Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 270713, 19 pages
doi:10.1155/2008/270713

Research Article

Congruences for Generalized *q*-Bernoulli Polynomials

Mehmet Cenkci and Veli Kurt

Department of Mathematics, Akdeniz University, 07058 Antalya, Turkey

Correspondence should be addressed to Mehmet Cenkci, cenkci@akdeniz.edu.tr

Received 9 December 2007; Accepted 15 February 2008

Recommended by Andrea Laforgia

In this paper, we give some further properties of p-adic q-L-function of two variables, which is recently constructed by Kim (2005) and Cenkci (2006). One of the applications of these properties yields general classes of congruences for generalized q-Bernoulli polynomials, which are q-extensions of the classes for generalized Bernoulli numbers and polynomials given by Fox (2000), Gunaratne (1995), and Young (1999, 2001).

1. Introduction and primary concepts

For $n \in \mathbb{Z}$, $n \geq 0$, Bernoulli numbers B_n originally arise in the study of finite sums of a given power of consecutive integers. They are given by $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30, \ldots$, with $B_{2n+1} = 0$ for odd $n > 1$, and

$$B_n = -\frac{1}{n+1}\sum_{m=0}^{n-1}\binom{n+1}{m}B_m, \tag{1.1}$$

for all $n \geq 1$. In the symbolic notation, Bernoulli numbers are given recursively by

$$(B+1)^n - B_n = \delta_{n,1}, \tag{1.2}$$

with the usual convention about replacing B^j by B_j, where $\delta_{n,1}$ is the Kronecker symbol. The Bernoulli polynomials $B_n(z)$ can be expressed in the form

$$B_n(z) = (B+z)^n = \sum_{m=0}^{n}\binom{n}{m}B_m z^{n-m}, \tag{1.3}$$

for an indeterminate z. The generating functions of these numbers and polynomials are given, respectively, by

$$F(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$
$$F(z,t) = \frac{t}{e^t - 1} e^{zt} = \sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!}, \tag{1.4}$$

for $|t| < 2\pi$. One of the notable facts about Bernoulli numbers and polynomials is the relation between the Riemann and the Hurwitz (or generalized) zeta functions.

Theorem 1.1 (see [1, 2]). *For every integer $n \geq 1$,*

$$\zeta(1-n) = -\frac{B_n}{n}, \qquad \zeta(1-n,z) = -\frac{B_n(z)}{n}, \tag{1.5}$$

where $\zeta(s)$ and $\zeta(s,z)$ are the Riemann and the Hurwitz (or generalized) zeta functions, defined, respectively, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \qquad \zeta(s,z) = \sum_{m=0}^{\infty} \frac{1}{(m+z)^s}, \tag{1.6}$$

with $s \in \mathbb{C}$, $\Re(s) > 1$, and $z \in \mathbb{C}$ with $\Re(z) > 0$.

Among various generalizations of Bernoulli numbers and polynomials, generalization with a primitive Dirichlet character χ has a special case of attention.

Definition 1.2 (see [2, 3]). For a primitive Dirichlet character χ having conductor $f \in \mathbb{Z}$, $f \geq 1$, the generalized Bernoulli numbers $B_{n,\chi}$ and polynomials $B_{n,\chi}(z)$ associated with χ are defined by

$$F_\chi(t) = \sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!},$$
$$F_\chi(z,t) = \sum_{a=1}^{f} \frac{\chi(a) t e^{(a+z)t}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(z) \frac{t^n}{n!}, \tag{1.7}$$

respectively, for $|t| < 2\pi/f$.

When $\chi = 1$, the classical Bernoulli numbers and polynomials are obtained in that $B_{n,1} = (-1)^n B_n$ and $B_{n,1}(z) = (-1)^n B_n(-z)$. The generalized Bernoulli numbers and polynomials can be expressed in terms of Bernoulli polynomials as

$$B_{n,\chi} = f^{n-1} \sum_{a=1}^{f} \chi(a) B_n\left(\frac{a}{f}\right),$$
$$B_{n,\chi}(z) = f^{n-1} \sum_{a=1}^{f} \chi(a) B_n\left(\frac{a+z}{f}\right). \tag{1.8}$$

Given a primitive Dirichlet character χ having conductor f, the Dirichlet L-function associated with χ is defined by [1, 2]

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}, \tag{1.9}$$

where $s\in\mathbb{C}$, Re $(s)>1$. It is well known [2] that $L(s,\chi)$ may be analytically continued to the whole complex plane, except for a simple pole at $s=1$ when $\chi=1$, in which case it reduces to Riemann zeta function, $\zeta(s)=L(s,1)$. The generalized Bernoulli numbers share a particular relationship with the Dirichlet L-function in that

$$L(1-n,\chi)=-\frac{B_{n,\chi}}{n}, \tag{1.10}$$

for $n\in\mathbb{Z}$, $n\geq 1$.

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Q}_p$, normalized so that $|p|_p=p^{-1}$. Let $p^*=4$ if $p=2$ and $p^*=p$ otherwise. Note that there exist $\phi(p^*)$ distinct solutions, modulo p^*, to the equation $x^{\phi(p^*)}-1=0$, and each solution must be congruent to one of the values $a\in\mathbb{Z}$, where $1\leq a\leq p^*-1$, $(a,p)=1$. Thus, by Hensel's lemma, given $a\in\mathbb{Z}$ with $(a,p)=1$, there exists a unique $w(a)\in\mathbb{Z}_p$ such that $w(a)\equiv a(\bmod\ p^*\mathbb{Z}_p)$. Letting $w(a)=0$ for $a\in\mathbb{Z}$ such that $(a,p)\neq 1$, it can be seen that w is actually a Dirichlet character having conductor $f_w=p^*$, called the Teichmüller character. Let $x=w(x)\langle x\rangle$. Then $\langle x\rangle\equiv 1(\bmod\ p^*\mathbb{Z}_p)$. In the sense of product of characters, let $\chi_n=\chi w^{-n}$. This implies that $f_{\chi_n}\mid fp^*$. Since $\chi=\chi_n w^n$, $f\mid f_{\chi_n}p^*$ is also true. Thus, f and f_{χ_n} differ by a factor that is a power of p.

During the development of p-adic analysis, researches were made to derive a meromorphic function, defined over the p-adic number field, that would interpolate the same, or at least similar values as the Dirichlet L-function at nonpositive integers. In [4], Kubota and Leopoldt proved the existence of such a function, considered as p-adic equivalent of the Dirichlet L-function.

Proposition 1.3 (see [3, 4]). *There exists a unique p-adic meromorphic (analytic if $\chi\neq 1$) function $L_p(s,\chi)$, $s\in\mathbb{Z}_p$, for which*

$$L_p(1-n,\chi)=\left(1-\chi_n(p)p^{n-1}\right)L(1-n,\chi_n), \tag{1.11}$$

for $n\in\mathbb{Z}$, $n\geq 1$.

By (1.10), this function yields the values

$$L_p(1-n,\chi)=-\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right)B_{n,\chi_n}, \tag{1.12}$$

for $n\in\mathbb{Z}$, $n\geq 1$. Since the time of the work of Kubota and Leopoldt, many mathematicians have derived the existence and generalizations of $L_p(s,\chi)$ by various means [5–12]. In particular, Washington [11] derived the function by elementary means and expressed it in an explicit form.

Let D denote the region

$$D=\left\{s\in\mathbb{C}_p:|s-1|_p<|p|_p^{1/(p-1)}|p^*|_p^{-1}\right\}. \tag{1.13}$$

Theorem 1.4 (see [11]). *Let F be a positive integer multiple of p^* and f, and let*

$$L_p(s,\chi)=\frac{1}{s-1}\frac{1}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{m=0}^{\infty}\binom{1-s}{m}\left(\frac{F}{a}\right)^{m}B_m. \tag{1.14}$$

Then, $L_p(s,\chi)$ is analytic for $s\in D$, when $\chi\neq 1$, and meromorphic for $s\in D$, with a simple pole at $s=1$, having residue $1-1/p$, when $\chi=1$. Furthermore, for each $n\in\mathbb{Z}$, $n\geq 1$,

$$L_p(1-n,\chi)=-\frac{1}{n}\left(1-\chi_n(p)p^{n-1}\right)B_{n,\chi_n}. \tag{1.15}$$

Thus, $L_p(s,\chi)$ vanishes identically if $\chi(-1)=-1$.

In [6], Fox derived a p-adic function $L_p(s,z,\chi)$, where $z\in\mathbb{C}_p$, $|z|_p\leq 1$, and $s\in D$, that interpolates the values

$$L_p(1-n,z,\chi)=-\frac{1}{n}\left(B_{n,\chi_n}(p^*z)-\chi_n(p)p^{n-1}B_{n,\chi_n}(p^{-1}p^*z)\right), \tag{1.16}$$

for positive integers n. By applying the method that Washington used to derive Theorem 1.4, Fox [7] obtained $L_p(s,z,\chi)$ by elementary means and expressed it in an explicit form.

Theorem 1.5 (see [7]). *Let F be a positive integer multiple of p^* and f, and let*

$$\begin{aligned}L_p(s,z,\chi)&=\frac{1}{s-1}\frac{\chi(-1)}{F}\sum_{\substack{a=1\\(a,p)=1}}^{F}\chi(a)\langle a-p^*z\rangle^{1-s}\\&\quad\times\sum_{m=0}^{\infty}\binom{1-s}{m}\left(\frac{F}{a-p^*z}\right)^{m}B_m.\end{aligned} \tag{1.17}$$

Then, $L_p(s,z,\chi)$ is analytic for $z\in\mathbb{C}_p$, $|z|_p\leq 1$, provided that $s\in D$, except for $s\neq 1$ when $\chi=1$. Also, if $z\in\mathbb{C}_p$, $|z|_p\leq 1$, this function is analytic for $s\in D$ when $\chi\neq 1$, and meromorphic for $s\in D$, with a simple pole at $s=1$, having residue $1-1/p$, when $\chi=1$. Furthermore, for each $n\in\mathbb{Z}$, $n\geq 1$,

$$L_p(1-n,z,\chi)=-\frac{1}{n}\left(B_{n,\chi_n}(p^*z)-\chi_n(p)p^{n-1}B_{n,\chi_n}(p^{-1}p^*z)\right). \tag{1.18}$$

In [12], Young gave p-adic integral representations for the two-variable p-adic L-function introduced by Fox. These representations leaded to generalizations of some formulas of Diamond [13, 14] and of Ferrero and Greenberg [15] for p-adic L-functions in terms of the p-adic gamma and log gamma functions. But, his work was restricted to character χ such that the conductor of χ_1 is not a power of p. The explicit formula given in Theorem 1.5 by Fox yielded to derive formulas similar to that obtained by Young, but for all primitive Dirichlet character χ.

In [16], Carlitz defined q-extensions of Bernoulli numbers and polynomials, and proved properties generalizing those satisfied by B_n and $B_n(z)$. When talking about q-extensions, q can be considered as an indeterminate, a complex number $q\in\mathbb{C}$ or a p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$, then it is assumed that $|q|<1$ and if $q\in\mathbb{C}_p$, then it is assumed that $|1-q|_p<p^{-1/(p-1)}$, so

that $q^x = \exp(\log_p q)$ for $|x|_p \leq 1$, where log is the Iwasawa p-adic logarithm function (see [3, Chapter 4]).

The q-Bernoulli numbers $\beta_{n,q}$, $n \in \mathbb{Z}$, $n \geq 0$, are usually defined by

$$\beta_{0,q} = \frac{q-1}{\log q}, \quad (q\beta_q + 1)^n - \beta_{n,q} = \delta_{n,1}, \tag{1.19}$$

where the usual convention about replacing β_q^j by $\beta_{j,q}$ in the binomial expansion is understood [8, 17–24]. It follows from (1.19) that

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{i}{[i]_q}, \tag{1.20}$$

where it is understood that for $i = 0$, the function $i/[i]_q = 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q}, \tag{1.21}$$

so that $\lim_{q\to 1}[x]_q = x$ for any $x \in \mathbb{C}$ in the complex case and $x \in \mathbb{C}_p$ with $|x|_p \leq 1$ in the p-adic case. In [8, 9], Kim defined q-Bernoulli polynomials $\beta_{n,q}(z)$, $n \in \mathbb{Z}$, $n \geq 0$, as

$$\begin{aligned} \beta_{n,q}(z) &= (q^z \beta_q + z)^n \\ &= \sum_{m=0}^{n} \binom{n}{m} q^{mz} \beta_{m,q} [z]_q^{n-m} \\ &= \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i q^{iz} \frac{i}{[i]_q}. \end{aligned} \tag{1.22}$$

Some basic properties of q-Bernoulli polynomials $\beta_{n,q}(z)$ similar to those of Bernoulli polynomials $B_n(z)$ can be deduced from (1.22) (see also [25]). For instance, we have

$$\beta_{n,q^{-1}}(1-z) = (-1)^n q^{n-1} \beta_{n,q}(z), \tag{1.23}$$

$$\beta_{n,q}(1+z) - \beta_{n,q}(z) = nq^z [z]_q^{n-1}, \tag{1.24}$$

$$\beta_{n,q}(z+\tau) = \sum_{m=0}^{n} \binom{n}{m} q^{mz} \beta_{m,q}(\tau) [z]_q^{n-m}. \tag{1.25}$$

Let χ be a Dirichlet character with conductor f. The generalized q-Bernoulli polynomials associated with χ, $\beta_{n,q,\chi}(z)$, $n \in \mathbb{Z}$, $n \geq 0$, are defined by [8, 9]

$$\beta_{n,q,\chi}(z) = [f]_q^{n-1} \sum_{a=1}^{f} \chi(a) \beta_{n,q^f}\left(\frac{a+z}{f}\right). \tag{1.26}$$

For $z = 0$, $\beta_{n,q,\chi}(0) = \beta_{n,q,\chi}$ are the generalized q-Bernoulli numbers,

$$\beta_{n,q,\chi} = [f]_q^{n-1} \sum_{a=1}^{f} \chi(a) \beta_{n,q^f}\left(\frac{a}{f}\right). \tag{1.27}$$

From (1.25), (1.26), and (1.27),

$$\beta_{n,q,\chi}(z) = \sum_{m=0}^{n} \binom{n}{m} q^{mz} \beta_{m,q,\chi} [z]_q^{n-m}. \tag{1.28}$$

An important property that the polynomials $\beta_{n,q,\chi}(z)$ satisfy is the following, which can be proved by using (1.24) and (1.26):

Proposition 1.6. *For $m \in \mathbb{Z}$, $m \geq 1$,*

$$\beta_{n,q,\chi}(mf + z) - \beta_{n,q,\chi}(z) = n \sum_{a=1}^{mf} \chi(a) q^{a+z} [a + z]_q^{n-1} \tag{1.29}$$

for all $n \in \mathbb{Z}$, $n \geq 1$.

Note that for $\chi = 1$ (i.e., $f = 1$), $z = 0$, and $q \to 1$, Proposition 1.6 reduces to

$$\sum_{a=1}^{m} a^{n-1} = \frac{1}{n} \left(B_{n,1}(m) - B_{n,1} \right), \tag{1.30}$$

which is the well-known property of Bernoulli numbers and polynomials.

Let $\mathbb{K}$ be an extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$. An infinite series $\sum a_n$, $a_n \in \mathbb{K}$, converges in $\mathbb{K}$ if and only if $|a_n|_p \to 0$, as $n \to \infty$. Let $\mathbb{K}[[x]]$ and $\mathbb{K}[x]$ be, respectively, the algebras of formal power series and of polynomials in x. Then, $A(x) = \sum a_n x^n \in \mathbb{K}[[x]]$ converges at $x = \eta$, $\eta \in \mathbb{C}_p$, if and only if $|a_n \eta^n|_p \to 0$, as $n \to \infty$. The following is a uniqueness property for power series found in [3].

Lemma 1.7. *Let $A(x), B(x) \in \mathbb{K}[[x]]$ such that each converges in a neighborhood of 0 in $\mathbb{C}_p$. If $A(\eta_n) = B(\eta_n)$ for a sequence $\{\eta_n\}$, $\eta_n \neq 0$, in $\mathbb{C}_p$ such that $\eta_n \to 0$, then $A(x) = B(x)$.*

Any positive integer n can be uniquely expressed in the form

$$n = \sum_{m=0}^{k} a_m p^m, \tag{1.31}$$

where $a_m \in \mathbb{Z}$, $0 \leq a_m \leq p - 1$, for $m = 0, 1, \ldots, k$ and $a_k \neq 0$. For such n, let

$$s_p(n) = \sum_{m=0}^{k} a_m \tag{1.32}$$

be the sum of the p-adic digits of n with $s_p(0) = 0$. For any $n \in \mathbb{Z}$, let $v_p(n)$ be the highest power of p dividing n. The function v_p is additive and relates s_p by means of

$$v_p(n!) = \frac{n - s_p(n)}{p - 1} \tag{1.33}$$

for all $n \geq 0$. For $n \geq 1$, (1.33) implies that

$$v_p(n!) \leq \frac{n - 1}{p - 1}. \tag{1.34}$$

We denote a particular subring of $\mathbb{C}_p$ as

$$o = \{a \in \mathbb{C}_p : |a|_p < 1\}. \tag{1.35}$$

If $z \in \mathbb{C}_p$ such that $|z|_p \le |p|_p^m$, where $m \in \mathbb{Q}$, then $z \in p^m o$, and this can be also written as $z \equiv 0(\text{mod } p^m o)$. Let the set R be defined as

$$R = \{a \in \mathbb{C}_p : |a|_p < p^{-1/(p-1)}\}. \tag{1.36}$$

Obviously, $R \subset o$. Since $|1-q|_p < p^{-1/(p-1)}$ for $q \in \mathbb{C}_p$, we have $1-q \in R$, which implies that $q \equiv 1(\text{mod } R)$. Let $\langle a : q\rangle = [a]_q w^{-1}(a)$. For the context in the sequel, an extension of $\langle a : q\rangle$ is needed. Since w can be considered as a Dirichlet character of conductor p^*, $w(a+p^*z) = w(a)$ for $a \in \mathbb{Z}$ with $(a,p) = 1$. Thus, $\langle a + p^*z : q\rangle$ can be defined by

$$\langle a + p^*z : q\rangle = \frac{[a+p^*z]_q}{w(a)}. \tag{1.37}$$

If $z \in \mathbb{C}_p$ such that $|z|_p \le 1$, then for any $a \in \mathbb{Z}$,

$$[a+p^*z]_q = [a]_q + q^a[p^*z]_q \equiv [a]_q(\text{mod } R). \tag{1.38}$$

Thus, $\langle a + p^*z : q\rangle \equiv 1(\text{mod}\, p^*R)$.

Let F be a positive integer multiple of f and p^*. In [9], Kim defined p-adic q-L-function of two variables $L_{p,q}(s,z,\chi)$ as follows:

$$\begin{aligned} L_{p,q}(s,z,\chi) = \frac{1}{s-1}\frac{1}{[F]_q}&\sum_{\substack{a=1\\(a,p)=1}}^{F} \chi(a)\langle a + p^*z : q\rangle^{1-s} \\ &\times \sum_{m=0}^{\infty}\binom{1-s}{m}\beta_{m,q^F} q^{(a+p^*z)m}\left[\frac{F}{a+p^*z}\right]_{q^{a+p^*z}}^m. \end{aligned} \tag{1.39}$$

The analytic properties of $L_{p,q}(s,z,\chi)$ are given by the following theorem.

Theorem 1.8 (see [9]). *Let F be a positive multiple of f and p^* and let $L_{p,q}(s,z,\chi)$ be as in* (1.39). *Then, $L_{p,q}(s,z,\chi)$ is analytic for $z \in \mathbb{C}_p$, $|z|_p \le 1$, provided that $s \in D$, except for $s = 1$ if $\chi \ne 1$. Moreover, if $z \in \mathbb{C}_p$, $|z|_p \le 1$, then this function is analytic for $s \in D$ if $\chi \ne 1$ and meromorphic for $s \in D$ with a simple pole at $s = 1$ with residue $(1/[F]_q)((q^F-1)/\log q)(1-1/p)$ if $\chi = 1$. Furthermore, for $n \in \mathbb{Z}$, $n \ge 1$,*

$$L_{p,q}(1-n,z,\chi) = -\frac{1}{n}\left(\beta_{n,q,\chi_n}(p^*z) - \chi_n(p)[p]_q^{n-1}\beta_{n,q^p,\chi_n}(p^{-1}p^*z)\right). \tag{1.40}$$

Kim [9] also gave a p-adic integral representation for the function $L_{p,q}(s,z,\chi)$ and derived a q-extension of the generalized Diamond-Ferrero-Greenberg formula for the two-variable p-adic L-function in terms of p-adic gamma and log-gamma functions. In [5], first author derived $L_{p,q}(s,z,\chi)$ by using convergent power series, a method developed by Iwasawa [3]. Resulting function from this derivation is in closed form but satisfies same properties of the function defined by (1.39).

The main motivation of this paper is to derive general classes of congruences for generalized q-Bernoulli polynomials by making use of the function $L_{p,q}(s,z,\chi)$. These classes are obtained as an application of the difference formula (see (2.12) for the p-adic q-L-function of two variables, which generalizes Proposition 1.6 and thus the well-known formula for Bernoulli numbers and polynomials (1.30).

2. Properties of $L_{p,q}(s,z,\chi)$

Recall that $L_{p,q}(s,z,\chi)$, $z\in\mathbb{C}_p$, $|z|_p\le 1$, interpolates the values

$$L_{p,q}(1-n,z,\chi)=-\frac{1}{n}b_n(z,q,\chi), \tag{2.1}$$

for $n\in\mathbb{Z}$, $n\ge 1$, where

$$b_n(z,q,\chi)=\beta_{n,q,\chi_n}(p^*z)-\chi_n(p)[p]_q^{n-1}\beta_{n,q^p,\chi_n}(p^{-1}p^*z). \tag{2.2}$$

Lemma 2.1. *For all $n\in\mathbb{Z}$, $n\ge 1$,*

$$b_n(-z,q^{-1},\chi)=\chi(-1)q^{n-1}b_n(z,q,\chi). \tag{2.3}$$

Proof. We use the method in [26, 27] for the proof. First, consider the case $\chi_n=1$, which implies $\chi=w^n$. Then

$$\begin{aligned}b_n(-z,q^{-1},\chi)&=\beta_{n,q^{-1},1}(-p^*z)-[p]_{q^{-1}}^{n-1}\beta_{n,q^{-p},1}(-p^{-1}p^*z)\\&=\beta_{n,q^{-1}}(1-p^*z)-\frac{1}{(q^{p-1})^{n-1}}[p]_q^{n-1}\beta_{n,q^{-p}}(1-p^{-1}p^*z).\end{aligned} \tag{2.4}$$

From (1.23), we have

$$\begin{aligned}b_n(-z,q^{-1},\chi)&=(-1)^nq^{n-1}\beta_{n,q}(p^*z)-\frac{[p]_q^{n-1}}{(q^{p-1})^{n-1}}(-1)^n(q^p)^{n-1}\beta_{n,q^p}(p^{-1}p^*z)\\&=(-1)^nq^{n-1}\Big(\beta_{n,q}(p^*z)-[p]_q^{n-1}\beta_{n,q^p}(p^{-1}p^*z)\Big).\end{aligned} \tag{2.5}$$

Using (1.24), we obtain

$$\begin{aligned}&b_n(-z,q^{-1},\chi)\\&=(-1)^nq^{n-1}\Big\{\beta_{n,q}(1+p^*z)-nq^{p^*z}[p^*z]_q^{n-1}-[p]_q^{n-1}\beta_{n,q^p}(1+p^{-1}p^*z)+[p]_q^{n-1}n(q^p)^{p^{-1}p^*z}[p^{-1}p^*z]_{q^p}^{n-1}\Big\}\\&=(-1)^nq^{n-1}\Big(\beta_{n,q}(1+p^*z)-[p]_q^{n-1}\beta_{n,q^p}(1+p^{-1}p^*z)\Big)\\&=(-1)^nq^{n-1}\Big(\beta_{n,q,1}(p^*z)-[p]_q^{n-1}\beta_{n,q^p,1}(p^{-1}p^*z)\Big)\\&=(-1)^nq^{n-1}b_n(z,q,\chi).\end{aligned} \tag{2.6}$$

Since $\chi=w^n$ and $w(-1)=1$, the lemma holds for $\chi_n=1$.

Now, suppose that $\chi_n \neq 1$. Then, from (1.26), we obtain

$$\begin{aligned}
b_n(-z, q^{-1}, \chi) &= \beta_{n,q^{-1},\chi_n}(-p^*z) - \chi_n(p)[p]_{q^{-1}}^{n-1}\beta_{n,q^{-p},\chi_n}(-p^{-1}p^*z) \\
&= [f_{\chi_n}]_{q^{-1}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(a)\beta_{n,q^{-f_{\chi_n}}}\left(\frac{a-p^*z}{f_{\chi_n}}\right) \\
&\quad - \chi_n(p)[p]_{q^{-1}}^{n-1}[f_{\chi_n}]_{q^{-p}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(a)\beta_{n,q^{-pf_{\chi_n}}}\left(\frac{a-p^{-1}p^*z}{f_{\chi_n}}\right) \\
&= [f_{\chi_n}]_{q^{-1}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(f_{\chi_n}-a)\beta_{n,q^{-f_{\chi_n}}}\left(\frac{f_{\chi_n}-a-p^*z}{f_{\chi_n}}\right) \\
&\quad - \chi_n(p)[p]_{q^{-1}}^{n-1}[f_{\chi_n}]_{q^{-p}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(f_{\chi_n}-a)\beta_{n,q^{-pf_{\chi_n}}}\left(\frac{f_{\chi_n}-a-p^{-1}p^*z}{f_{\chi_n}}\right) \\
&= [f_{\chi_n}]_{q^{-1}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(-a)\beta_{n,q^{-f_{\chi_n}}}\left(1-\frac{a+p^*z}{f_{\chi_n}}\right) \\
&\quad - \chi_n(p)[p]_{q^{-1}}^{n-1}[f_{\chi_n}]_{q^{-p}}^{n-1}\sum_{a=1}^{f_{\chi_n}}\chi_n(-a)\beta_{n,q^{-pf_{\chi_n}}}\left(1-\frac{a+p^{-1}p^*z}{f_{\chi_n}}\right).
\end{aligned} \tag{2.7}$$

Using (1.23), we have

$$\begin{aligned}
b_n(-z, q^{-1}, \chi) &= (-1)^n\left(q^{f_{\chi_n}}\right)^{n-1}[f_{\chi_n}]_{q^{-1}}^{n-1}\chi_n(-1)\sum_{a=1}^{f_{\chi_n}}\chi_n(a)\beta_{n,q^{f_{\chi_n}}}\left(\frac{a+p^*z}{f_{\chi_n}}\right) \\
&\quad - \chi_n(p)[p]_{q^{-1}}^{n-1}(-1)^n\left(q^{f_{\chi_n}}\right)^{n-1}[f_{\chi_n}]_{q^{-p}}^{n-1}\chi_n(-1)\sum_{a=1}^{f_{\chi_n}}\chi_n(a)\beta_{n,q^{pf_{\chi_n}}}\left(\frac{a+p^{-1}p^*z}{f_{\chi_n}}\right) \\
&= (-1)^n q^{n-1}\chi_n(-1)\beta_{n,q,\chi_n}(p^*z) - \chi_n(p)[p]_q^{n-1}(-1)^n q^{n-1}\chi_n(-1)\beta_{n,q^p,\chi_n}(p^{-1}p^*z) \\
&= (-1)^n q^{n-1}\chi_n(-1)b_n(z,q,\chi).
\end{aligned} \tag{2.8}$$

Note that $\chi_n(-1) = (-1)^n\chi(-1)$. Thus, the lemma holds for $\chi_n \neq 1$. Since the lemma holds for $\chi_n = 1$ and $\chi_n \neq 1$, the proof must be complete. □

Using this result, we can prove the following theorem.

Theorem 2.2. *Let $z \in \mathbb{C}_p$, $|z|_p \leq 1$, and $s \in D$, except for $s \neq 1$ if $\chi = 1$. Then*

$$L_{p,q^{-1}}(s, -z, \chi) = \chi(-1)q^{-s}L_{p,q}(s, z, \chi). \tag{2.9}$$

Proof. Let $z \in \mathbb{C}_p$, $|z|_p \leq 1$, and $n \in \mathbb{Z}$, $n \geq 1$. Since

$$L_{p,q}(1-n, z, \chi) = -\frac{1}{n}b_n(z, q, \chi). \tag{2.10}$$

Lemma 2.1 implies that

$$
\begin{aligned}
&L_{p,q^{-1}}(1-n,-z,\chi)\\
&\quad= -\frac{1}{n}b_n\left(-z,q^{-1},\chi\right) = -\frac{1}{n}\chi(-1)q^{n-1}b_n(z,q,\chi) = \chi(-1)q^{n-1}L_{p,q}(1-n,z,\chi),
\end{aligned}
\tag{2.11}
$$

and (2.9) holds for all $s = 1-n$, $n \in \mathbb{Z}$, $n \geq 1$. Since the negative integers have 0 as a limit point, Lemma 1.7 implies that Theorem 2.2 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (2.9). It is obvious that the domains, in the variable s, of the functions on both sides of (2.9) contain D, except for $s \neq 1$ if $\chi = 1$. This completes the proof. □

It is well known that the generalized Bernoulli polynomials associated with a Dirichlet character χ are important in regard to sums of consecutive integers, all of which raised to the same power. Proposition 1.6 represents a q-extension of this property. In this section, we will give an extension of Proposition 1.6 with the use of $L_{p,q}(s,z,\chi)$.

For the character χ, let $F_0 = 1\mathrm{cm}(f,p^*)$. Then, $f_{\chi_n} \mid F_0$ for each $n \in \mathbb{Z}$. Also, let F be a positive multiple of $p(p^*)^{-1}F_0$.

Theorem 2.3. *Let $z \in \mathbb{C}_p$, $|z|_p \leq 1$, and $s \in D$, except for $s \neq 1$ if $\chi = 1$. Then*

$$
L_{p,q}(s,z+F,\chi) - L_{p,q}(s,z,\chi) = -\sum_{\substack{a=1\\(a,p)=1}}^{p^*F} \chi_1(a)q^{a+p^*z}\langle a+p^*z:q\rangle^{-s}. \tag{2.12}
$$

Proof. Let $z \in \mathbb{C}_p$, $|z|_p \leq 1$, and let $n \in \mathbb{Z}$, $n \geq 1$. From (2.1), we have

$$
L_{p,q}(1-n,z+F,\chi) - L_{p,q}(1-n,z,\chi) = -\frac{1}{n}\left(b_n(z+F,q,\chi) - b_n(z,q,\chi)\right). \tag{2.13}
$$

Equation (2.2) then implies that

$$
\begin{aligned}
&b_n(z+F,q,\chi) - b_n(z,q,\chi)\\
&= \left(\beta_{n,q,\chi_n}(p^*z+p^*F) - \beta_{n,q,\chi_n}(p^*z)\right) - \chi_n(p)[p]_q^{n-1}\left(\beta_{n,q^p,\chi_n}(p^{-1}p^*z+p^{-1}p^*F) - \beta_{n,q^p,\chi_n}(p^{-1}p^*z)\right).
\end{aligned}
\tag{2.14}
$$

By Proposition 1.6, we can write

$$
\begin{aligned}
&b_n(z+F,q,\chi) - b_n(z,q,\chi)\\
&\quad= n\sum_{a=1}^{p^*F}\chi_n(a)q^{a+p^*z}\left[a+p^*z\right]_q^{n-1} - \chi_n(p)[p]_q^{n-1}n\sum_{a=1}^{p^{-1}p^*F}\chi_n(a)\left(q^p\right)^{a+p^{-1}p^*z}\left[a+p^{-1}p^*z\right]_{q^p}^{n-1}\\
&\quad= n\sum_{a=1}^{p^*F}\chi_n(a)q^{a+p^*z}\left[a+p^*z\right]_q^{n-1} - n\sum_{\substack{a=1\\p\mid a}}^{p^*F}\chi_n(a)q^{a+p^*z}\left[a+p^*z\right]_q^{n-1}\\
&\quad= n\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_n(a)q^{a+p^*z}\left[a+p^*z\right]_q^{n-1}.
\end{aligned}
\tag{2.15}
$$

Therefore,

$$L_{p,q}(1-n,z+F,\chi)-L_{p,q}(1-n,z,\chi)=-\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_n(a)q^{a+p^*z}\left[a+p^*z\right]_q^{n-1}. \tag{2.16}$$

Since $\chi_n=\chi_1 w^{-(n-1)}$, we can write

$$\chi_n(a)\left[a+p^*z\right]_q^{n-1}=\chi_1(a)w^{-(n-1)}(a)\left[a+p^*z\right]_q^{n-1}=\chi_1(a)\left\langle a+p^*z:q\right\rangle^{n-1}. \tag{2.17}$$

Thus,

$$L_{p,q}(1-n,z+F,\chi)-L_{p,q}(1-n,z,\chi)=-\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_1(a)q^{a+p^*z}\left\langle a+p^*z:q\right\rangle^{n-1}. \tag{2.18}$$

This implies that (2.12) is true for all $s=1-n$, $n\in\mathbb{Z}$, $n\geq 1$. Since negative integers have 0 as a limit point, Lemma 1.7 implies that Theorem 2.3 is true for all s in any neighborhood about 0 common to the domains of functions on both sides of (2.12).

The domains, with respect to s, of the functions on the left of (2.12) contain D, except for $s\neq 1$ if $\chi=1$. Consider the sum

$$\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_1(a)q^{a+p^*z}\left\langle a+p^*z:q\right\rangle^{-s}=\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_1(a)q^{a+p^*z}\left\langle a+p^*z:q\right\rangle^{-1}\left\langle a+p^*z:q\right\rangle^{1-s}. \tag{2.19}$$

This sum consists of the functions of the form $q^{a+p^*z}\left\langle a+p^*z:q\right\rangle^{1-s}$, $a\in\mathbb{Z}$, $(a,p)=1$. Thus, it is sufficient to show that each such function is analytic on D;

$\left\langle a+p^*z:q\right\rangle^{1-s}$ can be written as

$$\left\langle a+p^*z:q\right\rangle^{1-s}=e^{(1-s)\log_p\langle a+p^*z:q\rangle}=\sum_{m=0}^{\infty}\frac{1}{m!}(1-s)^m\left(\log_p\left\langle a+p^*z:q\right\rangle\right)^m. \tag{2.20}$$

Since $\langle a+p^*z:q\rangle\equiv 1(\mathrm{mod}\ p^*R)$ for all $a\in\mathbb{Z}$, $(a,p)=1$ and $z\in\mathbb{C}_p$, $|z|_p\leq 1$, we have $\log_p\langle a+p^*z:q\rangle\equiv 0(\mathrm{mod}\ p^*R)$, which implies that

$$\left|\log_p\left\langle a+p^*z:q\right\rangle\right|_p<\left|p^*\right|_p|p|_p^{1/(p-1)}. \tag{2.21}$$

Now, by (1.34) and the definition of the domain D,

$$\begin{aligned}\left|q^{a+p^*z}\frac{1}{m!}(1-s)^m\left(\log_p\left\langle a+p^*z:q\right\rangle\right)^m\right|_p&<|p|_p^{1/(p-1)}|p|_p^{(m-1)/(p-1)}\left|p^*\right|_p^{-m}|p|_p^{m/(p-1)}\left|p^*\right|_p^{m}|p|_p^{m/(p-1)}\\&=|p|_p^{3m/(p-1)}\longrightarrow 0,\end{aligned} \tag{2.22}$$

as $m\to\infty$. So, whenever $s\in D$, the power series converges. Thus, the functions on either side of (2.12) have domains which contain D, except possibly for $s\neq 1$ if $\chi=1$. This completes the proof. □

Corollary 2.4. *For $s\in D$, except for $s\neq 1$ if $\chi=1$. Then*

$$L_{p,q}(s,F,\chi)-L_{p,q}(s,\chi)=-\sum_{\substack{a=1\\(a,p)=1}}^{p^*F}\chi_1(a)q^{a}\langle a:q\rangle^{-s}. \tag{2.23}$$

3. Congruences for generalized q-Bernoulli polynomials

Congruences related to classical and generalized Bernoulli numbers have found an amount of interest. One of the most celebrated examples is the Kummer congruences for classical Bernoulli numbers (cf. [2]):

$$p^{-1}\Delta_c \frac{B_n}{n} \in \mathbb{Z}_p, \tag{3.1}$$

where $c \in \mathbb{Z}$, $c \geq 1$, $c \equiv 0(\bmod\ (p-1))$, and $n \in \mathbb{Z}$ is positive, even, and $n \not\equiv 0(\bmod\ (p-1))$. Here, Δ_c is the forward difference operator which operates on a sequence $\{x_n\}$ by

$$\Delta_c x_n = x_{n+c} - x_n. \tag{3.2}$$

The powers Δ_c^k of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}. \tag{3.3}$$

More generally, it can be shown that

$$p^{-k}\Delta_c^k \frac{B_n}{n} \in \mathbb{Z}_p, \tag{3.4}$$

where $k \in \mathbb{Z}$, $k \geq 1$, and c and n are as above, but with $n > k$.

Kummer congruences for generalized Bernoulli numbers $B_{n,\chi}$ were first regarded by Carlitz [28].

For positive $c \in \mathbb{Z}$, $c \equiv 0(\bmod\ (p-1))$, $n, k \in \mathbb{Z}$, $n > k \geq 1$, and χ such that $f = f_\chi \neq p^m$, where $m \in \mathbb{Z}$, $m \geq 0$,

$$p^{-k}\Delta_c^k \frac{B_{n,\chi}}{n} \in \mathbb{Z}_p[\chi]. \tag{3.5}$$

Here, $\mathbb{Z}_p[\chi]$ denotes the ring of polynomials in χ, whose coefficients are in $\mathbb{Z}_p$.

Shiratani [29] applied the operator Δ_c^k to $-(1-\chi_n(p)p^{n-1})B_{n,\chi_n}/n$ for similar c and χ, and showed that Carlitz's congruence is still true without the restriction $n > k$, requiring only that $n \geq 1$. He also established that the divisibility conditions on c can be removed, and proved

$$(p^*)^{-k}\Delta_c^k (1-\chi_n(p)p^{n-1}) \frac{B_{n,\chi_n}}{n} \in \mathbb{Z}_p[\chi]. \tag{3.6}$$

As an extension of the Kummer congruence, Gunaratne [30, 31] showed that the value

$$p^{-k}\Delta_c^k (1-\chi_n(p)p^{n-1}) \frac{B_{n,\chi_n}}{n}, \tag{3.7}$$

modulo $p\mathbb{Z}_p$, is independent of n and

$$p^{-k}\Delta_c^k (1-\chi_n(p)p^{n-1}) \frac{B_{n,\chi_n}}{n} \equiv p^{-k'}\Delta_c^{k'} (1-\chi_{n'}(p)p^{n'-1}) \frac{B_{n',\chi_{n'}}}{n'} (\bmod\ p\mathbb{Z}_p), \tag{3.8}$$

if $p > 3$, $c, n, k \in \mathbb{Z}$ are positive, $\chi = \omega^h$, where $h \in \mathbb{Z}$, $h \not\equiv 0 (\bmod\ (p-1))$, $n', k' \in \mathbb{Z}$, $k \equiv k' (\bmod\ (p-1))$. Furthermore, by means of the binomial coefficient operator

$$\binom{p^{-1}\Delta_c}{k} x_n = \frac{1}{k!}\left(\prod_{j=0}^{k-1}\left(p^{-1}\Delta_c - j\right)\right) x_n, \tag{3.9}$$

it has been shown that for similar character χ,

$$\binom{p^{-1}\Delta_c}{k}\left(1 - \chi_n(p)p^{n-1}\right)\frac{B_{n,\chi_n}}{n} \in \mathbb{Z}_p, \tag{3.10}$$

and this value, modulo $p\mathbb{Z}_p$, is independent of n.

Fox [6] derived congruences similar to those above for the generalized Bernoulli polynomials without restrictions on the character χ.

We now consider how Corollary 2.4 can be utilized to derive a collection of congruences related to generalized q-Bernoulli polynomials. Let $F_0 = \mathrm{lcm}(f, p^*)$ and F be a positive integer multiple of $p(p^*)^{-1}F_0$. We incorporate the polynomial structure

$$B_n(z, q, \chi) = -\frac{1}{n}\left(\beta_{n,q,\chi_n}(p^*z) - \chi_n(p)[p]_q^{n-1}\beta_{n,q^p,\chi_n}\left(p^{-1}p^*z\right)\right) \tag{3.11}$$

and the set structure

$$R^* = \left\{x \in \mathbb{Z}_p : |x|_p < p^{-1/(p-1)}\right\} \tag{3.12}$$

to derive the Kummer congruences for generalized q-Bernoulli polynomials. Throughout, we assume that $q \in \mathbb{Z}_p$ with $|1-q|_p < p^{-1/(p-1)}$, so that $q \equiv 1 (\bmod\ R^*)$.

Theorem 3.1. *Let n, c, k be positive integers and $z \in p(p^*)^{-1}F_0R^*$. Then, the quantity*

$$(p^*)^{-k}\Delta_c^k B_n(z, q, \chi) - (p^*)^{-k}\Delta_c^k B_n(0, q, \chi) \in R^*[\chi], \tag{3.13}$$

and, modulo $p^*R^*[\chi]$, is independent of n.

Proof. Since Δ_c is a linear operator, Corollary 2.4 implies that

$$\Delta_c^k L_{p,q}(1-n, F, \chi) - \Delta_c^k L_{p,q}(1-n, \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{p^*F} \chi_1(a) q^a \Delta_c^k \langle a : q\rangle^{n-1}. \tag{3.14}$$

Thus,

$$\Delta_c^k B_n(F, q, \chi) - \Delta_c^k B_n(0, q, \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{p^*F} \chi_1(a) q^a \langle a : 1\rangle^{-1} \Delta_c^k \langle a : q\rangle^{n}. \tag{3.15}$$

Note that

$$\Delta_c^k \langle a : q\rangle^n = \sum_{m=0}^{k}\binom{k}{m}(-1)^{k-m}\langle a : q\rangle^{n+mc} = \langle a : q\rangle^n\left(\langle a : q\rangle^c - 1\right)^k. \tag{3.16}$$

Now, $\langle a : q\rangle \equiv 1(\mathrm{mod}\ p^* R^*)$, which implies that $\langle a : q\rangle^c \equiv 1(\mathrm{mod}\ p^* R^*)$, and thus $\Delta_c^k \langle a : q\rangle^n \equiv 0(\mathrm{mod}\ (p^*)^k R^*)$. Therefore,

$$\Delta_c^k B_n(F, q, \chi) - \Delta_c^k B_n(0, q, \chi) \equiv 0\left(\mathrm{mod}\ (p^*)^k R^*[\chi]\right), \tag{3.17}$$

and so

$$(p^*)^{-k} \Delta_c^k B_n(F, q, \chi) - (p^*)^{-k} \Delta_c^k B_n(0, q, \chi) \in R^*[\chi]. \tag{3.18}$$

Also, since $\langle a : q\rangle^c \equiv 1(\mathrm{mod}\ p^* R^*)$,

$$\Delta_c^k B_n(F, q, \chi) - \Delta_c^k B_n(0, q, \chi) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{p^* F} \chi_1(a) q^a \langle a : q\rangle^{n-1} \left(\frac{\langle a : q\rangle^c - 1}{p^*}\right)^k \tag{3.19}$$

implies that the value of $(p^*)^{-k} \Delta_c^k B_n(F, q, \chi) - (p^*)^{-k} \Delta_c^k B_n(0, q, \chi)$, modulo $p^* R^*[\chi]$, is independent of n.

Let $z \in p(p^*)^{-1} F_0 R^*$. Since the set of positive integers in $p(p^*)^{-1} F_0 \mathbb{Z}$ is dense in $p(p^*)^{-1} F_0 R^*$, there exists a sequence $\{z_j\}$ in $p(p^*)^{-1} F_0 \mathbb{Z}$ with $z_j > 0$ for each j, such that $z_j \to z$. Now, $B_n(z, q, \chi)$ is a polynomial, which implies that $B_n(z_j, q, \chi) \to B_n(z, q, \chi)$. Therefore,

$$\lim_{j\to\infty} \left(\Delta_c^k B_n(z_j, q, \chi) - \Delta_c^k B_n(0, q, \chi)\right) = \Delta_c^k B_n(z, q, \chi) - \Delta_c^k B_n(0, q, \chi). \tag{3.20}$$

The left side of this equality is 0 modulo $(p^*)^k R^*[\chi]$, which implies that

$$\Delta_c^k B_n(z, q, \chi) - \Delta_c^k B_n(0, q, \chi) \equiv 0\left(\mathrm{mod}\ (p^*)^k R^*[\chi]\right), \tag{3.21}$$

and so

$$(p^*)^{-k} \Delta_c^k B_n(z, q, \chi) - (p^*)^{-k} \Delta_c^k B_n(0, q, \chi) \in R^*[\chi]. \tag{3.22}$$

Furthermore, for a positive integer n',

$$\begin{aligned}
&\lim_{j\to\infty} \left\{\left((p^*)^{-k} \Delta_c^k B_n(z_j, q, \chi) - (p^*)^{-k} \Delta_c^k B_n(0, q, \chi)\right) - \left((p^*)^{-k} \Delta_c^k B_{n'}(z_j, q, \chi) - (p^*)^{-k} \Delta_c^k B_{n'}(0, q, \chi)\right)\right\} \\
&= \left\{\left((p^*)^{-k} \Delta_c^k B_n(z, q, \chi) - (p^*)^{-k} \Delta_c^k B_n(0, q, \chi)\right) - \left((p^*)^{-k} \Delta_c^k B_{n'}(z, q, \chi) - (p^*)^{-k} \Delta_c^k B_{n'}(0, q, \chi)\right)\right\}.
\end{aligned} \tag{3.23}$$

Since $z_j \in p(p^*)^{-1}F_0\mathbb{Z}$ for all j, the quantity on the left must be 0 modulo $p^*R^*[\chi]$. Therefore, the value $(p^*)^{-k}\Delta_c^k B_n(z,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi)$, modulo $p^*R^*[\chi]$, is independent of n. □

Theorem 3.2. *Let n, c, k, k' be positive integers with $k \equiv k' (\mathrm{mod}\ (p-1))$ and let $z \in p(p^*)^{-1}F_0R^*$. Then*

$$(p^*)^{-k}\Delta_c^k B_n(z,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi) \equiv (p^*)^{-k'}\Delta_c^{k'} B_n(z,q,\chi) - (p^*)^{-k'}\Delta_c^{k'} B_n(0,q,\chi) (\mathrm{mod}\ pR^*[\chi]). \tag{3.24}$$

Proof. Let k and k' be positive integers such that $k \equiv k' (\mathrm{mod}\ (p-1))$. Without loss of generality, assume that $k \geq k'$. From (3.19),

$$\begin{aligned}
&\left((p^*)^{-k}\Delta_c^k B_n(F,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi)\right) - \left((p^*)^{-k'}\Delta_c^{k'} B_n(F,q,\chi) - (p^*)^{-k'}\Delta_c^{k'} B_n(0,q,\chi)\right) \\
&\quad = -\sum_{\substack{a=1 \\ (a,p)=1}}^{p^*F} \chi_1(a) q^a \langle a:q\rangle^{n-1} \left\{ \left(\frac{\langle a:q\rangle^c - 1}{p^*}\right)^k - \left(\frac{\langle a:q\rangle^c - 1}{p^*}\right)^{k'} \right\} \\
&\quad = -\sum_{\substack{a=1 \\ (a,p)=1}}^{p^*F} \chi_1(a) q^a \langle a:q\rangle^{n-1} \left(\frac{\langle a:q\rangle^c - 1}{p^*}\right)^{k'} \left\{ \left(\frac{\langle a:q\rangle^c - 1}{p^*}\right)^{k-k'} - 1 \right\}.
\end{aligned} \tag{3.25}$$

If a such that

$$\langle a:q\rangle^c - 1 \not\equiv 0 (\mathrm{mod}\ pp^*R^*), \tag{3.26}$$

then, since $k-k' \equiv 0 (\mathrm{mod}\ (p-1))$, we have

$$\left(\frac{\langle a:q\rangle^c - 1}{p^*}\right)^{k-k'} - 1 \equiv 0 (\mathrm{mod}\ pR^*). \tag{3.27}$$

Thus,

$$\begin{aligned}
&(p^*)^{-k}\Delta_c^k B_n(F,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi) \\
&\quad \equiv (p^*)^{-k'}\Delta_c^{k'} B_n(F,q,\chi) - (p^*)^{-k'}\Delta_c^{k'} B_n(0,q,\chi) (\mathrm{mod}\ pR^*[\chi]).
\end{aligned} \tag{3.28}$$

Now, let $z \in p(p^*)^{-1}F_0R^*$. Then, there exists a sequence $\{z_j\}$ in $p(p^*)^{-1}F_0\mathbb{Z}$ with $z_j > 0$ for each j, such that $z_j \to z$. Consider

$$\begin{aligned}
&\lim_{j\to\infty}\left\{\left((p^*)^{-k}\Delta_c^k B_n(z_j,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi)\right) - \left((p^*)^{-k'}\Delta_c^{k'} B_n(z_j,q,\chi) - (p^*)^{-k'}\Delta_c^{k'} B_n(0,q,\chi)\right)\right\} \\
&= \left\{\left((p^*)^{-k}\Delta_c^k B_n(z,q,\chi) - (p^*)^{-k}\Delta_c^k B_n(0,q,\chi)\right) - \left((p^*)^{-k'}\Delta_c^{k'} B_n(z,q,\chi) - (p^*)^{-k'}\Delta_c^{k'} B_n(0,q,\chi)\right)\right\}.
\end{aligned} \tag{3.29}$$

Since the left side of this equality must be 0 modulo $pR^*[\chi]$, the proof follows. □

The binomial coefficient operator $\binom{T}{k}$ associated to an operator T is defined by writing the binomial coefficients

$$\binom{X}{k} = \frac{X(X-1)\cdots(X-k+1)}{k!}, \tag{3.30}$$

for $k \geq 0$ as a polynomial in X, and replacing X by T.

In the proof of next theorem, we need special numbers, namely, the Stirling numbers of the first kind $s(n,k)$, which are defined by means of the generating function

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{t^n}{n!}, \tag{3.31}$$

for $k \in \mathbb{Z}$, $k \geq 0$. Since there is no constant term in the expansion of $\log(1+t)$, $s(n,k) = 0$ for $0 \leq n < k$. Also, $s(n,n) = 1$, for all $n \geq 0$. The numbers $s(n,k)$ are integers and satisfy the following relation related to binomial coefficients:

$$\binom{x}{k} = \frac{1}{n!}\sum_{k=0}^{n} s(n,k)x^k. \tag{3.32}$$

For further information for Stirling numbers, we refer to [32].

Theorem 3.3. *Let n, c, k be positive integers and $z \in p(p^*)^{-1}F_0R^*$. Then, the quantity*

$$\binom{(p^*)^{-1}\Delta_c}{k} B_n(z,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \in R^*[\chi], \tag{3.33}$$

and, modulo $p^*R^*[\chi]$, is independent of n.

Proof. Since the binomial coefficients operator is a linear operator, Corollary 2.4 implies that

$$\binom{(p^*)^{-1}\Delta_c}{k} L_{p,q}(1-n,F,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} L_{p,q}(1-n,\chi) = -\sum_{\substack{a=1\\(a,p)=1}}^{p^*F} \chi_1(a)q^a \binom{(p^*)^{-1}\Delta_c}{k} \langle a:q\rangle^{n-1}. \tag{3.34}$$

Then,

$$\binom{(p^*)^{-1}\Delta_c}{k} B_n(F,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) = -\sum_{\substack{a=1\\(a,p)=1}}^{p^*F} \chi_1(a)q^a\langle a:q\rangle^{-1} \binom{(p^*)^{-1}\Delta_c}{k} \langle a:q\rangle^{n}. \tag{3.35}$$

Utilizing (3.32), we can write

$$\begin{aligned}\binom{(p^*)^{-1}\Delta_c}{k}\langle a:q\rangle^n &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)(p^*)^{-m}\Delta_c^m\langle a:q\rangle^n \\ &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)(p^*)^{-m}\langle a:q\rangle^n(\langle a:q\rangle^c-1)^m\end{aligned} \tag{3.36}$$

which follows from (3.16). Thus,

$$\begin{aligned}
&\binom{(p^*)^{-1}\Delta_c}{k} B_n(F,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \\
&\quad = -\sum_{\substack{a=1\\(a,p)=1}}^{p^*F} \chi_1(a) q^a \langle a:q\rangle^{-1} \langle a:q\rangle^n \binom{(p^*)^{-1}(\langle a:q\rangle^c - 1)}{k}.
\end{aligned} \tag{3.37}$$

Since $(p^*)^{-1}(\langle a\rangle_q^c - 1) \in R^*$ for each $a \in \mathbb{Z}$ with $(a,p) = 1$, we see that

$$\langle a:q\rangle^n \binom{(p^*)^{-1}(\langle a:q\rangle^c - 1)}{k} \in R^*. \tag{3.38}$$

This then implies that

$$\binom{(p^*)^{-1}\Delta_c}{k} B_n(F,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \in R^*[\chi]. \tag{3.39}$$

Furthermore, since $\langle a:q\rangle^c \equiv 1 (\mathrm{mod}\ p^*R^*)$, the value of this quantity, modulo $p^*R^*[\chi]$, is independent of n.

Now, let $z \in p(p^*)^{-1}F_0R^*$, and let $\{z_j\}$ be a sequence in $p(p^*)^{-1}F_0\mathbb{Z}$, with $z_j > 0$ for each j, such that $z_j \to z$. Then,

$$\begin{aligned}
&\lim_{j\to\infty} \binom{(p^*)^{-1}\Delta_c}{k} B_n(z_j,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \\
&\quad = \binom{(p^*)^{-1}\Delta_c}{k} B_n(z,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi)
\end{aligned} \tag{3.40}$$

must be in $R^*[\chi]$. Now, let $n' \in \mathbb{Z}$, $n' > 0$, and consider

$$\begin{aligned}
&\lim_{j\to\infty} \Bigg\{ \binom{(p^*)^{-1}\Delta_c}{k} B_n(z_j,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \\
&\qquad - \binom{(p^*)^{-1}\Delta_c}{k} B_{n'}(z_j,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_{n'}(0,q,\chi) \Bigg\} \\
&\quad = \Bigg\{ \binom{(p^*)^{-1}\Delta_c}{k} B_n(z,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi) \\
&\qquad - \binom{(p^*)^{-1}\Delta_c}{k} B_{n'}(z,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_{n'}(0,q,\chi) \Bigg\}.
\end{aligned} \tag{3.41}$$

The quantity on the left must be 0 modulo $p^*R^*[\chi]$, which implies that the value of

$$\binom{(p^*)^{-1}\Delta_c}{k} B_n(z,q,\chi) - \binom{(p^*)^{-1}\Delta_c}{k} B_n(0,q,\chi), \tag{3.42}$$

modulo $p^*R^*[\chi]$, is independent of n. □

Acknowledgment

This work was supported by Akdeniz University Scientific Research Project Unit.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 1976.

[2] L. C. Washington, *Introduction to Cyclotomic Fields*, vol. 83 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1997.

[3] K. Iwasawa, *Lectures on* p*-adic* L*-Functions*, vol. 74 of *Annals of Mathematics Studies*, Princeton University Press, Princeton, NJ, USA, 1972.

[4] T. Kubota and H.-W. Leopoldt, "Eine p-adische theorie der zetawerte. I. Einführung der p-adischen Dirichletschen L-funktionen," *Journal für die reine und angewandte Mathematik*, vol. 214-215, pp. 328–339, 1964.

[5] M. Cenkci, "p-adic q-L-function of two variables," Ph.D. thesis, Department of Mathematics, Akdeniz University, Antalya, Turkey, 2006.

[6] G. J. Fox, "A p-adic L-function of two variables," *L'Enseignement Mathématique*, vol. 46, no. 3-4, pp. 225–278, 2000.

[7] G. J. Fox, "A method of Washington applied to the derivation of a two-variable p-adic L-function," *Pacific Journal of Mathematics*, vol. 209, no. 1, pp. 31–40, 2003.

[8] T. Kim, "On p-adic q-L-functions and sums of powers," *Discrete Mathematics*, vol. 252, no. 1–3, pp. 179–187, 2002.

[9] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[10] N. Koblitz, "A new proof of certain formulas for p-adic L-functions," *Duke Mathematical Journal*, vol. 46, no. 2, pp. 455–468, 1979.

[11] L. C. Washington, "A note on p-adic L-functions," *Journal of Number Theory*, vol. 8, no. 2, pp. 245–250, 1976.

[12] P. T. Young, "On the behavior of some two-variable p-adic L-functions," *Journal of Number Theory*, vol. 98, no. 1, pp. 67–88, 2003.

[13] J. Diamond, "The p-adic log gamma function and p-adic Euler constants," *Transactions of the American Mathematical Society*, vol. 233, pp. 321–337, 1977.

[14] J. Diamond, "On the values of p-adic L-functions at positive integers," *Acta Arithmetica*, vol. 35, no. 3, pp. 223–237, 1979.

[15] B. Ferrero and R. Greenberg, "On the behavior of p-adic L-functions at $s = 0$," *Inventiones Mathematicae*, vol. 50, no. 1, pp. 91–102, 1978.

[16] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, no. 4, pp. 987–1000, 1948.

[17] M. Cenkci, M. Can, and V. Kurt, "p-adic interpolation functions and Kummer-type congruences for q-twisted and q-generalized twisted Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 2, pp. 203–216, 2004.

[18] M. Cenkci and M. Can, "Some results on q-analogue of the Lerch zeta function," *Advanced Studies in Contemporary Mathematics*, vol. 12, no. 2, pp. 213–223, 2006.

[19] T. Kim, "On explicit formulas of p-adic q-L-functions," *Kyushu Journal of Mathematics*, vol. 48, no. 1, pp. 73–86, 1994.

[20] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[21] T. Kim, "Sums of powers of consecutive q-integers," *Advanced Studies in Contemporary Mathematics*, vol. 9, no. 1, pp. 15–18, 2004.

[22] N. Koblitz, "On Carlitz's q-Bernoulli numbers," *Journal of Number Theory*, vol. 14, no. 3, pp. 332–339, 1982.

[23] Y. Simsek, D. Kim, and S.-H. Rim, "On the two-variable Dirichlet q-L-series," *Advanced Studies in Contemporary Mathematics*, vol. 10, no. 2, pp. 131–142, 2005.

[24] H. M. Srivastava, T. Kim, and Y. Simsek, "q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.

[25] T. Kim and S.-H. Rim, "A note on p-adic Carlitz's q-Bernoulli numbers," *Bulletin of the Australian Mathematical Society*, vol. 62, no. 2, pp. 227–234, 2000.

[26] T. Kim, "p-adic q-integrals associated with the Changhee-Barnes' q-Bernoulli polynomials," *Integral Transforms and Special Functions*, vol. 15, no. 5, pp. 415–420, 2004.

[27] T. Kim and S.-H. Rim, "A note on the q-integral and q-series," *Advanced Studies in Contemporary Mathematics*, vol. 2, pp. 37–45, 2000.

[28] L. Carlitz, "Arithmetic properties of generalized Bernoulli numbers," *Journal für die Reine und Angewandte Mathematik*, vol. 202, pp. 174–182, 1959.

[29] K. Shiratani, "Kummer's congruence for generalized Bernoulli numbers and its application," *Memoirs of the Faculty of Science. Kyushu University. Series A*, vol. 26, pp. 119–138, 1972.

[30] H. S. Gunaratne, "A new generalisation of the Kummer congruence," in *Computational Algebra and Number Theory*, vol. 325 of *Mathematics and Its Applications*, pp. 255–265, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1995.

[31] H. S. Gunaratne, "Periodicity of Kummer congruences," in *Number Theory*, vol. 15 of *CMS Conference Proceedings*, pp. 209–214, American Mathematical Society, Providence, RI, USA, 1995.

[32] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansion*, D. Reidel Publishing, Dordrecht, The Netherlands, 1974.