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Research Article

Note on q-Extensions of Euler Numbers and Polynomials of Higher Order

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In 2007, Ozden et al. constructed generating functions of higher-order twisted (h,q)-extension of Euler polynomials and numbers, by using p-adic, q-deformed fermionic integral on $\mathbb{Z}_p$. By applying their generating functions, they derived the complete sums of products of the twisted (h,q)-extension of Euler polynomials and numbers. In this paper, we consider the new q-extension of Euler numbers and polynomials to be different which is treated by Ozden et al. From our q-Euler numbers and polynomials, we derive some interesting identities and we construct q-Euler zeta functions which interpolate the new q-Euler numbers and polynomials at a negative integer. Furthermore, we study Barnes-type q-Euler zeta functions. Finally, we will derive the new formula for "sums of products of q-Euler numbers and polynomials" by using fermionic p-adic, q-integral on $\mathbb{Z}_p$.

1. Introduction and notations

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. In this paper, we use the following notation:

$$[x]_q = [x:q] = \frac{1-q^x}{1-q} \tag{1.1}$$

(cf. [1–5, 22]).

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned} \mathrm{X} &= \varprojlim_{N}\left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \\ \mathrm{X}^* &= \bigcup_{\substack{0<a<d_p \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \left\{x \in \mathrm{X} \mid x \equiv a \left(\bmod dp^N\right)\right\}, \end{aligned} \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N ,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q} \tag{1.3}$$

is known to be a distribution on X (cf. [1–20]). From this distribution, we derive the p-adic, q-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{\left[p^N\right]_q}\sum_{x=0}^{p^N-1} q^x f(x), \quad f \in UD(\mathbb{Z}_p), \tag{1.4}$$

see [1–23].

Higher-order twisted Bernoulli and Euler numbers and polynomials are studied by many authors (see for detail [1–21]). In [14] Ozden et al. constructed generating functions of higher-order twisted (h,q)-extension of Euler polynomials and numbers, by using p-adic, q-deformed fermionic integral on $\mathbb{Z}_p$. By applying their generating functions, they derived the complete sums of products of the twisted (h,q)-extension of Euler polynomials and numbers, see [14, 15]. In this paper, we consider the new q-extension of Euler numbers and polynomials to be different which is treated by Ozden et al. From our q-Euler numbers and polynomials, we derive some interesting identities and we construct q-Euler zeta functions which interpolate the new q-Euler numbers and polynomials at a negative integer. Furthermore, we study Barnes-type q-Euler zeta functions. Finally, we will derive the new formula for "sums of products of q-Euler numbers and polynomials" by using fermionic p-adic, q-integral on $\mathbb{Z}_p$.

2. q-extension of Euler numbers

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Now we consider the q-extension of Euler polynomials as follows:

$$F_q(x,t) = \frac{[2]_q}{qe^t + 1}e^{xt} = \sum_{n=0}^{\infty}\frac{E_{n,q}(x)}{n!}t^n, \quad |t + \log q| < \pi. \tag{2.1}$$

Note that

$$\lim_{q\to 1} F_q(x,t) = F(x,t) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}\frac{E_n(x)}{n!}t^n. \tag{2.2}$$

In the special case $x = 0$, the q-Euler polynomial $E_{n,q}(0) = E_{n,q}$ will be called q-Euler numbers. It is easy to see that $F_q(x,t)$ is analytic function in $\mathbb{C}$. Hence we have

$$\sum_{n=0}^{\infty}\frac{E_{n,q}(x)}{n!}t^n = \frac{[2]_q}{qe^t+1}e^{xt} = [2]_q\sum_{n=0}^{\infty}(-1)^n q^n e^{(n+x)t}. \tag{2.3}$$

If we take the kth derivative at $t = 0$ on both sides in (2.3), then we have

$$E_{k,q}(x) = [2]_q\sum_{n=0}^{\infty}(-1)^n q^n (n+x)^k. \tag{2.4}$$

From (2.4) we can define q-zeta function which interpolating q-Euler numbers at negative integer as follows.

For $s \in \mathbb{C}$, we define

$$\zeta_q(s,x) = [2]_q\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(n+x)^s}, \quad s \in \mathbb{C}. \tag{2.5}$$

Note that $\zeta_q(s,x)$ is analytic in complex s-plane. If we take $s = -k\,(k \in \mathbb{Z}_+)$, then we have $\zeta_q(-k,x) = E_{k,q}(x)$.

By (2.4) and (2.5), we obtain the following.

Theorem 2.1. *For $k \in \mathbb{Z}_+$,*

$$E_{k,q}(x) = [2]_q\sum_{n=0}^{\infty}(-1)^n q^n (n+x)^k. \tag{2.6}$$

Let $F_q(0,t) = F_q(t)$. Then

$$\begin{aligned}[2]_q\sum_{k=0}^{n-1}(-1)^k q^k e^{kt} &= \frac{[2]_q}{1+qe^t} - [2]_q\frac{(-1)^n q^n e^{nt}}{1+qe^t}\\ &= F_q(t) - (-1)^n q^n F_q(n,t).\end{aligned} \tag{2.7}$$

From (2.7), *derive*

$$\sum_{k=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l q^l l^k\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\left(E_{k,q} - (-1)^n q^n E_{k,q}(n)\right)\frac{t^k}{k!}. \tag{2.8}$$

By comparing the coefficients on both sides in (2.8), we obtain the following.

Theorem 2.2. *Let $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$. If $n \equiv 0 \pmod 2$, then*

$$E_{k,q} - q^n E_{k,q}(n) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l l^k. \tag{2.9}$$

If $n \equiv 1 \pmod 2$, then

$$E_{k,q} + q^n E_{k,q}(n) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l l^k. \tag{2.10}$$

For $w_1, w_2, \ldots, w_r \in \mathbb{C}$, consider the multiple q-Euler polynomials of Barnes-type as follows:

$$\begin{aligned} F_q^r(w_1, w_2, \ldots, w_r \mid x, t) &= \frac{[2]_q^r e^{xt}}{(qe^{w_1 t}+1)(qe^{w_2 t}+1)\cdots(qe^{w_r t}+1)} \\ &= \sum_{n=0}^{\infty} E_{n,q}(w_1, \ldots, w_r \mid x)\frac{t^n}{n!}, \quad \textit{where } \max_{1\le i\le r} |w_i t + \log q| < \pi. \end{aligned} \tag{2.11}$$

For $x = 0$, $E_{n,q}(w_1, \ldots, w_r \mid 0) = E_{n,q}(w_1, \ldots, w_r)$ will be called the multiple q-Euler numbers of Barnes type. It is easy to see that $F_q^r(w_1, w_2, \ldots, w_r \mid x, t)$ is analytic function in the given region. From (2.11), we derive

$$[2]_q^r \sum_{n_1,\ldots,n_r=0}^{\infty} (-q)^{\sum_{i=1}^r n_i} e^{(\sum_{i=1}^r n_i w_i + x)t} = \sum_{n=0}^{\infty} E_{n,q}(w_1, \ldots, w_r \mid x)\frac{t^n}{n!}. \tag{2.12}$$

By the kth differentiation on both sides in (2.12), we see that

$$[2]_q^r \sum_{n_1,\ldots,n_r=0}^{\infty} (-q)^{\sum_{i=1}^r n_i} \left(\sum_{i=1}^r n_i w_i + x\right)^k = E_{k,q}(w_1, \ldots, w_r \mid x). \tag{2.13}$$

From (2.12), we can derive the following Barnes-type multiple q-Euler zeta function as follows.

For $s \in \mathbb{C}$, define

$$\zeta_{r,q}(w_1, w_2, \ldots, w_r \mid s, x) = [2]_q^r \sum_{n_1,\ldots,n_r=0}^{\infty} \frac{(-1)^{n_1+\cdots+n_r} q^{n_1+\cdots+n_r}}{(n_1 w_1 + n_2 w_2 + \cdots + n_r w_r + x)^s}. \tag{2.14}$$

By (2.13) and (2.14), we obtain the following.

Theorem 2.3. *For $k \in \mathbb{Z}_+, w_1, w_2, \ldots, w_r \in \mathbb{C}$,*

$$\zeta_{r,q}(w_1, w_2, \ldots, w_r \mid -k, x) = E_{k,q}(w_1, w_2, \ldots, w_r \mid x). \tag{2.15}$$

Let χ be the primitive Drichlet character with conductor f (=odd) $\in \mathbb{N}$. Then we consider generalized Euler numbers attached to χ as follows:

$$F_{\chi,q}(t) = \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q}\frac{t^n}{n!}, \tag{2.16}$$

where $|\log q + t| < \pi/f$. The numbers $E_{n,\chi,q}$ will be called the generalized q-Euler numbers attached to χ. From (2.16), note that

$$\begin{aligned} F_{\chi,q}(t) &= \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft} + 1} \\ &= [2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) \sum_{n=0}^{\infty} q^{nf} (-1)^n e^{(a+nf)t} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{a=0}^{f-1} (-1)^{a+nf} q^{a+nf} \chi(a+nf) e^{(a+nf)t} \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \chi(n) e^{nt} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \end{aligned} \tag{2.17}$$

Thus,

$$E_{k,\chi,q} = \frac{d^k}{dt^k} F_{\chi,q}(t)\bigg|_{t=0} = [2]_q \sum_{n=1}^{\infty} (-1)^n q^n \chi(n) n^k, \quad (k \in \mathbb{N}). \tag{2.18}$$

Therefore, we can define the Dirichlet-type l-function which interpolates at negative integer as follows.

For $s \in \mathbb{C}$, we define $l_q(s,\chi)$ as

$$l_q(s,\chi) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \chi(n)}{n^s}. \tag{2.19}$$

By (2.18) and (2.19), we obtain the following.

Theorem 2.4. *For $k \in \mathbb{Z}_+$,*

$$l_q(-k,\chi) = E_{k,\chi,q}. \tag{2.20}$$

From (2.1) *and the definition of q-Euler numbers, derive*

$$\begin{aligned} F_q(t,x) &= \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} \sum_{l=0}^{\infty} \frac{x^l}{l!} t^l \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} E_{n,q} \binom{m}{n} x^{m-n} \right) \frac{t^m}{m!}. \end{aligned} \tag{2.21}$$

By (2.21) *it is shown that*

$$E_{n,q}(x) = \sum_{m=0}^{n} E_{m,q} \binom{n}{m} x^{n-m}, \quad n \in \mathbb{Z}_+. \tag{2.22}$$

For f *(*=odd*)* $\in \mathbb{N}$, *note that*

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} &= \frac{[2]_q}{qe^t+1}e^{xt}\\ &= [2]_q\frac{1}{q^f e^{ft}+1}\sum_{a=0}^{f-1}(-1)^a q^a e^{((a+x)/f)ft}\\ &= \frac{[2]_q}{[2]_{q^f}}\sum_{a=0}^{f-1}(-1)^a q^a\left(\frac{[2]_{q^f}e^{((a+x)/f)ft}}{q^f e^{ft}+1}\right)\\ &= \frac{[2]_q}{[2]_{q^f}}\sum_{a=0}^{f-1}(-1)^a q^a\sum_{n=0}^{\infty}E_{n,q^f}\left(\frac{a+x}{f}\right)\frac{f^n t^n}{n!}.\end{aligned} \tag{2.23}$$

Thus, we have the distribution relation for q-Euler polynomials as follows.

Theorem 2.5. *For* f *(*=odd*)* $\in \mathbb{N}$,

$$E_{n,q}(x) = \frac{f^n[2]_q}{[2]_{q^f}}\sum_{a=0}^{f-1}(-1)^a q^a E_{n,q^f}\left(\frac{a+x}{f}\right). \tag{2.24}$$

For $k, n \in \mathbb{N}$ *with* $n \equiv 0 \pmod 2$*, it is easy to see that*

$$\begin{aligned}[2]_q\sum_{l=0}^{n-1}(-1)^{l-1}q^l l^k &= q^n E_{k,q}(n) - E_{k,q}\\ &= q^n\sum_{m=0}^{k}\binom{k}{m}n^{k-m}E_{m,q} - E_{k,q}\\ &= q^n\sum_{m=0}^{k-1}\binom{k}{m}E_{m,q}n^{k-m} + (q^n-1)E_{k,q}.\end{aligned} \tag{2.25}$$

Therefore, we obtain the following.

Theorem 2.6. *For* $k, n \in \mathbb{N}$ *with* $n \equiv 0 \pmod 2$*,*

$$[2]_q\sum_{l=0}^{n-1}(-1)^{l-1}q^l l^k = q^n\sum_{m=0}^{k-1}\binom{k}{m}E_{m,q}n^{k-m} + (q^n-1)E_{k,q}. \tag{2.26}$$

3. Witt-type formulae on $\mathbb{Z}_p$ in p-adic number field

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. g is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $g \in UD(\mathbb{Z}_p)$ if the difference quotient

$$F_g(x,y) = \frac{g(x)-g(y)}{x-y} \tag{3.1}$$

has a limit $f'(a)$ as $(x,y)\to(a,a)$. For $g \in UD(\mathbb{Z}_p)$, an invariant p-adic, q-integral is defined as

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}g(x)q^x. \tag{3.2}$$

The fermionic p-adic, q-integral is also defined as

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-1)^x q^x \tag{3.3}$$

(see [4]).

From (3.3), we have the integral equation as follows:

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0), \quad g_1(x) = g(x+1). \tag{3.4}$$

If we take $g(x) = e^{tx}$, then we have

$$I_q(e^{tx}) = \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t+1}. \tag{3.5}$$

From (3.5), we note that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t+1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}. \tag{3.6}$$

By comparing the coefficient on both sides, we see that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = E_{n,q}, \quad n \in \mathbb{Z}_+. \tag{3.7}$$

By the same method, we see that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{3.8}$$

Hence, we have the formula of Witt's type for q-Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = E_{n,q}(x), \quad n \in \mathbb{Z}_+. \tag{3.9}$$

For $n \in \mathbb{Z}_+$, let $g_n(x) = g(x+n)$. Then we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \tag{3.10}$$

If n is odd positive integer, then we have

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l g(l). \tag{3.11}$$

Let χ be the primitive Drichlet character with conduct f (=odd) $\in \mathbb{N}$ and let $g(x) = \chi(x)e^{xt}$. From (3.5) we derive

$$\begin{aligned} I_{-q}(\chi(x)e^{xt}) &= \int_X \chi(x)e^{tx} d\mu_{-q}(x) \\ &= \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft}+1} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \end{aligned} \tag{3.12}$$

Thus, we have the Witt formula for generalized q-Euler numbers attached to χ as follows:

$$\int_X \chi(x) x^n d\mu_{-q}(x) = E_{n,\chi,q}, \quad n \geq 0. \tag{3.13}$$

4. Higher-order q-Euler numbers and polynomials

In this section we also assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Now we study on higher-order q-Euler numbers and polynomials and sums of products of q-Euler numbers. First, we try to consider the multivariate fermionic p-adic, q-integral on $\mathbb{Z}_p$ as follows:

$$\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}} e^{(a_1x_1+a_2x_2+\cdots+a_rx_r)t}e^{xt}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \frac{[2]_q^r}{(qe^{a_1t}+1)(qe^{a_2t}+1)\cdots(qe^{a_rt}+1)}e^{xt}, \tag{4.1}$$

where $a_1, a_2, \ldots, a_r \in \mathbb{Z}_p$.

From (4.1) we consider the multiple q-Euler polynomials as follows:

$$\frac{[2]_q^r}{(qe^{a_1t}+1)(qe^{a_2t}+1)\cdots(qe^{a_rt}+1)}e^{xt} = \sum_{n=0}^{\infty}E_{n,q}(a_1, a_2, \ldots, a_r \mid x)\frac{t^n}{n!}. \tag{4.2}$$

In the special case $(a_1, a_2, \ldots, a_r) = (1, 1, \ldots, 1)$, we write

$$E_{n,q}(\underbrace{a_1, \ldots, a_r}_{r\text{ times}} \mid x) = E_{n,q}^{(r)}(x). \tag{4.3}$$

For $x = 0$, the multiple q-Euler polynomials will be called as q-Euler numbers of order r .

From (4.2) we note that

$$E_{n,q}(a_1, a_2, \ldots, a_r \mid x) = \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}}(a_1x_1+\cdots+a_rx_r+x)^n\prod_{j=1}^{r}d\mu_{-q}(x_j). \tag{4.4}$$

It is easy to check that

$$E_{n,q}(a_1, a_2, \ldots, a_r \mid x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}E_{l,q}(a_1, a_2, \ldots, a_r), \tag{4.5}$$

where $E_{n,q}(a_1, a_2, \ldots, a_r) = E_{n,q}(a_1, a_2, \ldots, a_r \mid 0)$. Multinomial theorem is well known as follows:

$$\left(\sum_{j=1}^{r}x_j\right)^n = \sum_{\substack{l_1,\ldots,l_r\geq 0\\ l_1+\cdots+l_r=n}}\binom{n}{l_1, \ldots, l_r}\prod_{a=1}^{r}x_a^{l_a}, \tag{4.6}$$

where

$$\binom{n}{l_1, \ldots, l_r} = \frac{n!}{l_1!l_2!\cdots l_r!}. \tag{4.7}$$

By (4.2) and (4.6) we easily see that

$$E_{n,q}^{(r)}(x) = \sum_{m=0}^{n}\sum_{\substack{l_1,\ldots,l_r\geq 0\\ l_1+\cdots+l_r=m}}\binom{n}{m}\binom{m}{l_1, \ldots, l_r}x^{n-m}\prod_{j=1}^{r}E_{l_j,q}. \tag{4.8}$$

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