w(y)g(y)dy \\
&\quad = h(x)\int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy - \int_D \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy.
\end{aligned} \tag{2.18}$$

Multiplying the left sides and right sides of (2.18), we get

$$\begin{aligned}
&\left(\int_D w(y)dy\right)^2 f(x)g(x) - \left(\int_D w(y)dy\right)f(x)\left(\int_D w(y)g(y)dy\right)\\
&\quad - \left(\int_D w(y)dy\right)g(x)\left(\int_D w(y)f(y)dy\right) + \left(\int_D w(y)f(y)dy\right)\left(\int_D w(y)g(y)dy\right)\\
&\quad = h^2(x)\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy\\
&\quad + \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy\\
&\quad - h(x)\int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy\\
&\quad - h(x)\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy.
\end{aligned} \tag{2.19}$$

Multiplying both sides of (2.19) by $w(x)$ and integrating the resulting identity with respect to x over D, we get

$$\begin{aligned}
&\left(\int_D w(y)dy\right)^2 \int_D w(x)f(x)g(x)dx - \left(\int_D w(y)dy\right)\left(\int_D w(x)f(x)dx\right)\left(\int_D w(y)g(y)dy\right)\\
&- \left(\int_D w(y)dy\right)\left(\int_D w(x)g(x)dx\right)\left(\int_D w(y)f(y)dy\right)\\
&+ \left(\int_D w(x)dx\right)\left(\int_D w(y)f(y)dy\right)\left(\int_D w(y)g(y)dy\right)\\
&= \int_D \left(w(x)h^2(x)\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy\right)dx\\
&+ \int_D \left(w(x)\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy\right)dx\\
&- \int_D \left(w(x)h(x)\int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)h(y)dy\right)dx\\
&- \int_D \left(w(x)h(x)\int_D \frac{\sum_{i=1}^n (\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(c)/\partial x_i)(x_i - y_i)} w(y)dy \cdot \int_D \frac{\sum_{i=1}^n (\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^n (\partial h(d)/\partial x_i)(x_i - y_i)} w(y)h(y)dy\right)dx.
\end{aligned} \tag{2.20}$$

From (2.20), it is easy to arrive at inequality (2.17). The proof of Theorem 2.5 is completed. □

Remark 2.6. Taking $n = 1$, we have $D = [a_1, b_1]$ and

$$\frac{\sum_{i=1}^{n}(\partial f(c)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(c)/\partial x_i)(x_i - y_i)} = \frac{f'(c)}{h'(c)}, \qquad \frac{\sum_{i=1}^{n}(\partial g(d)/\partial x_i)(x_i - y_i)}{\sum_{i=1}^{n}(\partial h(d)/\partial x_i)(x_i - y_i)} = \frac{g'(d)}{h'(d)}, \tag{2.21}$$

where $c = y_1 + \alpha(x_1 - y_1)$, $0 < \alpha < 1$, and $d = y_1 + \beta(x_1 - y_1)$, $0 < \beta < 1$. In this case, (2.20) becomes the following inequality which was given by Pachpatte in [8]:

$$|G[w, f, g]| \leq \left| \int_a^b w(x)h^2(x)dx - \frac{\left(\int_a^b w(x)h(x)dx\right)^2}{\int_a^b w(x)dx} \right| \left\| \frac{f'}{g'} \right\|_\infty \left\| \frac{g'}{h'} \right\|_\infty, \tag{2.22}$$

where $f(x), g(x), h(x) : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b), $w : [a, b] \to [0, \infty)$ is an integrable function with $\int_a^b w(x)dx > 0$, and

$$G[w, p, q] := \int_a^b w(x)p(x)q(x)dx - \frac{\left(\int_a^b w(x)p(x)dx\right)\left(\int_a^b w(x)q(x)dx\right)}{\int_a^b w(x)dx}. \tag{2.23}$$

Remark 2.7. If $h(x) = \sum_{i=1}^{n} x_i$, then (2.5) becomes

$$f(x) - f(y) = \sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}(x_i - y_i), \quad g(x) - g(y) = \sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i}(x_i - y_i). \tag{2.24}$$

Multiplying the left and right sides of (2.24), we get

$$f(x)g(x) - f(x)g(y) - g(x)f(y) + f(y)g(y) = \left[\sum_{i=1}^{n} \frac{\partial f(c)}{\partial x_i}(x_i - y_i)\right]\left[\sum_{i=1}^{n} \frac{\partial g(d)}{\partial x_i}(x_i - y_i)\right]. \tag{2.25}$$

Integrating both sides of (2.25) with respect to y on D, we have the following inequality which was established by Pachpatte in [6]:

$$\begin{aligned} &\left| f(x)g(x) - f(x)\left(\frac{1}{M}\int_D g(y)dy\right) - g(x)\left(\frac{1}{M}\int_D f(y)dy\right) + \frac{1}{M}\int_D f(y)g(y)dy \right| \\ &\quad \leq \frac{1}{M}\int_D \left\{ \left[\sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i|\right]\left[\sum_{i=1}^{n} \left\| \frac{\partial g}{\partial x_i} \right\|_\infty |x_i - y_i|\right] \right\} dy, \end{aligned} \tag{2.26}$$

where $M = \text{mes}D = \prod_{i=1}^{n}(b_i - a_i)$.

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